Examples of Wightman fields and algebraic quantum field theory

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Mathematical Quantum Field Theory

- Quantum mechanics: Hilbert space, $[Q_j, P_k] = \mathrm{i}\hbar\delta_{j,k}$. $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$, Q_j : multiplication by x_j, $P_k = -\mathrm{i}\partial_k$. Hamiltonian e.g. $H = \frac{m}{2}P^2 + \frac{1}{2}Q^2 + \frac{1}{4}Q^4$.
- Quantum field theory (QFT): $[\phi(x), \Pi(y)] = \mathrm{i}\hbar\delta(x-y)$... What is $\phi(x)$?
- Axiomatic approaches: **Wightman**, Osterwalder-Schrader, **Araki-Haag-Kastler**.
- Examples: free fields, $\mathcal{P}(\phi)_2$ models (and more "(super)renormalizable" models), some gauge theories in $d = 1+1$, ϕ^4_3 model, **conformal field theories (CFT) in $d = 1+1$, integrable models in $d = 1+1$.**
- No known interacting example in $d = 3+1$ (cf. triviality of ϕ^4_4). Constructing the Yang-Mills theory is a Millenium problem. QED is even more difficult.
- Research topics: Constructing examples, studying representations, calculating entanglement measures, curved spacetime...

What is a quantum field?

- A (scalar) **classical** field ϕ is a function on the Minkowski space. Together with its momentum Π, it satisfies a certain equation of motion.
- A **quantum** field should be an **operator-valued** object on the Minkowski space, acting on a certain Hilbert space, satisfying the "same" equation of motion.
- In the simplest case, they should also satisfy the equal time canonical commutation relations (CCR)

$$[\phi(x), \Pi(y)] = \mathrm{i}\hbar\delta(x - y)$$

  so they must be operator-valued **distributions**. For a test function f, $\phi(f)$ is an (unbounded) operator.
- Hamiltonian (of the free field) $\mathcal{H} = \frac{1}{2}\Pi(0,x)^2 + \frac{1}{2}\nabla\phi(0,x)^2 + m^2\phi(0,x)^2$
- $\phi(x)^n$ do not make sense directly ($\Longrightarrow$ renormalization)

Wightman axioms

A **Wightman field theory** on a Hilbert space $\mathcal{H}$ consists of a (family of) operator-valued distribution(s) ϕ on a dense common invariant domain $\mathscr{D}$, a unitary representation U of the Poincaré group and a vacuum $\Omega \in \mathcal{H}$ satisfying

- Locality: $[\phi(f), \phi(g)] = 0$ if f, g have spacelike separated supports.
- Covariance: $U(g)\phi(x)U(g)^* = \phi(g \cdot x)$.
- Positive energy: The spectrum of $U|_{\mathbb{R}^{d+1}}$ is contained in the future lightcone.
- Vacuum: Ω is unique s.t. $U(g)\Omega = \Omega$ and $\phi(f_1) \cdots \phi(f_n)\Omega$ span $\mathcal{H}$.

Wightman axioms

Examples from Constructive QFT:

- free fields in all d
- $\mathcal{P}(\phi)_2$ models,
  $\mathcal{H} = \frac{1}{2}\Pi(0,x)^2 + \frac{1}{2}\nabla\phi(0,x)^2 + m^2\phi(0,x)^2 + \mathcal{P}(\phi(0,x))$
- ϕ^4_3 model
- Yukawa model $d = 2, 3$
- some gauge fields $d = 2$

Some of these examples were first constructed on the **Euclidean space** and then analitically continued to the Minkowski space (the **Osterwalder-Schrader** reconstruction).

Araki-Haag-Kastler axioms

An **Araki-Haag-Kastler net** consists of a family of von Neumann algebras $\{\mathcal{A}(O)\}$, a unitary representation U of the Poincaré group and a vacuum $\Omega \in \mathcal{H}$ satisfying

- Isotony: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: If O_1 and O_2 are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.
- Covariance: $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(g \cdot O)$.
- Positive energy: The spectrum of $U|_{\mathbb{R}^{d+1}}$ is contained in the future lightcone.
- Vacuum: Ω is unique s.t. $U(g)\Omega = \Omega$ and $\bigcup_O \mathcal{A}(O)\Omega$ span $\mathcal{H}$.

Araki-Haag-Kastler axioms

Assume that a Wightman field $\phi(x)$ satisfies a technical condition (linear energy bounds). For spacetime regions O, define

$$\mathcal{A}(O) = \{\mathrm{e}^{\mathrm{i}\phi(f)} : \mathrm{supp}\, f \subset O\}'',$$

the smallest von Neumann algebra containing $\{\mathrm{e}^{\mathrm{i}\phi(f)}, \mathrm{supp}\, f \subset O\}$.

Here, for a set M of bounded operators, M' is called the **commutant** of M and it is the set of all bounded operators on $\mathcal{H}$ commuting with all elements of M. M'' is the double commutant.
Then $\mathcal{A}, U, \Omega$ satisfy the AHK axioms.

