Problem 10469
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Proposed by J. Anglesio (France).

Let P be a point in the interior of the triangle ABC and let the lines AP, BP, CP meet the sides BC, CA, AB respectively at the points D, E, F. Let the circles on diameters BC and AD intersect at points a, a'; the circles on diameters CA and BE intersect at points b, b'; and the circles on diameters AB and CF intersect at points c, c'. Show that a, a', b, b', c, c' lie on a circle.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

Lemma. *Let Q be a point in the interior of the triangle FGH and let the lines FQ, GQ meet the sides GH, FH respectively at the points V, U. Let L and M be two points such that*

$$|HL| = |FG|,\ |FL| = |GH| \text{ and } |HM| = |FG|,\ |GM| = |FH|.$$

Let R and S be two points such that:

$$|FR| = |FL|,\ |VR| = |VM| \text{ and } |GS| = |GM|,\ |US| = |UL|.$$

Then $|QR| = |QS|$.
Proof. We can assume, without loss of generality, that all the points are in the complex plane with $F = -1$, $G = 1$ and $H = w$ with $\mathrm{Im}(w) > 0$. Then

$$U = \alpha w + (1-\alpha)(-1),\ V = \beta w + (1-\beta)(1),\ L = w-2,\ M = w+2,$$

where $\alpha, \beta \in]0, 1[$ and, since the point Q is the intersection of the lines FV and GU, it is easy to verify that

$$Q = sV + (1-s)F = tU + (1-t)G \text{ with } s = \frac{\alpha}{\alpha+\beta-\alpha\beta},\ t = \frac{\beta}{\alpha+\beta-\alpha\beta}.$$

Since $s = \frac{|FQ|}{|FV|}$, by Carnot's theorem applied first to the triangle FQR and then to the triangle FVR, we find that (this is Stewart's theorem[1])

$$\begin{aligned}|QR|^2 &= |FQ|^2 + |FR|^2 - 2|FQ||FR|\frac{|FV|^2+|FR|^2-|VR|^2}{2|FV||FR|} \\ &= s|VR|^2 + (1-s)|FR|^2 - s(1-s)|FV|^2.\end{aligned}$$

Hence, since $s\beta = s + t - 1$, we obtain by the preceding equation:

$$\begin{aligned}|QR|^2 &= s|VM|^2 + (1-s)|GH|^2 - s(1-s)|FV|^2 \\ &= s|(1+\beta) + (1-\beta)w|^2 + (1-s)|1-w|^2 - s(1-s)|(2-\beta) + \beta w|^2 \\ &= [s(1-\beta)^2 + (1-s) - s(1-s)\beta^2]|w|^2 + \\ &\quad 2[s(1-\beta^2) - (1-s) - s(1-s)(2-\beta)\beta]\mathrm{Re}(w) + \\ &\quad [s(1+\beta)^2 + (1-s) - s(1-s)(2-\beta)^2] \\ &= (s+t-2)^2|w|^2 + 2(s^2-t^2)\mathrm{Re}(w) + (s-t)^2 + 4(s+t-1).\end{aligned}$$

In the same way, since $t = \frac{|GQ|}{|GU|}$ and $t\alpha = s + t - 1$, we can find that:

$$\begin{aligned}|QS|^2 &= t|US|^2 + (1-t)|GS|^2 - t(1-t)|GU|^2 \\ &= t|UL|^2 + (1-t)|FH|^2 - t(1-t)|GU|^2 \\ &= t|(1+\alpha) - (1-\alpha)w|^2 + (1-t)|1+w|^2 - t(1-t)|(2-\alpha) - \alpha w|^2 \\ &= (s+t-2)^2|w|^2 - 2(t^2-s^2)\mathrm{Re}(w) + (s-t)^2 + 4(s+t-1).\end{aligned}$$

[1]H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, M.A.A. 1967, p. 6.

So $|QR| = |QS|$. $\square$

Solution of the proposed problem. Since the segments AD, BE and CF are concurrent, by Ceva's theorem[2]). , applied to the triangle ABC:

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1.$$

Let A', B', C', D', E' and F' be the midpoints respectively of the segments BC, CA, AB, AD, BE and CF. Therefore, $D' \in B'C'$, $E' \in C'A'$ and $F' \in A'B'$. Moreover,

$$\frac{|A'F'|}{|F'B'|}\frac{|B'D'|}{|D'C'|}\frac{|C'E'|}{|E'A'|} = 1$$

because $|A'F'| = \frac{1}{2}|FB|$, $|F'B'| = \frac{1}{2}|AF|$, $|B'D'| = \frac{1}{2}|DC|$, $|D'C'| = \frac{1}{2}|BD|$, $|C'E'| = \frac{1}{2}|EA|$ and $|E'A'| = \frac{1}{2}|CE|$. Hence, by Ceva's theorem applied to the triangle $A'B'C'$, the segments $A'D'$, $B'E'$ and $C'F'$ pass through one point, say P'.

Now, we show that a, a', b, b', c, c' lie on a circle with center at P'. Since a and a' are the intersections of two circles centered respectively at A' and at D', then $|aP'| = |a'P'|$ because $P' \in A'D'$. The same can be deduced for the couples of points b, b' and c, c'. Finally, the Lemma applied first to the triangle $A'B'C'$ and then to the triangle $B'C'A'$ with interior point P', yields that $|aP'| = |bP'| = |cP'|$. $\square$

[2] *Ibidem*, p. 4.