Problem 10523
(American Mathematical Monthly, Vol.103, May 1996)

Proposed by M. Bencze (Romania).

Find all sets of distinct integers $1 < a < b < c < d$ *such that* $abcd - 1$ *is exactly divisible by* $(a-1)(b-1)(c-1)(d-1)$.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

We begin by defining the map $R :]1, +\infty[^4 \to \mathbb{R}$ as

$$R(a,b,c,d) = \frac{abcd - 1}{(a-1)(b-1)(c-1)(d-1)}$$

and by proving the following lemma:

Lemma. If $1 < a' \leq a$, $1 < b' \leq b$, $1 < c' \leq c$, $1 < d' \leq d$ then

$$1 < R(a,b,c,d) \leq R(a',b',c',d').$$

Proof. The first inequality is straight:

$$R(a,b,c,d) = \frac{(a-1)bcd + bcd - 1}{(a-1)(b-1)(c-1)(d-1)} > \frac{bcd}{(b-1)(c-1)(d-1)} > 1.$$

By the Mean Value Theorem, there exists $(x,y,z,w) \in]1, +\infty[^4$ such that

$$R(a,b,c,d) - R(a',b',c',d') = \nabla R(x,y,z,w) \cdot (a-a', b-b', c-c', d-d') \leq 0$$

because the gradient of R

$$\nabla R(x,y,z,w) = \frac{R(x,y,z,w)}{xyzw-1}\left(\frac{1-yzw}{x-1}, \frac{1-xwz}{y-1}, \frac{1-xyw}{z-1}, \frac{1-xyz}{w-1}\right)$$

has all negative components.$\square$

In our problem, if $R(a,b,c,d) \in \mathbb{N}$ with $1 < a < b < c < d$ integers then only two cases are possible:
i) a,b,c,d are all even and $R(a,b,c,d) = 3$,
ii) a,b,c,d are all odd and $R(a,b,c,d) = 2$.
In fact, if there are at least an even number and an odd number among a,b,c,d then $abcd-1$ is odd and $(a-1)(b-1)(c-1)(d-1)$ is even and therefore the quotient $R(a,b,c,d)$ cannot be integer. Moreover, if a,b,c,d are all even then, since $3 < R(2,4,6,8) < 4$ and $abcd-1$ is odd, by the lemma, $R(a,b,c,d) = 3$. Otherwise, if a,b,c,d are all odd then, since $2 < R(3,5,7,9) < 3$, by the lemma, $R(a,b,c,d) = 2$.

Now we can analyse the two cases separately.

i) a, b, c, d are all even and $R(a, b, c, d) = 3$.
Since $R(4, 6, 8, 10) < 3$, by the lemma $a = 2$. Since $R(2, 8, 10, 12) < 3$, by the lemma either $b = 4$ or $b = 6$. But 3 does not divide the numerator of $R(2, 6, c, d)$ and therefore $b = 4$. It is easy to verify that

$$R(2, 4, c, d) = \frac{8cd - 1}{3(c-1)(d-1)} = 3$$

implies that

$$d = \frac{9c - 10}{c - 9} = 9 + \frac{71}{c - 9}.$$

So, since 71 is prime, c and d are positive integers iff $c = 10$ and $d = 80$.

ii) a, b, c, d are all odd and $R(a, b, c, d) = 2$.
Since $R(5, 7, 9, 11) < 2$, by the lemma $a = 3$. Moreover, one more by the lemma, since $2 < R(3, 9, 11, 13) < R(3, 9, 11, 15) < 3$ then either $b = 5$ or $b = 7$. But 3 divides the denominator of $R(2, 7, c, d)$ whereas it does not divide the numerator of $R(2, 7, c, d)$ and therefore $b = 5$. It is easy to verify that

$$R(3, 5, c, d) = \frac{15cd - 1}{8(c-1)(d-1)} = 2$$

implies that

$$d = \frac{16c - 17}{c - 16} = 16 + \frac{239}{c - 16}.$$

So, since 239 is prime, c and d are positive integers iff $c = 17$ and $d = 255$.

Hence the wanted sets are $\{2, 4, 10, 80\}$ and $\{3, 5, 17, 255\}$. □