Problem 10543
(American Mathematical Monthly, Vol.103, October 1996)

Proposed by Y. Diao (USA).

Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of those intervals. Show that S is a finite union of disjoint intervals of total length at least 1.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

If we have an odd number n of intervals, each of unit length, then S is a finite union of disjoint intervals of total length given by the following formula:

$$|S| = \sum_{i=1}^{2n} (-1)^i x_i \tag{1}$$

where $x_1 \leq x_2 \leq \ldots \leq x_{2n}$ are the extreme points of the n intervals.
Now we prove that $|S| \geq 1$ by induction on the number n of intervals. For $n = 1$ it is obvious. Assume that $|S| \geq 1$ for an odd number N and take $n = N + 2$ intervals each of unit length. Denote by $(a, a + 1)$ and $(b, b + 1)$ the two intervals such that $a \leq b$ are the two smaller left extreme points, and let $x_1 \leq x_2 \leq \ldots \leq x_{2N}$ be the extreme points of the other N intervals.
If $a + 1 \leq b$ then the result is trivial. Assume that $b < a + 1$. In this case, the list of the extreme points is:

$$a \leq b \leq x_1 \leq \ldots \leq x_s \leq a + 1 \leq x_{s+1} \leq \ldots \leq x_t \leq b + 1 \leq x_{t+1} \leq \ldots \leq x_{2N}$$

fos some $1 \leq s \leq t \leq 2N$. Let

$$y_1 = \sum_{i=1}^{s} (-1)^i x_i \ , \quad y_2 = \sum_{i=s+1}^{t} (-1)^i x_i \ , \quad y_3 = \sum_{i=t+1}^{2N} (-1)^i x_i \ ,$$

then, by (**??**) and by the induction hypothesis $y_1 + y_2 + y_3 \geq 1$.
We conclude by distinguishing four cases:

1) if s and t are even then, by (**??**),

$$|S| = -a + b + y_1 - (a + 1) - y_2 + (b + 1) + y_3 = 2[b - a - y_2] + y_1 + y_2 + y_3 \geq 1$$

because[1] $(b + 1) - (a + 1) \geq y_2 = \sum_{i=s+1}^{t} (-1)^i x_i$;

2) if s is even and t is odd then, by (**??**),

$$|S| = -a + b + y_1 - (a + 1) - y_2 - (b + 1) + y_3 = 2[-y_2 - (a + 1)] + y_1 + y_2 + y_3 \geq 1$$

because[1] $x_t - (a + 1) \geq y_2 + x_t = \sum_{i=s+1}^{t-1} (-1)^i x_i$;

[1]Note that, if an interval $I = (a, b)$ contains $I_1 = (x_1, x_2), \ldots, I_n = (x_{2n-1}, x_{2n})$ intervals such that $x_i \leq x_j$ for $i \leq j$ then $|I| \geq |I_1| + \ldots + |I_n|$, i.e. $b - a \geq \sum_{i=1}^{2n} (-1)^i x_i$.

3) if s is odd and t is even then, by (**??**),

$$|S| = -a + b + y_1 + (a+1) - y_2 + (b+1) + y_3 = 2[(b+1) - y_2] + y_1 + y_2 + y_3 \geq 1$$

because[1] $(b+1) - x_{s+1} \geq -x_{s+1} + y_2 = \sum_{i=s+2}^{t} (-1)^i x_i$;

4) if s and t are odd then, by (**??**),

$$|S| = -a + b + y_1 + (a+1) - y_2 - (b+1) + y_3 = 2[-y_2] + y_1 + y_2 + y_3 \geq 1$$

because[1] $x_t - x_{s+1} \geq -x_{s+1} - y_2 + x_t = \sum_{i=s+2}^{t-1} (-1)^i x_i$. $\square$