Problem 10362
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Proposed by H. Liebeck and A. Osborne (UK).

Let A be a real orthogonal matrix without eigenvalue 1. Let B be obtained from A by replacing one of its rows or one of its columns by its negative. Show that B has 1 as an eigenvalue.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

Let's consider the euclidean space $\mathbb{R}^n$ with the canonical scalar product $< \cdot, \cdot >$ and define for $u \in \mathbb{R}^n$ such that $||u|| = < u, u >^{\frac{1}{2}} = 1$ the linear trasformation

$$S_u(x) \stackrel{\mathrm{d}}{=} x - 2 < u, x > u \quad \forall x \in \mathbb{R}^n.$$

It represents a reflection with respect to the hyperplane passing through the origin and orthogonal to u.

Theorem. *If A is an orthogonal trasformation without eigenvalue 1 then the trasformations $S_u \circ A$ and $A \circ S_u$ have 1 as an eigenvalue.*

Proof. Since 1 is not an eigenvalue of A, then $A - I$ is non singular. Let's define $x_o \stackrel{\mathrm{d}}{=} (A - I)^{-1} u \in \mathbb{R}^n \setminus \{0\}$ and prove that x_0 is a fixed point of $S_u \circ A$, hence an eigenvector of $S_u \circ A$ relative to the eigenvalue 1. From the definition of S_u and x_0, we have that

$$S_u A x_0 = A x_0 - 2 < (A - I)x_0, A x_0 > (A - I)x_0. \qquad (*)$$

Since A is orthogonal, we have that

$$\begin{aligned} 2 < (A - I)x_0, Ax_0 > &= < (A - I)x_0, Ax_0 > + < x_0, (A - I)^t A x_0 > \\ &= < (A - I)x_0, Ax_0 > + < x_0, (I - A)x_0 > \\ &= < (A - I)x_0, Ax_0 > + < (A - I)x_0, -x_0 > \\ &= < (A - I)x_0, (A - I)x_0 > = < u, u > = 1. \end{aligned}$$

Then, going on from $(*)$

$$S_u A x_0 = A x_0 - (A - I)x_0 = x_0.$$

Moreover, $y_0 = A x_o \in \mathbb{R}^n \setminus \{0\}$ is a fixed point and eigenvector of $S_u \circ A$ relative to the eigenvalue 1:

$$A S_u y_0 = A(S_u A x_0) = A x_0 = x_0.$$

Now, we can easily solve the proposed problem by the theorem: if we change sign to the kth row (column) of A just take $u = e_k$, the kth element of the canonical orthonormalized base of $\mathbb{R}^n$, and the new matrix B is equal to $S_{e_k} \circ A$ ($A \circ S_{e_k}$). □