Construction of Haag-Kastler nets for factorizing S-matrices with poles

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Algebraic quantum field theory

- A classical field is a function on the spacetime $\mathbb{R}^{d+1}$ (or a section in a principal bundle...)
- A quantum field is an operator-valued distribution $\Phi(x)$: for a test function f, $\Phi(f)$ gives an (unbounded) operator.
- For spacetime region O, let $\mathcal{A}(O) = \{e^{i\Phi(f)} : \mathrm{supp}\, f \subset O\}''$. $\Rightarrow$ Haag-Kastler net.

- The "vaccum" state $\Rightarrow$ the geometric modular action
- DHR representation of a net $\Rightarrow$ subfactor
- Entropy $\Rightarrow$ relative modular objects

Constructive QFT: old and new

- **Constructing examples is hard**. Cf. groups, von Neumann algebras.
- (Glimm, Jaffe... '70$\sim$) Start with the free field, take the interacting Hamiltonian, define the new dynamics and take a new representation of local algebras.
- 2d conformal field theory (Buchholz-Mach-Todorov '88$\sim$).
- (Lechner '08) Start with the (factorizing) S-matrix, twist the Fock space, construct first observables in wedge regions. The existence of local observables follow from the split property.

- The class of (the two-particle elastic) S-matrices treated by Lechner is analytic in the strip $\mathbb{R} + i(0, \pi)$.
- We extend the programme to certain meromorphic S-matrices. Poles should correspond to bound states.
- **Unbounded operators, analytic functions, modular theory**, quantum group symmetry?

Minkowski space

In relativistic theory, we equip $\mathbb{R}^2 \ni (x_0, x_1)$ with the indefinite metric $(x-y, x-y) = (x_0 - y_0)^2 - (x_1 - y_1)^2$.

The group of transformations of $\mathbb{R}^2$ that preserve this metric is called the Poincaré group. In the case of $\mathbb{R}^2$, its connected component of the unit element $\mathcal{P}_+^\uparrow$ is $\mathbb{R}^2 \rtimes \mathbb{R}$, where $\mathbb{R}^2$ is the group of translations and $\mathbb{R}$ is called the Lorentz boosts.

We say that two points x, y are spacelike separated if $(x-y, x-y) < 0$. The set $V_+ = \{x \in \mathbb{R}^2 : (x, x) > 0, x_0 > 0\}$ is called the future light cone.

AQFT axioms

Definition

A **Haag-Kastler net** on $\mathbb{R}^2$ is $(\mathcal{A}, U, \Omega)$, where $\{\mathcal{A}(O)\}$ is a family of von Neumann algebras parametrized by open regions in $\mathbb{R}^2$ such that

- Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: O_1 and O_2 spacelike separated $\Rightarrow [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.
- Poincaré covariance: U is a unitary representation of the Poincaré group such that $\mathrm{Ad}\, U(g)(\mathcal{A}(O)) = \mathcal{A}(gO)$.
- Positive energy: the restriction of U to translations has the spectrum contained in $\overline{V}_+$.
- Vacuum: there is a unique (up to a scalar) unit vector Ω such that $U(g)\Omega = \Omega$ and cyclic for each $\mathcal{A}(O)$.

Massive free field

The simplest example of Haag-Kastler net.
Let $m > 0$ (called the mass). An irreducible representation of the Poincaré group $\mathcal{P}_+^\uparrow$ is given on $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, for $(a, \tau) \in \mathcal{P}_+^\uparrow$,

$$(U_m(a,\tau)\Psi)(\theta) = e^{im(a_0 \cosh\theta - a_1 \sinh\theta)}\Psi(\theta - \tau).$$

$\mathcal{H}_1$ is called the one-particle space. The Hilbert space for the free field is the (bosonic) Fock space $\mathcal{F}(\mathcal{H}_1) = \bigoplus_n P_n \mathcal{H}_1^{\otimes n}$, where P_n is the symmetrization

$$(P_n\Psi_n)(\theta_1, \cdots, \theta_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \Psi_n(\theta_{\sigma(1)}, \cdots, \theta_{\sigma(n)})$$

On the Fock space $\mathcal{F}(\mathcal{H}_1)$, $\psi \in \mathcal{H}_1$, one has the creation and annihilation operators such that for $\Psi = (\Psi_0, \Psi_1, \cdots,)$

$$(a(\psi)^\dagger\Psi)_{n+1} = \sqrt{n+1}P_{n+1}(\psi\otimes\Psi_n), a(\psi) = (a(J\psi)^\dagger)^*,$$

where $J\psi(\theta) = \overline{\psi(\theta)}$.

