Problem 10483
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Proposed by S. Rabinowitz (USA).

Given an odd positive integer n, let $A_1, A_2, \ldots, A_n$ be a regular n-gon with circumcircle Γ. A circle O_i with radius r is drawn externally tangent to Γ at A_i for $i = 1, 2, \ldots, n$. Let P be any point on Γ between A_n and A_1. A circle C (with any radius) is drawn externally tangent to Γ at P. Let t_i be the length of the common external tangent between the circles C and O_i. Prove that

$$\sum_{i=1}^{n}(-1)^i t_i = 0.$$

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

First, we prove the following Lemma:

Lemma. *Let Γ be a circle of radius s and center O. Draw two circles C_P and C_Q externally tangent to Γ respectively at P and at Q. Then the common external tangent between the circles C_P and C_Q is*

$$t = 2\sqrt{(s+r)(s+R)}\,|\sin(\frac{1}{2}P\widehat{O}Q)|$$

where r and R are the radii respectively of C_P and C_Q.

Proof. We can assume without loss of generality that $R \geq r$. Let A, B and D, C be the centers and the tangential points of the circles C_P and C_Q. In the trapezoid $ABCD$ the angles $B\widehat{C}D$ and $C\widehat{D}A$ are right angles and therefore

$$t^2 = |AB|^2 - (|BC| - |DA|)^2 = |AB|^2 - (R-r)^2.$$

Moreover, if we apply the Carnot's Theorem to the triangle AOB, we obtain

$$\begin{aligned} |AB|^2 &= |AO|^2 + |OB|^2 - 2|AO||OB|\cos(P\widehat{O}Q) \\ &= (s+r)^2 + (s+R)^2 - 2(s+r)(s+R)\cos(P\widehat{O}Q). \end{aligned}$$

Hence, it is easy to verify that

$$t^2 = 2(s+r)(s+R)(1-\cos(P\widehat{O}Q)) = 4(s+r)(s+R)\sin^2(\frac{1}{2}P\widehat{O}Q).$$

□

Going back to our problem, assume that the circle Γ is centered in O with radius s and the circle C has radius R. Then, applying the Lemma to each couple of circles O_i and C for $i = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n}(-1)^i t_i = 2\sqrt{(s+r)(s+R)}\sum_{i=1}^{n}(-1)^i|\sin(\frac{1}{2}P\widehat{O}A_i)|.$$

This means that it suffices to solve the proposed problem when $r = R = 0$ and $s = 1$, i. e. we have to verify that

$$\sum_{i=1}^{n}(-1)^i|\sin(\frac{1}{2}P\widehat{O}A_i)| = 0.$$

Assume that $A_1, A_2, \ldots, A_n$ are the nth-complex roots of 1, hence $A_1\widehat{O}A_i = \frac{2\pi}{n}(i-1)$ for $i = 1, 2, \ldots, n$. Let $\alpha = P\widehat{O}A_1$ and $n = 2m+1$ then $0 < \alpha < \frac{2\pi}{n}$ and

$$\begin{aligned}
\sum_{i=1}^{n}(-1)^i|\sin(\frac{1}{2}P\widehat{O}A_i)| &= \sum_{k=1}^{m}\sin(\frac{1}{2}P\widehat{O}A_{2k}) - \sum_{k=0}^{m}\sin(\frac{1}{2}P\widehat{O}A_{2k+1}) \\
&= \sum_{k=1}^{m}\sin(\frac{\alpha}{2}+\frac{\pi}{n}(2k-1)) + \sum_{k=0}^{m}\sin(\frac{\alpha}{2}+\frac{\pi}{n}2k+\pi) \\
&= \sum_{k=1}^{m}\sin(\frac{\alpha}{2}-\frac{\pi}{n}+\frac{2\pi}{n}k) + \sum_{k=0}^{m}\sin(\frac{\alpha}{2}-\frac{\pi}{n}+\frac{2\pi}{n}(k+m+1)) \\
&= \sum_{k=1}^{n}\sin(\frac{\alpha}{2}-\frac{\pi}{n}+\frac{2\pi}{n}k) = 0.
\end{aligned}$$

The last sum corresponds to the imaginary part of the sum of the complex numbers $A_1, A_2, \ldots, A_n$ rotated by $\frac{\alpha}{2} - \frac{\pi}{n}$ which is zero. □