For an AHK net one can consider

- states as normalized positive functionals
- representations for states (charged, thermal)
- the Tomita-Takesaki theory (modular operator, relative entropy)

Two-dimensional chiral conformal field theory

- Models with a **large symmetry** could be more tractable.
- A relativistic QFT has the Poincaré symmetry, which preserves the Lorentz metric.
- We may consider **two-dimensional** models (one space and one time dimensions) having $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$-symmetry, acting on the lightrays $x_0 \pm x_1 = 0$.
- $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ is the conformal group (transformations of $\mathbb{R}^2$ which preserve the metric up to a function). Such models are called **conformal field theory (CFT)**.
- There are some observables invariant under $\mathrm{Diff}(\mathbb{R}) \times \iota$ or $\iota \times \mathrm{Diff}(\mathbb{R})$ (chiral observables). These $\mathcal{A}_\pm$ can be considered as QFT on $\mathbb{R}$.
- The full CFT $\mathcal{A}$ is a certain extension of a pair of chiral components $\mathcal{A}_+ \otimes \mathcal{A}_-$.

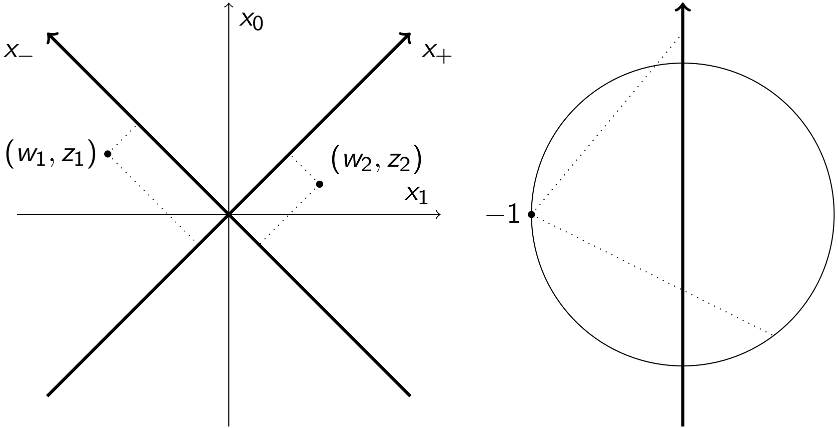


Figure: The lightray decomposition of $\mathbb{R}^{1+1}$, the stereographic projection of $S^1 \subset \mathbb{C}$ to $\mathbb{R}$,

Full 2d CFT (Bischoff, Kawahigashi, Longo, Rehren...)

- A full 2d CFT contains a left and a right chiral components, $\mathcal{A}_\pm$.
- A full theory is an **extension** $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{A}$.
- Consider the case where $\mathcal{A}_+ = \mathcal{A}_-$. Take a family Δ of irreducible representation that is closed under fusion and decomposition. In a nice situation, $\mathcal{A}$ should be constructed in such a way that the extension is given on

$$\mathcal{H} = \bigoplus_{\lambda \in \Delta} \mathcal{H}_\lambda \otimes \mathcal{H}_{\bar{\lambda}},$$

where $\lambda, \bar{\lambda}$ are "representations" of $\mathcal{A}$.

Examples: the U(1)-current

- The derivative of the 2d massless scalar field decomposes into the left and right chiral components: the $\mathrm{U}(1)$-current.
  $[J(x), J(y)] = i\delta'(x-y)$, or $[J_m, J_n] = m\delta_{m+n}$, $m, n \in \mathbb{Z}$. An **infinite-dimensional Lie algebra**.
- This algebra has the **vacuum representation** $\mathcal{H}_0$ (Bosonic Fock space):
  - the vacuum $\Omega \in \mathcal{H}_0$
  - $J_n\Omega = 0$ for all $n \geq 0$
  - $\mathcal{H}_0$ is spanned by $J_{-k_1} \cdots J_{-k_n}\Omega$, $k_j > 0$.
  - a scalar product with $J_n^* = J_{-n}$.
- In the vacuum representation, $J(z) = \sum_n z^{-n-1} J_n$, $z \in S^1 \cong \mathbb{R} \cup \{\infty\}$, or $J(f) = \sum_n f_n J_n$, $f \in C^\infty(S^1)$ defines a **Wightman field on** S^1 (operator-valued distribution).
- There is a Virasoro field (stress-energy tensor) $L(z) = \sum_n L_n z^{-2-n}$, $L_n = \frac{1}{2}\sum_k :J_{n-k}J_k:$, satisfying
  $$[L_m, L_n] = (m+n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m,-n}.$$
- $\Longrightarrow \mathrm{Diff}(S^1)$-covariance.