Massive free field

$\mathcal{H}_1 = L^2(\mathbb{R}, d\theta), \mathcal{F}(\mathcal{H}_1) = \bigoplus_n P_n \mathcal{H}_1^{\otimes n}$.

For $f \in \mathscr{S}(\mathbb{R}^2)$, we define $f^+(\theta) = \tilde{f}(m\cosh\theta, m\sinh\theta)$, the restriction of the Fourier transform of f to the mass shell $(m\cosh\theta, m\sinh\theta)$.

The operator $\phi(f) = a^\dagger(f^+) + a(f^+)$ is called the free massive field. Together with the second quantization $U(g) = \bigoplus_n U_m(g)^{\otimes n}$ and the vacuum vector $\Omega \in \mathcal{H}_0$, it satisfies, among other things,

- Locality. $[\phi(f), \phi(g)] = 0$ (actually they commute strongly) if $\mathrm{supp}\, f$ and $\mathrm{supp}\, g$ are spacelike separated.
- Poincaré covariance. $\mathrm{Ad}\, U(g)(\phi(f)) = \phi(f_g)$, where $f_g(x) = f(g^{-1} \cdot x)$.

One can now construct the Haag-Kastler net, together with U and Ω, by

$$\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}''.$$

Some interacting 2d QFTs (Lechner '08)

Take a certain analytic function $S(\theta)$ called the **two-particle S-matrix**, $S : \mathbb{R} + i(0, \pi) \to \mathbb{C}$, satisfying

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) = S(\theta + \pi i), \ \ \theta \in \mathbb{R}.$$

Take a different symmetrization $P_n \Psi_n = \Psi_n$ such that

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

We can construct an analogue of Fock space $\mathcal{F}(\mathcal{H}_1)$, creation and annihilation operators $z^\dagger, z$ and the field $\phi(f) = z^\dagger(f^+) + z(f^+)$. But locality $[\phi(f), \phi(g)]$ is no longer satisfied.

We can introduce another set of creation and annihilation operators $z'^\dagger, z'$ by $(z'^\dagger(\psi)\Psi)_{n+1} = \sqrt{n+1}P_{n+1}(\Psi \otimes \psi)$.
Then, with $\phi'(g) = z'^\dagger(g^+) + z'(g^+)$, it holds that $[\phi(f), \phi'(g)] = 0$ if $\mathrm{supp}\, f$ and $\mathrm{supp}\, g$ are spacelike separated and f is on the left of g.

(Wedge-)locality

z and z' commute because they act from the left and the right. Similarly for $z^\dagger$ and $z'^\dagger$.

It holds that $Jf^+(\theta) = f^+(\theta + i\pi)$, and as S is analytic,

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) = \\ -\int d\theta\, (f^+(\theta)g^-(\theta)S(\theta_1 - \theta) - f^+(\theta + \pi i)g^-(\theta + \pi i)S(\theta_1 - \theta + \pi i)) \\ \times \Psi_1(\theta_1)$$

This vanishes by the Cauchy theorem. It works for general states.

