Problem 10533
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On a parallelogram P construct exterior squares on the sides. The centers of these squares form a square Q_E. On the same parallelogram construct the interior squares on the sides. The centers of these squares form another square Q_I.
(a) Show that Area(Q_E) − Area(Q_I) = 2Area(P).
(b) Is there a generalization when P is replaced by an arbitrary convex quadrilateral?

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We will face directly the general case of a convex quadrilateral Q.
First note that if ABC is a counter-clockwise oriented triangle in $\mathbb{C}$ then $\text{Area}(ABC) = \frac{1}{2}\text{Re}\{i(B-A)\overline{(C-B)}\}$ $(\star)$.
Let $Q = ABCD$ be a convex counter-clockwise oriented quadrilateral in $\mathbb{C}$. If

$$z_1 = B - A, \quad z_2 = C - B, \quad z_3 = D - C, \quad z_4 = A - D,$$

then the vertices of the exterior quadrilateral $Q_E = P_1P_2P_3P_4$ and the interior quadrilateral $Q_I = P_1'P_2'P_3'P_4'$ are:

$$\begin{cases} P_1 = \frac{1-i}{2}z_1 \\ P_2 = z_1 + \frac{1-i}{2}z_2 \\ P_3 = z_1 + z_2 + \frac{1-i}{2}z_3 \\ P_4 = z_1 + z_2 + z_3 + \frac{1-i}{2}z_4 \end{cases} \qquad \begin{cases} P_1' = \frac{1+i}{2}z_1 \\ P_2' = z_1 + \frac{1+i}{2}z_1 \\ P_3' = z_1 + z_2 + \frac{1+i}{2}z_1 \\ P_4' = z_1 + z_2 + z_3 + \frac{1+i}{2}z_1 \end{cases}$$

By $(\star)$, the area of the counter-clockwise oriented quadrilateral Q_E is

$$\begin{aligned} \text{Area}(Q_E) &= \frac{1}{2}\text{Re}\{i[(P_2-P_1)\overline{(P_3-P_2)} + (P_4-P_3)\overline{(P_1-P_4)}]\} \\ &= \frac{1}{8}\text{Re}\{i[(1+i)z_1 + (1-i)z_2]\overline{[(1+i)z_2 + (1-i)z_3]} + \\ &\quad +i[(1+i)z_3 + (1-i)z_4]\overline{[(1+i)z_4 + (1-i)z_1]}\} = \\ &= \frac{1}{4}\text{Re}\{i[z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_4} + z_4\overline{z_1}]\} + \frac{1}{4}(|z_2|^2 + |z_4|^2 - 2\text{Re}\{z_1\overline{z_3}\}) \\ &= \text{Area}(Q) + \frac{1}{8}(|z_1 - z_3|^2 + |z_2 - z_4|^2). \end{aligned}$$

In a similar way, it is possible to check that the area of the clockwise oriented quadrilateral Q_I is

$$\text{Area}(Q_I) = -\text{Area}(Q) + \frac{1}{8}(|z_1 - z_3|^2 + |z_2 - z_4|^2).$$

Therefore, the following equality holds

$$\text{Area}(Q_E) - \text{Area}(Q_I) = 2\text{Area}(Q).$$

which generalizes the particular case (a) when $z_1 = -z_3$ and $z_2 = -z_4$. □