Problem 10476
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Let X be a countable compact Hausdorff space. Prove that every continuous map $f : X \to X$ has a periodic point.

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Since X is a non-empty compact Hausdorff space, there is a non-empty closed subset X_0 such that $f(X_0) = X_0$ and the dynamical system (X_0, f) is *minimal* [1], that is for all $x \in X_0$ the forward orbit $O^+(x) = \{f^k(x) : k \in \mathbb{N}\}$ is dense in X_0. Assume by contradiction that f has not a periodic point then X_0 is not finite and therefore it is countable.

Now, we will show that there is a one-to-one map Φ of the set of sequences $\{0,1\}^{\mathbb{N}}$ into X_0 against the fact that X_0 is countable. Let $x \in X_0$ then $k \neq k'$ implies $f^k(x) \neq f^{k'}(x)$. Moreover, since X_0 is a compact Hausdorff space, any neighborhood of a point contains a closed neighborhood of the same point. We construct, by induction on the lenght $n \geq 1$, for every finite sequence $s \in \{0,1\}^n$, a closed neighborhood U_s in X_0 such that:

(1) if lenght(s) =lenght(s') and $s \neq s'$ then $U_s \cap U_{s'} = \emptyset$;

(2) if lenght(s) >lenght(s') and $s \supset s'$ then $U_s \subset U_{s'}$;

(3) $f^{k_s}(x)$ belongs to the interior of U_s for some $k_s \in \mathbb{N}$.

Here is the costruction. Since $f(x) \neq x$ there are two closed neighborhoods, U_0 of x and U_1 of $f(x)$, such that $U_0 \cap U_1 = \emptyset$. Let $k_0 = 0$ and $k_1 = 1$. Let assume that we have already constructed the closed neighborhood U_s for every sequence s of lenght $n \geq 1$ and that for this purpose we have used points of the set $\{f^{k_s}(x) : s \in \{0,1\}^n\}$. We denote by $s0$ and $s1$ the finite sequence of lenght $n+1$ such that the first n elements correspond to the n-sequence s and the last element is respectively 0 or 1. By (3), $f^{k_s}(x)$ belongs to to the interior of U_s and since by minimality the forward orbits of $f^{k_s+1}(x)$ is dense in X_0 we can choose $k > k_s$ such that $f^k(x)$ is another point of the interior of U_s. Hence there are two closed neighborhoods, U_{s0} of $f^{k_s}(x)$ and U_{s1} of $f^k(x)$ such that they are contained in U_s and $U_{s0} \cap U_{s1} = \emptyset$. Let $k_{s0} = k_s$ and $k_{s1} = k$.

The map Φ is defined as follows. Let σ be a sequence in $\{0,1\}^{\mathbb{N}}$ then the intersection $U_\sigma = \cap\{U_s : s \subset \sigma\}$ is nested by (2) and, since each U_s is a non-empty closed subset of the compact set X_0, U_σ is non-empty. By the Axiom of Choice we can assign to $\Phi(\sigma)$ an element of non-empty set U_σ. The map Φ is injective because if $\sigma \neq \sigma'$ then there are two different finite sequences s and s' of the same lenght such that $s \subset \sigma$, $s' \subset \sigma'$. Hence, by (1), $U_s \cap U_{s'} = \emptyset$ and therefore $\Phi(s) \neq \Phi(s')$.
□

[1] J. Brown, *Topological Dynamics*, Academic Press 1976, p. 46.