Problem 10415
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Proposed by E. Kitchen (USA).

Let $\mathcal{A}$ be a triangle whose centroid is at the origin. Choose $k \in \mathbb{R}$, $k > 1$, and dilate one of the Napoleon triangles of $\mathcal{A}$ by a factor of $-k$ and the other by a factor of $k/(1-k)$. Prove that $\mathcal{A}$ is (simultaneously) perspective with both dilated triangles.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

Suppose that the triangle $\mathcal{A} = \triangle ABC$ is positively oriented, then:
1) the outer Napoleon triangle $\triangle A'B'C'$ is equilateral, positively oriented, with the center in the origin;
2) the inner Napoleon triangle $\triangle A''B''C''$ is equilateral, negatively oriented, with the center in the origin.
If we dilate $\triangle A'B'C'$ by $-k$ and $\triangle A''B''C''$ by $k/(1-k)$, we obtain two equilateral triangles, respectively $\triangle LMN$ and $\triangle L'M'N'$, oppositely oriented and concentric in the origin.

Let $t = \frac{1}{k} \in]0,1[$, then

$$tL + (1-t)L' = \frac{1}{k}(-kA') + (1-\frac{1}{k})(\frac{k}{1-k}A'') = -(A' + A'') = -2H = A \tag{1}$$

where H is the midpoint of the side CB. The last equality in (**??**) follows from the fact that, since the origin O is the centroid of the triangle $\mathcal{A}$, it divides the segment AH in such a way that $|AO| = 2|OH|$. Of course we can establish similar relations for the other vertices:

$$tM + (1-t)M' = B \ \ , \ \ tN + (1-t)N' = C. \tag{2}$$

We can assume that $L \neq L'$, $M \neq M'$ and $N \neq N'$, otherwise the conclusion is trivial (e.g. if $L = L'$ then, by (**??**), $A = L = L'$). Now, in order to complete the proof, we use a result proved by John E. Wetzel in his article *Converses of Napoleon's theorem* (in A. M. M., 99(1992), 339-351).

In fact, by Corollary 3 of this paper, the lines LL', MM' and NN' lie in a pencil (i.e. they are concurrent or parallel) hence, by (**??**) and (**??**), the triangles $\triangle LMN$, $\triangle ABC$ and $\triangle L'M'N'$ are simultaneously perspective. $\square$