Problem 10542
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Let $\mathcal{C}$ be the circumcircle of a triangle $A_0B_0C_0$ and $\mathcal{I}$ the incircle. It is known that, for each point A on $\mathcal{C}$, there is a triangle ABC having $\mathcal{C}$ for circumcircle and $\mathcal{I}$ for incircle.

Show that the locus of the centroid G of triangle ABC is a circle that is traversed three times by G as A traverses $\mathcal{C}$ once, and determine the center and radius of this circle.

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Assume that the circumcircle $\mathcal{C}$ has center $O = (0,0)$ and radius $R = 1$ and the incircle $\mathcal{I}$ has center $I = (d,0)$ and radius $r > 0$.
Let ABC be a triangle having $\mathcal{C}$ for circumcircle and $\mathcal{I}$ for incircle. First, we will prove that its centroid G belongs to the circle $\mathcal{C}_{\rm G}$ centered in $O_G = (\frac{2}{3}d, 0)$ and of radius $r_G = \frac{1}{3}d^2$.
Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ then, since $G = \frac{1}{3}(A + B + C)$, we have to verify that

$$(x_1 + x_2 + x_3 - 2d)^2 + (y_1 + y_2 + y_3)^2 = 9|GO_G|^2 = d^4.$$

We start by working out the left side of the equality

$$9|GO_G|^2 = 9 - a^2 - b^2 - c^2 - 4d(x_1 + x_2 + x_3) + 4d^2,$$

where $a = |AB|$, $b = |BC|$ and $c = |CA|$.
By Ceva's theorem [2], since I is the incentre of the triangle ABC, the following equalities hold

$$\begin{cases} ax_1 + bx_2 + cx_3 = d(a + b + c) \\ ay_1 + by_2 + cy_3 = 0 \end{cases}.$$

If we sum the first equation multiplied by $(x_1+x_2+x_3)$ and the second one multiplied by $(y_1+y_2+y_3)$ we obtain that

$$d(x_1 + x_2 + x_3) = 3 - \frac{a^2(b + c) + b^2(c + a) + c^2(a + b)}{2(a + b + c)}.$$

Therefore

$$9|GO_G|^2 = -3 + \frac{-a^3 - b^3 - c^3 + a^2(b + c) + b^2(c + a) + c^2(a + b)}{(a + b + c)} + 4d^2.$$

By the law of sines ([1] p.2) and Heron's formula ([1] p.58), the area (ABC) of the triangle can be written in the following two ways

$$\begin{cases} 8(ABC) = 2abc = 4r(a + b + c) \\ 16(ABC)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 4r^2(a + b + c)^2 \end{cases}.$$

If we sum these two equations and divide the sum by $(a + b + c)$, then

$$4r + 4r^2 = \frac{-a^3 - b^3 - c^3 + a^2(b + c) + b^2(c + a) + c^2(a + b)}{(a + b + c)}.$$

Therefore, since by Euler's formula $d^2 = 1 - 2r$ ([1] p.29),

$$9|GO_G|^2 = -3 + 4r + 4r^2 + 4d^2 = -3 + 2(1 - d^2) + (1 - d^2)^2 + 4d^2 = d^4.$$

As the point A rotates around $\mathcal{C}$, the centroid $G(A)$ of the triangle ABC having $\mathcal{C}$ for circumcircle and $\mathcal{I}$ for incircle moves continuously on $\mathcal{C}_{\rm G}$ (ABC exists by Poncelet'theorem).
Therefore, $G(\widehat{P_1P_2})$ is a connected subset of $\mathcal{C}_{\rm G}$ and since

$G(P_1) = G_1 = (\frac{1}{3}(2d + d^2), 0)$, $G(P_2) = G_2 = (\frac{1}{3}(2d - d^2), 0)$
and $G(\cdot)$ is injective[1] in the arc $\widehat{P_1P_2}$, then $G(\widehat{P_1P_2})$ is equal to the upper half of the circle $\mathcal{C}_{\mathrm{G}}$. By simmetry

$G(\widehat{P_1P_2}) = -G(\widehat{P_2P_3}) = G(\widehat{P_3P_4}) = -G(\widehat{P_4P_5}) = G(\widehat{P_5P_6}) = -G(\widehat{P_6P_1})$,

hence, as A traverses $\mathcal{C}$ once, $\mathcal{C}_{\mathrm{G}}$ is traversed exactly three times by $G(A)$.
Note that, by a theorem due to Euler ([1] p.19), the circumcenter O the centroid G and the orthocenter H of the triangle ABC are collinear and $OH = 3OG$. Hence also the locus of orthocenter H is a circle with center $O_H = 3O_G$ and radius $r_H = 3r_G$. □

References

[1] H.S.M. Coxeter, S.L. Greitzer, *Geometry Revisited*, Math. Ass. of America, New York, 1967.
[2] S. Landy, *A generalization of Ceva's theorem to higher dimensions*, Amer. Math. Monthly, 95 (1988), 936-939.

[1]If $G \in \mathcal{C}_{\mathrm{G}}$ then $G(P) = G$ iff P is one of the three intersection points of the parametric curve $x(t) = 3x_G - 2(r + d\cos t)\cos t$, $y(t) = 3y_G - 2(r + d\cos t)\sin t$ with $t \in [0, 2\pi[$ (*cardioid*) and the circle $\mathcal{C}$.