Representations of the stabilizer subgroup at the point of infinity in Diff(S^1)

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Introduction

Conformal symmetry

What is the conformal symmetry?

- Poincaré group + dilation and special conformal group (preserving the angle).
- conformal symmetry appears in string theory, statistical mechanics, massless particles.

A traditional set of mathematical objects is

- Wightman field (operator valued distribution on Minkowski space)
- Unitary representation of conformal symmetry group with spectrum condition
- the vacuum vector

Introduction

Diff(S^1) symmetry

Why do we consider Diff(S^1) symmetry?

1. consider a conformal field on 2 dimensional space parametrized (t, x).
2. some important observables (e.g. stress energy tensor) decompose into components depending only on $t + x$ or $t - x$.
3. under the assumption of dilation symmetry and spectrum condition, each component of stress energy tensor can be extended to S^1 and has a certain commutation relations (Lüscher-Mack theorem).
4. the commutation relations are same as the Lie algebra of Diff(S^1).
5. the component of stress energy tensor is Diff(S^1) covariant.

Introduction

So our minimal mathematical objects are

- Projective unitary irreducible representations of $\mathrm{Diff}(S^1)$ with spectrum condition.

We need to include representations without vacuum, since they appear in charged sectors.
Moreover, if we drop the existence of vacuum, the extension of stress energy tensor to S^1 (Lüscher-Mack theorem) is no longer true in general.

Introduction

So our minimal mathematical objects are

- Projective unitary irreducible representations of the stabilizer subgroup B_0 of the point at infinity in $\mathrm{Diff}(S^1)$ with spectrum condition.

And our main result is

Theorem

There are representations of B_0 which does not extend to $\mathit{Diff}(S^1)$.

representations of Diff(S^1)

Let U be a representation of Diff(S^1).
Diff(S^1) includes important one-parameter subgroups:

Möbius group

$$
\begin{aligned}
\text{rotation } \rho_s(z) &= e^{is}z, \ \text{for } z \in S^1 \subset \mathbb{C} \\
\text{translation } \tau_s(x) &= x + s, \ \text{for } x \in \mathbb{R} \\
\text{dilation } \delta_s(x) &= e^s x, \ \text{for } x \in \mathbb{R}
\end{aligned}
$$

Spectrum condition
$\Longleftrightarrow$ the spectrum of translation is positive
$\Longleftrightarrow$ the spectrum of rotation is positive
The rotation group is compact $\Longrightarrow$ the rotation group has the lowest eigenvector with eigenvalue h.

representations of Diff(S^1)

U is a projective representation.
$\iff U(g)U(h) = c(g,h)U(gh)$ where $c(g,h) \in S^1 \subset \mathbb{C}$
The form of $c(g,h)$ is very restricted (second cohomology).

fact

the second cohomology of Diff(S^1) is one dimensional.

$c(g,h)$ is determined by a real number c.

representations of Diff(S^1)

fact

projective unitary, positive energy, irreducible representations of Diff(S^1) are completely classified by c and h. There exists such a representation if and only if there exist natural numbers m, r, s such that

$$
\begin{aligned}
c &= 1 - \frac{6}{(m+2)(m+3)}, 0 \leq m \\
h &= \frac{\{(m+3)r - (m+2)s\}^2 - 1}{4(m+2)(m+3)}, 1 \leq s \leq r \leq m+1,
\end{aligned}
$$

or $c \geq 1$ and $h \geq 0$.

representations of Möbius group

- Möb: generated by rotation, translation, dilation
- $\mathcal{P}$: generated by translation, dilation

$$
\begin{array}{ccc}
\mathrm{Diff}(S^1) & \supset & B_0 \\
\cup & & \cup \\
\text{Möb} & \supset & \mathcal{P}
\end{array}
$$

representations of Möb $\supset \mathcal{P}$

Again Möb includes the rotation group, so any positive energy representation has a lowest energy h.

fact

the second cohomology of Möb is trivial.

fact

projective unitary, positive energy, irreducible representations of Möb is classified by the lowest eigenvalue h of rotation, and any value of $h \geq 0$ is possible.

fact

projective unitary, positive energy, irreducible representations of $\mathcal{P}$ is unique. Any restriction of irreducible representations of Möb is irreducible.

restrictions of representations of Diff(S^1) to B_0

$$\begin{array}{ccc} \mathrm{Diff}(S^1) & \supset & B_0 \\ \cup & & \cup \\ \text{Möb} & \supset & \mathcal{P} \end{array}$$

Theorem (Weiner '07)

Any restriction of representations $\pi_h^c|_{B_0}$ of Diff(S^1) is irreducible.

Theorem (Weiner '07)

Two restricted representations $\pi_h^c|_{B_0}, \pi_{h'}^{c'}|_{B_0}$ are equivalent only if $c = c'$. For $c \leq 1$, all possible $\{\pi_h^1|_{B_0}\}$ are inequivalent. For $c > 2$, some examples $\{\pi_{h_1}^c|_{B_0}, \pi_{h_2}^c|_{B_0}\}$ of equivalent restrictions are exhibited.

What about nonextendable representations of B_0?

Lie algebra of $\mathrm{Diff}(S^1)$

group/algebra	elements	operation
$\mathrm{Diff}(S^1)$	C^∞ diffeomorphisms of S^1	$f \circ g$ (composition)
$\mathrm{Vect}(S^1)$	C^∞ vector fields on S^1	$[f, g] := fg' - f'g$
Witt	$L_n(\theta) = ie^{in\theta}$	$[L_m, L_n] = (m-n)L_{m+n}$

fact

Witt is simple. In particular, the first cohomology of Witt on $\mathbb{C}$ is trivial (Witt has no nontrivial one dimensional representation).
The second cohomology of Vir on $\mathbb{C}$ is one dimensional (Witt has the unique central extension Vir).