Examples: extensions of the U(1)-current

- The $\mathrm{U}(1)$-current admits a family of representations $\mathcal{H}_\alpha$ parametrized by $\alpha \in \mathbb{R}$, where $J_0 = \alpha$.
- Consider $\hat{\mathcal{H}} := \bigoplus_\alpha \mathcal{H}_\alpha$. $\hat{J}_n := \bigoplus_\alpha J_n$, $\hat{L}_n := \bigoplus_\alpha L_n$.
- For $\beta \in \mathbb{R}$, there is a **non-local** field $Y_\beta(z)$ acting on $\bigoplus_\alpha \mathcal{H}_\alpha$, where $Y_\beta(z) : \mathcal{H}_\alpha^{\mathrm{fin}} \mapsto \mathcal{H}_{\alpha+\beta}^{\mathrm{fin}}$.
- $E^\pm(\beta, z) = \exp\left(\mp \sum_{n>0} \frac{\beta \hat{J}_{\pm n}}{n} z^{\mp n}\right)$, $Y_\beta(z) = c_\beta E^-(z) E^+(z) z^{\beta J_0}$, where c_β is the unitary shift $\mathcal{H}_\beta \to \mathcal{H}_{\alpha+\beta}$, $c_\beta \Omega_\alpha = \Omega_{\alpha+\beta}$. βJ_0 on $\mathcal{H}_\alpha$ gives $\alpha \cdot \beta$, fields on $S^1 \setminus \{-1\} \cong \mathbb{R}$
- On $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$, we consider the product field

$$\tilde{Y}_\beta(z, w) = Y_\beta(z) \otimes Y_\beta(w).$$

  Restricts to $\bigoplus_{\alpha \in \mathbb{R}} \mathcal{H}_\alpha \otimes \mathcal{H}_\alpha$.
- $\tilde{Y}_\beta(z, w)$ is a two-dimensional conformal **Wightman field**, generates a conformal Araki-Haag-Kastler net. (Adamo-Giorgetti-T. CMP 2023, more construction arXiv:2506.01008)

More results

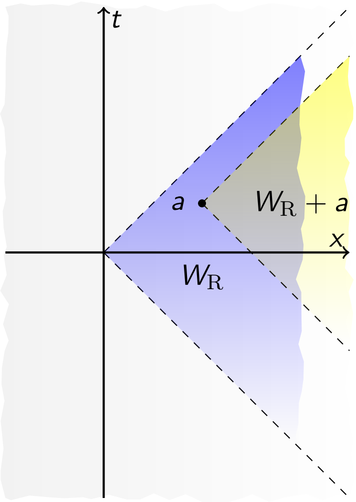
- Relations between conformal Araki-Haag-Kastler nets and **Vertex Operator Algebras** (VOA). Carpi-Kawahigashi-Longo-Weiner, Gui, Tener... **Many examples** of chiral fields.
- Relations between CFT and **subfactors**, **classification** of some classes of CFT. Böckenhauer, Evans, Doplicher, Fredenhagen, Haag, Kawahigashi, Longo, Müger, Rehren, Roberts, Schroer, Wassermann, Xu...
- From full VOA to **Osterwalder-Schrader axioms**. Adamo-Moriwaki-T. arXiv:2407.18222.

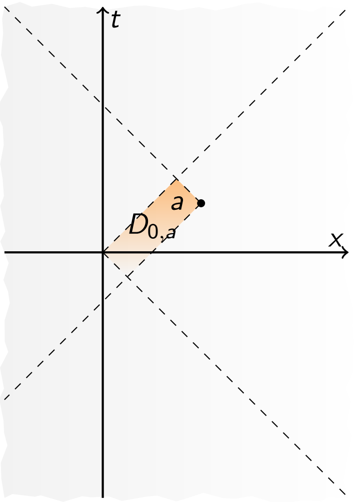
Integrable QFT

(Alazzawi, Bostelmann, Buchholz, Cadamuro, Lechner, Schroer, T., based on the form factor programme (Babujian, Karowski, Smirnov...))

- Some massive 2d QFT (sine/sinh-Gordon, Gross-Neveu, Thirring...) are believed to be integrable, and the S-matrix S is conjectured.
- One can construct the Fock space **twisted by** S, creation and annhilation operators $z^\dagger, z$.
- **non-local field**: $\phi(f) = z^\dagger(f^+) + z(f^+)$.
- $\mathcal{A}(W_\mathrm{R}) = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_\mathrm{R}\}}^{\mathrm{vN}}$ algebra for wedges.
- Prove that $\mathcal{A}(O) = \mathcal{A}(W_\mathrm{R} + a) \cap \mathcal{A}(W_\mathrm{L} + b)$ is large.
- For nice S (including those conjectured for the sinh-Gordon model), $\mathcal{A}(O)$ is a Araki-Haag-Kastler net (Lechner CMP 2008).
- (more algebraic construction T. FoM Sigma 2013).
- For most S, **no corresponding Wightman fields are known**.

Standard wedge and double cone





Outlook/open problems

- Construct the standard model?
- Construct 4d Yang-Mills? The methods of Bałaban-Dimock?