Overview of the strategy

- $W_{\mathrm{R}} := \{a \in \mathbb{R}^2 : a_1 > |a_0|\}$.
- $\phi(f)$ generate the algebra $\mathcal{A}(W_{\mathrm{L}})$, $\phi'(f)$ generate the algebra $\mathcal{A}(W_{\mathrm{R}})$,
- For a general region $D_{a,b}$ take the intersection

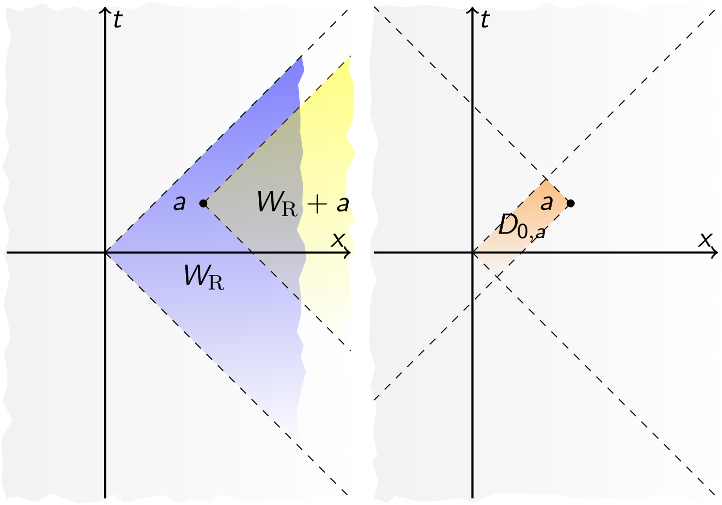
$$\mathcal{A}(D_{a,b}) = U(a)\mathcal{A}(W_{\mathrm{R}})U(a)^* \cap U(b)\mathcal{A}(W_{\mathrm{L}})U(b)^*$$

- The Haag-Kastler axioms are automatic, except that Ω is cyclic for $\mathcal{A}(D)$.
- $\mathcal{A}(D)$ is large enough if modular nuclearity or wedge-splitting holds.
- This has been done if S is analytic, $S(0) = -1$ and satisfies a certain regularity condition (Lechner '08, Alazzawi-Lechner '15).
- Examples:

$$S_\varepsilon(\theta) = \frac{\tanh \frac{1}{2}(\theta - i\epsilon)}{\tanh \frac{1}{2}(\theta + i\epsilon)}$$

with $0 < \epsilon < \pi$.

Standard wedge and double cone



S-matrix with poles

If S has a pole:

$$[\phi(f), \phi'(g)]\Psi_1(\theta_1) =$$
$$-\int d\theta \left(f^+(\theta)g^-(\theta)S(\theta_1 - \theta) - f^+(\theta + \pi i)g^-(\theta + \pi i)S(\theta_1 - \theta + \pi i)\right)$$
$$\times \Psi_1(\theta_1)$$

obtains the **residue** of S and does not vanish.

- Example (the Bullough-Dodd model): poles at $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, residues $-R, R$

$$S_\varepsilon(\theta) = \frac{\tanh\frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh\frac{1}{2}\left(\theta - \frac{(1-\varepsilon)\pi}{3}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{(1-\varepsilon)\pi i}{3}\right)} \frac{\tanh\frac{1}{2}\left(\theta - \frac{(1+\varepsilon)\pi i}{3}\right)}{\tanh\frac{1}{2}\left(\theta + \frac{(1+\varepsilon)\pi i}{3}\right)},$$

where $0 < \varepsilon < \frac{1}{2}$. $S_\varepsilon(\theta) = S_\varepsilon\left(\theta + \frac{\pi i}{3}\right) S_\varepsilon\left(\theta - \frac{\pi i}{3}\right)$.

New wedge-local field?

The one-particle bound state operator

- $\xi(\zeta)$: analytic in $\mathbb{R} + i(0, \pi)$, $\overline{\xi(\theta + \pi i)} = \xi(\theta)$ ("real").
- $\mathcal{H}_1 = L^2(\mathbb{R})$
- $\mathscr{D}_0 = H^2(-\frac{\pi}{3}, 0)$: L^2-analytic functions in $\mathbb{R} + i(-\frac{\pi}{3}, 0)$

- $(\chi_1(\xi))\Psi_1(\theta) := \xi(\theta + \frac{\pi i}{3})\Psi_1(\theta - \frac{\pi i}{3})$
- $\chi_1(\xi)$ is a symmetric operator.

