Problem 10317
(American Mathematical Monthly, Vol.100, June/July 1993)

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Let $\triangle ABC$ be inscribed in the circle Γ and let A', B', C' be the midpoints of the arcs BC, CA, AB respectively.

a) Prove that the incenter of $\triangle ABC$ is the orthocentre of $\triangle A'B'C'$.

b) Prove that the pedal triangle $\triangle A'B'C'$ is homothetic to $\triangle ABC$.

Solution proposed by Roberto Tauraso, Scuola Normale Superiore, piazza dei Cavalieri, 56100 Pisa, Italy.

Let $\angle BAC = 2\alpha$; $\angle ABC = 2\beta$; $\angle BCA = 2\gamma$; O the centre of the circle Γ; O' the incentre of $\triangle ABC$. Moreover AA' and $B'C'$ meet at A''; BB' and $C'A'$ meet at B''; CC' and $A'B'$ meet at C''.

We divide the proof into 3 steps.

1) Since A' is the midpoint of the arc BC and the angle $\angle BAC$ is inscribed in the arc $\Gamma \setminus BC$, we have

$$\angle BOA' = \angle A'OC = \frac{1}{2}\angle BOC = \frac{1}{2}4\alpha = 2\alpha$$

Then the angle $\angle BAA'$ inscribed in the arc $\Gamma \setminus BA'$ is equal to α and the angle $\angle A'AC$ inscribed in the arc is $\Gamma \setminus A'C$ is equal to α. This means that AA' is the bisector of the angle $\angle BAC$; analogously BB' and CC' are the bisectors of the angles $\angle ABC$ and $\angle BCA$. Therefore AA', BB', CC', bisectors of $\triangle ABC$, meet at O' incentre of $\triangle ABC$.

2) The angles $\angle A'AC$ and $\angle A'C'C$ are inscribed in the same arc $\Gamma \setminus A'C$ then, for the first step, the angle $\angle A'C'C = \alpha$. With the same procedure we obtain:

$$\angle A'C'C = \angle BB'A' = \alpha \quad \angle B'A'A = \angle CC'B' = \beta \quad \angle C'B'B = \angle AA'C' = \gamma$$

Now $\angle A'A''C' = 180° - (\alpha + \beta + \gamma) = 90°$, therefore AA' is orthogonal to $B'C'$ and $A'A''$ is the altitude of $\triangle A'B'C'$ with respect to the side $C'B'$. in teh same way, $B'B''$ and $C'C''$ are the altitudes of $\triangle A'B'C'$ with respect to the sides $C'A'$ and $A'B'$. We have proved that O' is the orthocentre of $\triangle A'B'C'$.

3) Since the points A, A', B, B', C, C' are on the same circle Γ and AA', BB', CC' meet at O', then we have

$$|AO'||A'O'| = |BO'||B'O'| = |CO'||C'O'|$$

Moreover, from the preceding steps, $\triangle A'C''O'$ and $\triangle A''C'O'$ are similar, $\triangle B'C''O'$ and $\triangle B''C'O'$ are similar, $\triangle A'B''O'$ and $\triangle A''B'O'$ are similar then we obtain

$$|A'O'||A''O'| = |B'O'||B''O'| = |C'O'||C''O'|$$

(O' is always inside $\triangle A'B'C'$ because each angle of $\triangle A'B'C'$ is less than $\alpha + \beta + \gamma = 90°$). From these two equalities we obtain:

$$\frac{|AO'|}{|A''O'|} = \frac{|BO'|}{|B''O'|} = \frac{|CO'|}{|C''O'|}$$

This means that $\triangle ABC$ and $\triangle A''B''C''$ (the pedal triangle of $\triangle A'B'C'$) are homothetic with centre O'. $\square$