Problem 10454
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Proposed by H. Tamvakis (USA).

We say that a natural number n is amenable if there exist integers $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \ldots + a_n = a_1 a_2 \cdots a_n = n.$$

Find all amenable numbers.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

Theorem. *Let $\mathcal{A}$ be the set of all amenable numbers then*

$$\mathcal{A} = \{n \in \mathbb{N}^* \ : \ [n]_4 = [1]_4 \text{ or } [n]_4 = [0]_4\} \setminus \{4\}.$$

Note that $\mathcal{A}$ is closed under multiplication.

Proof. Let $n \in \mathbb{N}$.

(a) If $[n]_4 = 1$ then $n \in \mathcal{A}$ because it suffices to choose

$$a_1 = n \ \text{ and } \begin{cases} a_{1+i} = -1 & \text{for } i = 1, \ldots, \frac{n-1}{2} \\ a_{1+\frac{n-1}{2}+i} = 1 & \text{for } i = 1, \ldots, \frac{n-1}{2} \end{cases}.$$

(b) If $[n]_4 = 0$ and $n > 4$ then $n \in \mathcal{A}$. If $\frac{n}{4}$ is even then choose

$$a_1 = \frac{n}{2}, \ a_2 = 2 \ \text{ and } \begin{cases} a_{2+i} = -1 & \text{for } i = 1, \ldots, \frac{n}{4} \\ a_{2+\frac{n}{4}+i} = 1 & \text{for } i = 1, \ldots, \frac{3n}{4} - 2 \end{cases}.$$

Whereas, if $\frac{n}{4}$ is odd then choose

$$a_1 = \frac{n}{2}, \ a_2 = -2 \text{ and } \begin{cases} a_{2+i} = -1 & \text{for } i = 1, \ldots, \frac{n}{4} - 2 \\ a_{\frac{n}{4}+i} = 1 & \text{for } i = 1, \ldots, \frac{3n}{4} \end{cases}.$$

(c) $4 \notin \mathcal{A}$: if $4 = a_1 a_2 a_3 a_4$ then it is easy to verify that $a_1 + a_2 + a_3 + a_4 \neq 4$.

Now, we assume that $n \in \mathcal{A}$. Define the sets

$$\mathcal{I}^+ = \{1 \le i \le n \ : \ a_i > 0\} \ \text{ and } \ \mathcal{I}^- = \{1 \le i \le n \ : \ a_i < 0\}$$

and let u, v be their cardinalities.

Since the equation of the statement holds, then v is even and

$$\sum_{i \in \mathcal{I}^+} a_i + \sum_{i \in \mathcal{I}^-} a_i = u + v = n.$$

Hence

$$\sum_{i \in \mathcal{I}^+} a_i - u + \sum_{i \in \mathcal{I}^-} a_i + v = 2v$$

and this means that

$$[\sum_{i \in \mathcal{I}^+} (a_i - 1) + \sum_{i \in \mathcal{I}^-} (a_i + 1)]_4 = [0]_4.$$

Now, we use the above equation to complete the proof.

(d) If $[n]_4 = 2$ then $n \notin \mathcal{A}$: let $n = a_1 a_2 \cdots a_n$ then one and only one of the factors a_i is even, therefore

$$[\sum_{i \in \mathcal{I}^+} (a_i - 1) + \sum_{i \in \mathcal{I}^-} (a_i + 1)]_4 \in \{[1]_4, [3]_4\}$$

contradicting the previous equation.

(e) If $[n]_4 = 3$ then $n \notin \mathcal{A}$: let $n = a_1 a_2 \cdots a_n$ then all the factors are odd and for an odd number of them, $[a_i]_4 = 3$, thus

$$[\sum_{i \in \mathcal{I}^+} (a_i - 1) + \sum_{i \in \mathcal{I}^-} (a_i + 1)]_4 = [2]_4$$

contradicting the previous equation. □