Problem 10560
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Proposed by E. Alkan (Turkey).

Consider a convex quadrilateral $ABCD$, and choose points P, Q, R and S on sides AB, BC, CD, and DA, respectively, with

$$\frac{|PA|}{|PB|} = \frac{|RD|}{|RC|} \quad \textit{and} \quad \frac{|QB|}{|QC|} = \frac{|SA|}{|SD|}.$$

Let K denote the area of $ABCD$, and let K_A, K_B, K_C, and K_D denote the areas of SAP, PBQ, QCR, and RDS, respectively.

Show that $K^4 \geq 2^{12} K_A K_B K_C K_D$ and determine a necessary and sufficient condition for equality.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

By hypothesis, there are $\lambda, \mu \in [0,1]$ such that

$$|PA| = \lambda |AB|, \;\; |RD| = \lambda |CD|, \;\; |QB| = \mu |BC|, \;\text{ and }\; |SA| = \mu |DA|.$$

Therefore,

$$K = \frac{1}{4}\left(|DA||AB| \sin\widehat{A} + |AB||BC| \sin\widehat{B} + |BC||CD| \sin\widehat{C} + |CD||DA| \sin\widehat{D}\right)$$

and

$$2^{12} K_A K_B K_C K_D = (16\mu(1-\mu)\lambda(1-\lambda)|AB||BC||CD||DA|)^2 \sin\widehat{A} \sin\widehat{B} \sin\widehat{C} \sin\widehat{D}.$$

Since $4x(1-x) \leq 1$ for $x \in [0,1]$ and equality holds iff $x = \frac{1}{2}$, it is enough to prove that

$$\frac{1}{4}\left(|DA||AB| \sin\widehat{A} + |AB||BC| \sin\widehat{B} + |BC||CD| \sin\widehat{C} + |CD||DA| \sin\widehat{D}\right)$$

is greater or equal to

$$\left(|AB|^2 |BC|^2 |CD|^2 |DA|^2 \sin\widehat{A} \sin\widehat{B} \sin\widehat{C} \sin\widehat{D}\right)^{\frac{1}{4}}.$$

This inequality holds by the Arithmetical-Geometrical-Mean inequality

$$\frac{1}{n}\sum_{i=1}^{n} s_i \geq \left(\prod_{i=1}^{n} s_i\right)^{\frac{1}{n}}.$$

where $n = 4$, $s_1 = |DA||AB| \sin\widehat{A}$, $s_2 = |AB||BC| \sin\widehat{B}$, $s_3 = |BC||CD| \sin\widehat{C}$ and $s_4 = |CD||DA| \sin\widehat{D}$. Note that s_1, s_2, s_3 and s_4 are all positive because the quadrilateral $ABCD$ is convex. Moreover equality holds iff $\lambda = \mu = \frac{1}{2}$ and $s_1 = s_2 = s_3 = s_4$, that is, when Area(AOB) = Area(COD) and Area(BOC) = Area(DOA), where O is the intersection point of the two diagonals AC and BD. These conditions are equivalent to $|AO| = |OC|$ and $|BO| = |OD|$, i.e. the convex quadrilateral is a parallelogram. □