The bound state operator

S: two-particle S-matrix, poles $\theta = \frac{\pi i}{3}, \frac{2\pi i}{3}$, $S(\theta) = S\left(\theta + \frac{\pi i}{3}\right) S\left(\theta - \frac{\pi i}{3}\right)$

P_n: S-symmetrization, $\mathcal{H} = \bigoplus P_n \mathcal{H}_1^{\otimes n}$, $\mathcal{H}_1 = L^2(\mathbb{R})$,

$$\mathrm{Dom}(\chi_1(\xi)) : \text{ to be defined}$$

$$(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi\left(\theta + \frac{\pi i}{3}\right)\Psi_1\left(\theta - \frac{\pi i}{3}\right), R = \mathrm{Res}_{\zeta = \frac{2\pi i}{3}} S(\zeta)$$

New observables :

$$\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\right) P_n,$$

$$\widetilde{\phi}(\xi) := \phi(\xi) + \chi(\xi) \qquad (= z^\dagger(\xi) + \chi(\xi) + z(\xi)),$$

$$\widetilde{\phi}'(\eta) := J\widetilde{\phi}(J_1\eta)J, \qquad \chi'(\eta) = J\chi(J_1\eta)J.$$

Theorem (Cadamuro-T. arXiv:1502.01313)

ξ: L^2 bounded analytic in $\mathbb{R} + i(0, \pi)$ "real", η: L^2 bounded analytic in $\mathbb{R} + i(-\pi, 0)$ "real", then $\langle \widetilde{\phi}(\xi)\Phi, \widetilde{\phi}'(\eta)\Psi \rangle = \langle \widetilde{\phi}'(\eta)\Phi, \widetilde{\phi}(\xi)\Psi \rangle$ on a dense domain.

Bound state operator

But we need that $\widetilde{\phi}(\xi)$ and $\widetilde{\phi}'(\eta)$ commute strongly ($[e^{is\widetilde{\phi}(\xi)}, e^{it\widetilde{\phi}'(\eta)}] = 0$.
Which is the right domain? Which is the right self-adjoint extension?

- $(\chi_1(\xi))\Psi_1(\theta) := \sqrt{2\pi|R|}\xi(\theta + \frac{\pi i}{3})\Psi_1(\theta - \frac{\pi i}{3})$

What are self-adjoint extensions of $\chi_1(\xi)$?

- If ξ has a zero in $\mathbb{R} + i(\frac{\pi i}{3}, \frac{2\pi i}{3})$, we may add to the domain functions with a pole there.
- **Many extensions**: $n_\pm(\chi_1(\xi)) =$ "half of the zeros" of ξ

Which is the right self-adjoint extension of $\chi_1(\xi)$?

- **Choose** $\xi = \xi_0^2$, no zeros, no singular part (Beurling decomposition). Set $\xi_+(\theta + \frac{\pi i}{3}) = \exp\left(\int d\theta\, P(\theta + \frac{2\pi i}{3}) \log\left|\xi(\theta + \frac{\pi i}{3})\right|\right)$, where $P(\theta)$ is the Poisson kernel for $\{\zeta : \frac{\pi}{3} < \mathrm{Re}\,\zeta < \frac{2\pi}{3}\}$.

  $\chi_1(\xi) := M_{\xi_+}^* \Delta_1^{\frac{1}{6}} M_{\xi_+}$ is positive, self-adjoint and a natural extension of the above, M_{ξ_+} is unitary, $(\Delta_1^{\frac{1}{6}}\Psi_1)(\theta) = \Psi_1(\theta - \frac{\pi i}{3})$.

A sufficient condition for strong commutativity

(cf. Nelson-Glimm-Jaffe "commutator theorem")

Theorem (Driessler-Fröhlich)

Let T be a positive self-adjoint operator, A, B symmetric operators on $\mathrm{Dom}(T)$ *such that for* $\Psi, \Phi \in \mathrm{Dom}(T)$

- $\|A\Psi\| \leq C\|T\Psi\|, \|B\Psi\| \leq C\|T\Psi\|$ *for* $\Psi \in \mathrm{Dom}(T)$.
- $|\langle A\Psi, T\Phi\rangle - \langle T\Psi, A\Phi\rangle| \leq C\|T\Psi\|\|\Phi\|$,
  $|\langle B\Psi, T\Phi\rangle - \langle T\Psi, B\Phi\rangle| \leq C\|T\Psi\|\|\Phi\|$.
- $|\langle A\Psi, T\Phi\rangle - \langle T\Psi, A\Phi\rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$,
  $|\langle B\Psi, T\Phi\rangle - \langle T\Psi, B\Phi\rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$.
- $\langle A\Psi, B\Phi\rangle = \langle B\Psi, A\Phi\rangle$

Then A and B strongly commute.

