Problem 10405
(American Mathematical Monthly, Vol.101, October 1994)

Proposed by H. Gülicher (Germany).

Let $A_1A_2A_3A_4A_5A_6$ *be a hexagon circumscribed about a conic, and form the intersectons* $P_i = A_iA_{i+2} \cap A_{i+1}A_{i+3}$ $(i = 1, \ldots, 6$ *all indices mod* $6)$*.*

Show that the P_i *are the vertices of a hexagon inscribed in a conic.*

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

By Brianchon's theorem, the three diagonals of the hexagon are concurrent in a point Q:

$$Q = A_1A_4 \cap A_2A_5 \cap A_3A_6$$

Passing to homogeneous coordinates, this means that there exist six real numbers $\alpha_1, \ldots, \alpha_6$ all different from zero such that

$$Q = \alpha_1 A_1 + \alpha_4 A_4 = \alpha_2 A_2 + \alpha_5 A_5 = \alpha_3 A_3 + \alpha_6 A_6 \tag{1}$$

From (1) we can easely define the points R, S, T as follows

$$\begin{cases} R = \alpha_2 A_2 - \alpha_6 A_6 = \alpha_3 A_3 - \alpha_5 A_5 \\ S = \alpha_3 A_3 - \alpha_1 A_1 = \alpha_4 A_4 - \alpha_6 A_6 \\ T = \alpha_4 A_4 - \alpha_2 A_2 = \alpha_5 A_5 - \alpha_1 A_1 \end{cases} \tag{2}$$

By the definition of the P_i, we have that: the points A_{i+1} and A_{i+3} belong to the line P_iP_{i+1} for $i = 1, \ldots, 6$ and all indices mod 6.

Then, by (2) we find

$$R = P_5P_6 \cap P_2P_3 \ , \ S = P_6P_1 \cap P_3P_4 \ , \ T = P_1P_2 \cap P_4P_5$$

that is R, S, T are the diagonal points of the hexagon $P_1P_2P_3P_4P_5P_6$.

But, again by (2), we have that in homogeneous coordinates

$$T = \alpha_4 A_4 - \alpha_2 A_2 = (\alpha_4 A_4 - \alpha_6 A_6) - (\alpha_2 A_2 - \alpha_6 A_6) = S - R$$

therefore R, S, T are collinear.

Hence, by the converse of the Pascal's theorem, we obtain that the hexagon $P_1P_2P_3P_4P_5P_6$ is inscribed in a conic. $\square$