Operator-algebras seminar at Tokyo, 2023.5.9

No.

Date.

Lattice Green functions (after Balaban–Dimock) joint with W. Dybalski, A. Stottmeister

arXiv: 2303.10754. cf. Balaban '81~'89, Dimock '13~'14.

1. Axiomatic / constructive QFT.

- Araki–Haag–Kastler net $\{\mathcal{A}(O)\}$ family of von Neumann algebras, $O \subset \mathbb{R}^d$ Minkowski
- Wightman fields $\phi(x)$ operator-valued distributions on $\mathbb{R}^d$

  Bisognano–Wichmann property, DHR theory $\to$ subfactors.

  Many examples in $d = 2, 3$. $P(\phi)_2$, ϕ^4_3, CFT_2, ...
- Schwinger functions (Osterwalder–Schrader axioms) on $\mathbb{R}^d$ Euclidean $x = (\mathbf{x}, t) \in \mathbb{R}^d$

  $S_{\neq}(\mathbb{R}^{nd}) = \{f : \text{test functions on } \mathbb{R}^d : f(x_1, \dots x_n) = 0 \text{ if } x_j = x_k \text{ for some } j \neq k\}$

  $S'_{\neq}(\mathbb{R}^{nd})$ = dual of $S_{\neq}(\mathbb{R}^{nd})$. distributions.

  $S_+(\mathbb{R}^{nd}) = \{f : \text{test functions on } \mathbb{R}^d : f(x_1, \dots x_n) \neq 0 \text{ only if } t_1 < t_2 < \dots < t_n\}$

  Consider a family $\{S_n\}$. $S_n \in S'_{\neq}(\mathbb{R}^{nd})$ s.t.

(OS0) Regularity. $|S_n(f_1 \otimes \cdots \otimes f_n)| \leq \alpha (n!)^\beta \prod_j \|f_j\|$, for some $\alpha, \beta > 0$, $\|\cdot\|$ Schwartz norm

(OS1) $\overline{S_n(f)} = S_n(\Theta f^*)$, $(\Theta f)(x_1, \dots x_n) = f((\mathbf{x}_1, -t_1), \dots (\mathbf{x}_n, -t_n))$.

(OS2) Invariance. $S_n(f) = S_n(f_\gamma)$, $f_\gamma(x_1, \dots x_n) = f(\gamma^{-1}x_1, \dots \gamma^{-1}x_n)$. $\gamma \in \mathbb{R}^d \rtimes SO(d)$

(OS3) Reflection positivity. For $\{f_\ell\}$, $f_\ell \in S_+(\mathbb{R}^{\ell d})$, $\sum_{m,n} S_{m+n}((\Theta f_m^*) \otimes f_n) \geq 0$.

(OS4) Locality. $S_n(f) = S_n(f_\pi)$, $f_\pi(x_1, \dots x_n) = f(x_{\pi^{-1}(1)}, \dots x_{\pi^{-1}(n)})$.

(OS5) (Clustering)

OS $\Rightarrow$ Wightman, Wightman + (linear energy bounds) $\Rightarrow$ AHK.

<u>Examples</u> $P(\phi)_2$, ϕ^4_3, $U(1)$-gauge. Schwinger functions: probability (commutative).

Operator-algebraic Euclidean theory? Modular theory? Schlingemann, Adamo, Neeb.

2. Lattice construction.

Take $\mathbb{T}_{M,N} = L^{-N}\mathbb{Z}^d / L^M \mathbb{Z}^d$. L: large, M: fixed (volume). $N \to \infty$ (L^{-N} spacing).

We want to construct for each N, $\rho_N : \mathbb{R}^{\mathbb{T}_{M,N}} \to \mathbb{R}_+$, $\phi \in \mathbb{R}^{\mathbb{T}_{M,N}}$ $\phi(x) \in \mathbb{R}$ for $x \in$

$Z^{M,N} = \int \rho_N(\phi)\, d\phi$ partition function

$S_n^{M,N}(x_1, \dots x_n) = \frac{1}{Z^{M,N}} \int \rho_N(\phi)\, \phi(x_1) \cdots \phi(x_n)\, d\phi$. correlation functions.

$$S_n^{M,N}(f_1 \otimes \cdots \otimes f_n) = \sum_{x_1, \dots x_n \in \mathbb{T}_{M,N}} f(x_1) \cdots f(x_n)\, S_n^{M,N}(x_1, \dots x_n).$$

We want to find a sequence $\{\rho_N\}$, s.t. $\{S_n^{M,N}\}$ converges as distribution.