Towards proof of strong commutativity

Note: $\chi_1(\xi) = M^*_{\xi_+}\Delta_1^{\frac{1}{6}}M_{\xi_+}$ **have different domains for different** ξ.

$$\chi(\xi) := \bigoplus \chi_n(\xi), \qquad \chi_n(\xi) = nP_n\left(\chi_1(\xi)\otimes \mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_n$$
$$= nM^{*\otimes n}_{\xi_+}P_n\left(\Delta_1^{\frac{1}{6}}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)P_nM^{\otimes n}_{\xi_+}.$$

If $\chi(\xi)+\chi'(\eta)$ **is self-adjoint**, then...

- $\chi(\xi)+\chi'(\eta)+cN$ is self-adjoint.
- $T(\xi,\eta) := \widetilde{\phi}(\xi)+\widetilde{\phi}'(\eta)+cN$ is self-adjoint by Kato-Rellich. $(=\chi(\xi)+\chi'(\eta)+cN+\phi(\xi)+\phi'(\eta))$
- $[T(\xi,\eta),\widetilde{\phi}(\xi)] = [cN,\widetilde{\phi}(\xi)] = [cN,\phi(\xi)]$ is small, $\|\widetilde{\phi}(\xi)\Psi\| \le \|T(\xi,\eta)\Psi\|$.
- use Driessler-Fröhlich theorem (weak $\Rightarrow$ **strong commutativity**: $[e^{i\widetilde{\phi}(\xi)}, e^{i\widetilde{\phi}'(\eta)}]=0$) with $T(\xi,\eta)$ as the reference operator.

Self-adjointness of $\chi_n(\xi) + \chi'_n(\eta)$

We exhibit the proof for

$$\chi_2(\xi) \cong P_2(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})P_2 \quad \subset \quad \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_S(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_S^* \quad \text{on } \mathcal{H}_1 \otimes \mathcal{H}_1.$$

Domain: L^2-functions $\Psi(\theta_1, \theta_2)$ analytic in θ_1 in $\mathbb{R} + i(-\frac{\pi i}{3}, 0)$ and s.t. $S(\theta_1 - \theta_2)\Psi(\theta_1, \theta_2)$ analytic in θ_2 in $\mathbb{R} + i(-\frac{\pi i}{3}, 0)$.

Lemma (Kato-Rellich+)

If $A, B, A+B$ are self-adjoint, and assume that there is $\delta > 0$ such that $\mathrm{Re}\,\langle A\Psi, B\Psi\rangle > (\delta - 1)\|A\Psi\|\|B\Psi\|$ for $\Psi \in \mathrm{Dom}(A+B)$. If T is a symmetric operator such that $\mathrm{Dom}(A) \subset \mathrm{Dom}(T)$ and $\|T\Psi\|^2 < \delta\|A\Psi\|^2$, then $A + B + T$ is self-adjoint.

$\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}}$ is self-adjoint, Domain: L^2-functions $\Psi(\theta_1, \theta_2)$ both analytic in θ_1 and in θ_2.

Self-adjointness of $\chi_n(\xi) + \chi'_n(\eta)$

Start with $\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}}$, self-adjoint on the domain: L^2-functions $\Psi(\theta_1, \theta_2)$ both analytic in θ_1 and in θ_2.

Let x be an invertible element in $\mathcal{B}(\mathcal{H})$, A be a self-adjoint operator on $\mathcal{H}$ and assume that Ax^* is densely defined. Then xAx^* is self-adjoint.