The Virasoro algebra has the following commutation relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{Cn(n^2-1)}{12}\delta_{m,-n}$$

Lowest weight representations of Vir

fact

For any $c, h \in \mathbb{R}$ there is a representation of Vir with a contravariant sesquilinear form $\langle \cdot, \cdot \rangle$ and a lowest weight vector v such that

$$Cv = cv, L_0 v = hv, L_n v = 0 \text{ for } n > 0.$$

The sesquilinear form is positive semidefinite if and only if there exist natural numbers m, r, s such that

$$\begin{aligned} c &= 1 - \frac{6}{(m+2)(m+3)}, 0 \leq m \\ h &= \frac{\{(m+3)r - (m+2)s\}^2 - 1}{4(m+2)(m+3)}, 1 \leq s \leq r \leq m+1, \end{aligned}$$

or $c \geq 1$ and $h \geq 0$. And in these cases it integrates to a representation of $\mathrm{Diff}(S^1)$.

Lie algebra of B_0

group/algebra	elements	operation
B_0	diffeomorphisms stabilizing $\theta = 0$	$f \circ g$ (composition)
$Vect(S^1)_0$	vector fields f with $f(0) = 0$	$[f, g] := fg' - f'g$
$\mathcal{K}_0$	$K_n(\theta) = i(1 - e^{in\theta})$	Restriction of Vir

Theorem (T. '09)

The ideal structure of $\mathcal{K}_0$ is determined as an infinite sequence of ideals

$$\mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots$$

and an exceptional ideal $\mathcal{K}_{1,3} \supset \mathcal{K}_3$ and it holds that $[\mathcal{K}_n, \mathcal{K}_n] = \mathcal{K}_{2n+1}$. In particular, $\mathcal{K}_1 = [\mathcal{K}_0, \mathcal{K}_0]$ has codimension one in $\mathcal{K}_0$ and $\mathcal{K}_0$ has one dimensional representation.
The second cohomology of $\mathcal{K}_0$ on $\mathbb{C}$ is one dimensional ($\mathcal{K}_0$ has the unique central extension $\mathcal{K}$).

several representations of B_0

We can consider a factor map

$$\mathcal{K}_0 \longmapsto \mathcal{K}_0/\mathcal{K}_2$$

- $\mathcal{K}_0/\mathcal{K}_2 \simeq$ the Lie algebra of $\mathcal{P}$.
- $\mathcal{P}$ has the unique positive energy representation.
- the composition π gives rise to a positive energy representation of B_0.

π has a big kernel.

Theorem (T. in preparation)

For $c \leq 1$, $\pi \otimes \pi_h^c$ is irreducible.

These representations are not associated with stress energy tensor.

Verma modules of $\mathcal{K}$

We can define a generalization of Verma modules for $\mathcal{K}$.

Theorem (T. '09, T. in preparation)

For any $c \in \mathbb{R}$, $h, \lambda \in \mathbb{C}$, there is a representation of $\mathcal{K}$ with a contravariant sesquilinear form $\langle \cdot, \cdot \rangle$ and a lowest weight vector v such that

$$Cv = cv, K_n v = (h + n\lambda)v \text{ for } n > 0.$$

(Note that $K_n = L_0 - L_n$ in Vir. If $h \in \mathbb{R}, \lambda = 0$, this module reduces to a restriction of Vir module.)
The sesquilinear form is positive semidefinite for some values of $c \geq 1$, $h \in \mathbb{C} \setminus \mathbb{R}$, and $\lambda \in \mathbb{C}$.

These modules are expected to integrate to representations of B_0 but not to $\mathrm{Diff}(S^1)$.

What about much smaller subgroups?
Consider $\mathrm{Diff}(\mathbb{R})_c$ = diffeomorphisms of $\mathbb{R}$ with compact supports.

- $\mathrm{Diff}(\mathbb{R})_c$ is simple (Thurston '74, Mather '74).
- There is a notion of positivity of energy (Fewster and Hollands '05).

Is there any representation of $\mathrm{Diff}(\mathbb{R})_c$ other than restrictions of π_h^c?

Consider U(1)-current $J(z)$ on S^1 with $[J(z_1), J(z_2)] = i\delta'(z_1 - z_2)$ in vacuum representation.
The Fourier components of $T(z) = \frac{1}{2} : J^2 : (z)$ satisfy the Virasoro commutation relations with $c = 1, h = 0$.

representations of Diff$(\mathbb{R})_c$

Buchholz-Mack-Todorov automorphisms '90

For a smooth real function $\rho(z)$, $J(z) \mapsto J(z) + \rho(z)$ is an automorphism. If $q := \int \frac{dz}{2\pi iz}\rho(z) \neq 0$, $T(z)$ is mapped to the representation $\pi^c_{q^2}$.

Theorem (T. in preparation)

If ρ is a function on S^1 smooth except the point of infinity, divergent sufficently strongly at infinity, then the transformed $T(z)$ integrates to a representation of Diff$(\mathbb{R})_c$, but not to Diff(S^1).

summary and outlook

- several representations of B_0 and $\mathrm{Diff}(\mathbb{R})_c$ exist.
- some are not associated with stress energy tensor.

- what about vacuum representation?
- integrability and inequivalence of Verma modules? good invariant?
- characterization of extendability to $\mathrm{Diff}(S^1)$?