Towards lattice construction of quantum field theory

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1 Introduction

Quantum Mechanics: operators on a Hilbert space $\{Q_j\}, \{P_j\}$ $1 \le j \le n$.
$[Q_j, P_k] = i\delta_{j,k}$. Canonical Commutation Relations. (CCR).
Heisenberg eq. $\frac{d}{dt} Q_j(t) = i[H, Q_j(t)]$, $\frac{d}{dt} P_j(t) = i[H, P_j(t)]$.
$\Rightarrow Q_j(t) = e^{itH} \cdot Q_j(0) \cdot e^{-itH}$.
$H = \Sigma_j P_j^2 + V(Q_1, \cdots, Q_n)$. ex). $V(Q_1, \cdots, Q_n) = \Sigma_j Q_j^4$.

Quantum Field Theory: $\{1, 2, \cdots, n\} \to \{x\}_{x \in \mathbb{R}^d}$.
$[\phi(x), \pi(y)] = i\delta(x-y)$. operator-valued distributions.
What does $\int \phi(x)^4 dx$ mean? UV / IR divergence.

2. Axiomatic / constructive QFT.

Mathematical definitions (Wightman / Araki-Haag-Kastler / Osterwalder-Schrader)
$\phi(x)$: operator-valued distributions on $\mathbb{R}^d$ satisfying Wightman axioms
(locality, Poincaré covariance, positivity of energy, vacuum Ω).
$W_n(x_1, \cdots, x_n) = \langle \Omega, \phi(x_1) \cdots \phi(x_n)\Omega \rangle$ correlation "functions".
$G_n(\mathfrak{x}_1, \cdots, \mathfrak{x}_n) = W_n(\rho\mathfrak{x}_1, \cdots \rho\mathfrak{x}_n)$. $\rho\mathfrak{x} = (ix^0, x^1, \cdots x^{d-1})$. analytic continuations
$\{G_n\}$ satisfy the OS axioms. (symmetry, Euclidean invariance, reflection positivity, cluster

$$\phi \underset{\text{GNS}}{\overset{\text{Expectation}}{\rightleftarrows}} W \underset{\text{Analytic cont. + "linear growth"}}{\overset{\text{analytic cont.}}{\rightleftarrows}} G$$

$G_n(\mathfrak{x}_1, \cdots, \mathfrak{x}_n) = G_n(\mathfrak{x}_{\sigma(1)}, \cdots \mathfrak{x}_{\sigma(n)})$. "commutative".
Easier to construct examples, using statistical mechanics.
Relations to conformal field theory (work in progress with Adamo, Moriwaki).

3. Lattice field construction.

L^M

discrete torus.

Let L, M, N be large integers. $\mathbb{T}_M^{-N} := L^{-N}\mathbb{Z}^d / L^M \mathbb{Z}^d$
Field. $\phi : \mathbb{T}_M^{-N} \to \mathbb{R}$
Action. $S^N(\phi) = \frac{1}{L^{Nd}} \Sigma_{x \in \mathbb{T}_M^{-N}} \left(\sum_{j=1}^d (\partial_j \phi(x))^2 + \frac{\mu_N^2}{2}\phi(x)^2 + \frac{\lambda_N}{4}\phi(x)^4 + \varepsilon_N \right)$.
with some parameters $\mu_N, \lambda_N, \varepsilon_N$. $\partial_j \phi(x) = \frac{1}{L^{-N}}(\phi(x + e_j L^{-N}) - \phi(x))$
Partition function $Z^N = \int \exp(-S^N(\phi)) \mathcal{D}\phi$. $\mathcal{D}\phi = \prod_{x \in \mathbb{T}_M^{-N}} d\phi(x)$.
$\exp(-S_N(\phi))/Z^N$ defines a probability distribution on $\phi \in \mathbb{R}^{\mathbb{T}_M^{-N}}$.
Correlation function $G_n^N(\mathfrak{x}_1, \cdots \mathfrak{x}_n) = \langle \phi(\mathfrak{x}_1) \cdots \phi(\mathfrak{x}_n) \rangle = \int \phi(\mathfrak{x}_1) \cdots \phi(\mathfrak{x}_n) \exp(-S(\phi)) \mathcal{D}\phi / Z_N$.

As $N \to \infty$, we want $\{G^N_n\}$ to converge and the limit to satisfy the OS axioms.
Actually, as $N \to \infty$, we have to change $\mu_N, \lambda_N, \varepsilon_N$. "counterterms"

- If $\lambda_N = 0$, then we can take $\mu_N = \mu$, but $\varepsilon_N \nearrow \infty$.
- If $\lambda_N = \lambda > 0$, then μ_N must depend on N. (mass renormalization).
- In more complicated models, λ_N cannot be fixed.

How can we choose $\mu_N, \lambda_N, \varepsilon_N$?

4. Renormalization

Instead of increasing N, one can try to see what happens as N decreases:

$$Q\phi(y) = \frac{1}{L^d} \sum_{x \in B_y} \phi(x) \qquad L\ \boxed{y\colon}\ B_y.$$

For nice models, the probability distribution "looks like" that for ϕ, with different parameters $\mu_{N-1}, \lambda_{N-1}, \varepsilon_{N-1}$.
One can define the flow $(\mu_N, \lambda_N, \varepsilon_N) \mapsto (\mu_{N-1}, \lambda_N, \varepsilon_N)$.
From a starting value $(\mu_0, \lambda_0, \varepsilon_0)$, one should go back the flow and hope that $\{G^N_n\}$ converge.

The S^N is called the ϕ^4_d-model. Triviality for $d \geq 4$.
Millenium problem: construct the Yang-Mills correlation functions in $d=4$.
The best result so far: Balaban ~'89 UV stability. (boundedness of Z_N).
State of art? Many people left.
Dimock 2011~ Methods of Balaban applied to simpler models.
Recent results (Dybalski – Stottmeister – T.)

- σ-model for $SU(2)$. first step of renormalization.
- convergence of Z^N in ϕ^4_2.

Formalization?