$C(\theta_2 - \theta_1)$: function with the same poles and zeros as S in $0 < \mathrm{Im}\,(\theta_2 - \theta_1) < \frac{\pi}{3}$, bounded above/below if $-\frac{\pi i}{3} < \mathrm{Im}\,(\theta_2 - \theta_1) < 0$.
$M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^* = M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})M_C^* + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint.

Self-adjointness of $\chi_n(\xi) + \chi'_n(\eta)$

$M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + \mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^* = M_C(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})M_C^* + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint. If ε (the coupling constant in S) is small enough, and K large enough,

$\Rightarrow M_C^{\frac{k}{K}}(\Delta_1^{\frac{1}{6}} \otimes \mathbb{1})M_C^{\frac{k}{K}*} + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint by KR+.

$\Rightarrow \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_C(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_C^*$ is self-adjoint by KR+.

$\Rightarrow \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_C M_{\mathcal{J}}^{\frac{k}{K}}(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_{\mathcal{J}}^{\frac{k}{K}*} M_C^*$ is self-adjoint by KR+, where $C(\theta)\mathcal{J}(\theta) = S(\theta)$.

$\Rightarrow \Delta_1^{\frac{1}{6}} \otimes \mathbb{1} + M_S(\mathbb{1} \otimes \Delta_1^{\frac{1}{6}})M_S^*$ is self-adjoint by KR+.

For a fixed ε, $\chi_{\varepsilon_2,2}(\xi)$ is a perturbation of $\chi_{\varepsilon_1,2}(\xi)$ if $\varepsilon_2 - \varepsilon_1$ is sufficiently small (by intertwining P_{ε_1} and P_{ε_2}).

Similar arguments work for n and $\chi_n(\xi) + \chi'_n(\eta)$ (as long as $\varepsilon_2 < \frac{\pi}{6}$)) (after computations of 30 pages long...).

(sample computations of crossing terms)

$$\left\langle M_{C_\varepsilon}^{\frac{k}{K}}(\Delta_1^{\frac{1}{6}}\otimes\mathbb{1})M_{C_\varepsilon}^{\frac{k}{K}*}\Psi,\ M_{C_\varepsilon}\left(\mathbb{1}\otimes\Delta_1^{\frac{1}{6}}\right)M_{C_\varepsilon}^{*}\Psi\right\rangle$$

$$= \int d\boldsymbol{\theta}\ \overline{C_\varepsilon\left(\theta_2-\theta_1\right)^{\frac{k}{K}}}\, C_\varepsilon\left(\theta_2-\theta_1-\frac{\pi i}{3}\right)^{\frac{k}{K}}\overline{\Psi\left(\theta_1-\frac{\pi i}{3},\theta_2\right)}$$

$$\times\ C_\varepsilon\left(\theta_2-\theta_1\right)\overline{C_\varepsilon\left(\theta_2-\theta_1+\frac{\pi i}{3}\right)}\Psi\left(\theta_1,\theta_2-\frac{\pi i}{3}\right)$$

$$= \int d\boldsymbol{\theta}\ C_\varepsilon\left(\theta_2-\theta_1\right)\overline{\Psi\left(\theta_1-\frac{\pi i}{6},\theta_2-\frac{\pi i}{6}\right)}$$

$$\times\ C_\varepsilon\left(\theta_2-\theta_1\right)^{\frac{k}{K}}\ \overline{C_\varepsilon\left(\theta_2-\theta_1-\frac{\pi i}{3}\right)^{\frac{k}{K}}}C_\varepsilon\left(\theta_2-\theta_1+\frac{\pi i}{3}\right)C_\varepsilon\left(\theta_2-\theta_1\right)^{-1}$$

$$\times\ \overline{C_\varepsilon\left(\theta_2-\theta_1\right)}\Psi\left(\theta_1-\frac{\pi i}{6},\theta_2-\frac{\pi i}{6}\right)+\text{residue}$$

and the factor in the middle has positive real part, the residue is small if ε is small...