<u>Examples</u> Free field.

$$\rho_N(\phi) = \exp\left(-\frac{1}{4} z_N L^{-Nd} \sum_{\langle x,y \rangle \in \mathbb{T}_{M,N}} (\phi(x) - \phi(y))^2 - \frac{\bar{\mu}_N}{2} L^{-Nd} \sum_{x \in \mathbb{T}_{M,N}} \phi(x)^2 \right)$$

To make the correlation functions converge, we should take $z_N = L^{-2N}$, $\bar{\mu}_N = m^2$.

$Z^{M,N} = \int \rho_N(\phi)\, d\phi$. ($\Rightarrow$ constant from Wick ordering).

Not easy to find other examples satisfying reflection positivity.

Interacting case (the ϕ^4_d-model)

$$\rho_N(\phi) = \exp\Big(-\tfrac{1}{4} z_N L^{-dk} \sum_{\langle x,y\rangle \in \mathbb{T}_{MN}} (\phi(x)-\phi(y))^2 - \frac{\mu_N}{2} L^{-d}\sum_x \phi(x)^2 - \frac{\lambda_N}{4} L^{-d} \sum_x \phi(x)^4 - \varepsilon_N\Big)$$

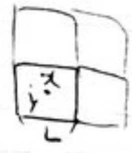
We need to fix $\mu_N, \lambda_N, z_N, \varepsilon_N$ in such a way that S_n^{MN} converge

$\Rightarrow$ Renormalization. cf. Triviality (Fröhlich, Aizenman, Duminil-Copin) $d \geq 4$.

3. Renormalization.

For fixed N, ρ_N defines a measure, $Z^{M,N}$, $S_n^{M,N}$ on $\mathbb{T}_{MN}$.

We choose the parameters in such a way that, "seen from a distance", it looks similar even if $N \to \infty$. $\begin{cases} \text{Block-spin renormalization (cf. Jones)} \\ \text{exponential averaging (Balaban, Dimock)} \end{cases}$



Let Φ_1 be a field on $\mathbb{T}_{M,N-1} = L^{-N+1}\mathbb{Z}^d / L^M \mathbb{Z}^d$.

$$\rho_1^N(\Phi_1) = \mathcal{N}_1 \int \exp\Big(-\frac{a}{2L^2}\|\Phi_1 - Q\phi\|^2\Big)\rho^N(\phi)\,d\phi, \text{ where}$$

$$(Q\phi)(x) = L^{-d}\sum_{B(x)} \phi(y).$$

Repeat this N times. $\rho_N^N(\Phi_N) \text{ "=" } \exp(-\widetilde{S}_N(\Phi_N) + E_N(\Phi_N))$

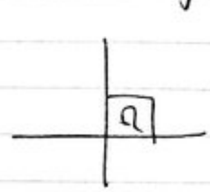
$\rho_N^N(\Phi_N)$ should look like $\rho'(\phi)$, with different parameters.

This gives the "flow" $\mu^N = \mu_0^N \mapsto \mu_1^N \mapsto - \mapsto \mu_N^N$ etc.

We should choose μ^N in such a way that μ_N^N are the interesting values

$\Rightarrow$ We need control on the flow.

4. A technical result.



Consider $L^{-N}\mathbb{Z}^d$ and a hypercube Ω with 0 on one corner

$\Omega_k = L^k\Omega \cap \Omega$. put $(Q_{\Omega,k} f)(x) = L^{-d}\sum_{y \in B(x)} f(y)$

Define $G_k(\Omega) = (-\Delta_\Omega + \bar{\mu}_k + a_k Q^*_{\Omega,k} Q_{\Omega,k})^{-1}$ on $\mathcal{L}^2(\Omega)$.

where $\bar{\mu}_k = L^{2k}\mu^N$, $a_k = \frac{1-L^{-2}}{1-L^{-2k}}$. Δ_Ω is the Laplacian with Neumann boundary cond.

<u>Thm (Balaban)</u> $|(G_k(\Omega)f)(x)| \leq C L^{2d} \exp(-C_1 d(x, \operatorname{supp} f))\|f\|_\infty$,

where C, C_1 are independent of k, L, Ω.

This is useful in controlling the flow.

$$\rho_1^N(\Phi_1) = \mathcal{N}_1^{-1}\int \exp(-(\text{quadratic}) - (\text{rest}))\,d\phi.$$

$$(\text{quadratic}) = (G_1^{-1}\phi - A)^2$$

Shift the integral and expand around $G_1 A$.

Cut G_1 into local pieces: $G_1 = \sum (h_w G_1(\Omega_w) h_w) - (\quad)$

"Random walk expansion"

$\Rightarrow$ Cluster expansion. $\rho_1^N(\Phi_1) = Z_1 \exp(-\sim)$, under the small field condition