Problem 10613
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Proposed by F. J. Flanigan (USA).

Fix a positive real number ν. Find all polynomials $P(x)$ with nonnegative real coefficients such that (a) $P(0) = 0$, $P(1) = 1$, and $P(x) \leq x^\nu$ for all $x \geq 0$. (b) $P(0) = 0$, $P(1) = 1$, and $P(x) \geq x^\nu$ for all $x \geq 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.

Let $P(x) = \sum_{i=m}^n a_i x^i$ be a polynomial with nonnegative real coefficients and assume that $a_n > 0$ and $a_m > 0$. The conditions $P(0) = 0$ and $P(1) = 1$ implies that $m \geq 1$ and $\sum_{i=m}^n a_i = 1$
(a). If the inequality $P(x) \leq x^\nu$ holds for all $x \geq 0$ then necessarily

$$\lim_{x\to+\infty} \frac{P(x)}{x^\nu} = \lim_{x\to+\infty} \frac{a_n x^n}{x^\nu} \leq 1 \text{ therefore } n \leq \nu;$$

$$\lim_{x\to0^+} \frac{P(x)}{x^\nu} = \lim_{x\to0^+} \frac{a_m x^m}{x^\nu} \leq 1 \text{ therefore } \nu \leq m.$$

Hence $m = \nu = n$ and condition (a) is satisfied if and only if ν is a positive integer and $P(x) = x^\nu$.
(b). If the inequality $P(x) \geq x^\nu$ holds for all $x \geq 0$ then necessarily the differentiable function $\phi(x) = P(x) - x^\nu$ has a minimum for $x = 1$ in the open set $]0, +\infty[$. Therefore

$$\prime\phi(1) = \sum_{i=m}^n i a_i - \nu = 0$$

that is ν is a convex combination of the integers $n, \ldots, m$. On the other hand, if P is a polynomial whose coefficients satisfy $\sum_{i=m}^n i a_i = \nu$ then, by a well known inequality, we have that for $x \geq 0$

$$P(x) = \sum_{i=m}^n a_i x^i \geq x^{\sum_{i=m}^n i a_i} = x^\nu.$$

So condition (b) is satisfied if and only if $\nu \geq 1$ and $P(x)$ is a convex combination of the monomials $x^n, \ldots, x$ with coefficients $a_n, \ldots, a_1$ such that $\sum_{i=m}^n i a_i = \nu$. □