Existence of local operators: modular nuclearity

- $\mathcal{N} \subset \mathcal{M}$: inclusion of von Neumann algebras, Ω: cyclic and separating for both, Δ: the modular operator for $\mathcal{M}$.
- **Modular nuclearity** (Buchholz-D'Antoni-Longo): if the map

$$\mathcal{N} \ni A \longmapsto \Delta^{\frac{1}{4}} A\Omega \in \mathcal{H}$$

  is nuclear, then the inclusion $\mathcal{N} \subset \mathcal{M}$ is split, that is, $N \vee M' \cong N \otimes M'$.
- (sketch of proof) By assumption, the map

$$\mathcal{N} \ni \mathcal{A} \longmapsto \langle JA\Omega, \cdot\, \Omega\rangle = \langle \Delta^{\frac{1}{4}} A^*\Omega, \Delta^{\frac{1}{4}} \cdot\, \Omega\rangle \in \mathcal{M}_*$$

  is nuclear. $\langle JBJ\Omega, A\Omega\rangle = \sum \varphi_{1,n}(A)\varphi_{2,n}(B)$ and one may assume that $\varphi_{k,n}$ are normal. This defines a normal state on $\mathcal{N} \otimes \mathcal{M}'$ which is equivalent to $\mathcal{N} \vee \mathcal{M}'$.
- If $\mathcal{N} \subset \mathcal{M}$ is split, the relative commutant is large.

Towards modular nuclearity

Bisognano-Wichmann property: for $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$, Δ^{it} is Lorentz boost (follows if one assumes strong commutativity)

Choose $\xi = \xi_0^2$ as before.

Consider $\mathcal{A}(W_{\mathrm{R}} + a) \subset \mathcal{A}(W_{\mathrm{R}})$, where $a = (0, a_1)$ and the vacuum Ω.
Modular nuclearity: $\mathcal{A}(W_{\mathrm{R}}) \ni A \mapsto \Delta^{\frac{1}{4}} U(a) A\Omega \in \mathcal{H}$,

$$(\Delta^{\frac{1}{4}} U(a) A\Omega)_n(\boldsymbol{\theta}) = e^{-ia_1 \sum_k \sinh(\theta_k - \frac{\pi i}{2})} (A\Omega)_n \left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right),$$

which contains a strongly damping factor $e^{-c \sum_k \cosh\theta_k}$.

- $A\Omega$ has a bounded analytic extension.
- Represent it as a Cauchy integral.

Towards modular nuclearity

- $A\Omega$ has a bounded analytic extension.

$A \in \mathcal{A}(W_{\mathrm{R}}) \Longrightarrow A\Omega \in \mathrm{Dom}(\widetilde{\phi}(\xi)) \Longrightarrow (A\Omega)_n \in \mathrm{Dom}(\chi_n(\xi))$, where $\chi_1(\xi) = M_{\xi_+}\Delta_1^{\frac{1}{6}} M_{\xi_+}^*$.

$$
\begin{aligned}
&\langle \chi_n(\xi)(A\Omega)_n, (A\Omega)_n\rangle = n\|(\Delta_1^{\frac{1}{12}} M_{\xi_+}^* \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \cdot (A\Omega)_n\|^2 \\
&= \langle (\widetilde{\phi}(\xi) - \phi(\xi))(A\Omega)_n, (A\Omega)_n\rangle \\
&= \langle (A\xi - \phi(\xi)A\Omega)_n, (A\Omega)_n\rangle \leq 3\sqrt{n+1}\|\xi\| \cdot \|A\Omega\|^2
\end{aligned}
$$

Towards modular nuclearity

Choose a **nice** ξ so that $|\xi_+(\theta + i\lambda)| > |e^{-ia_1 \sinh \frac{\theta}{2}}|$ for $\lambda > \delta > 0$.

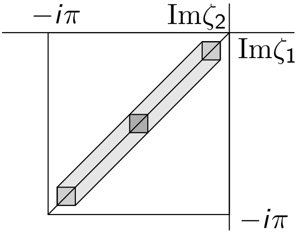
$\Longrightarrow$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{6}, \theta_2, \cdots, \theta_n\right)$ by $\|A\|$

$\Longrightarrow$ By S-symmetry and the flat tube theorem, $(U(\frac{a}{2})A\Omega)_n$ has an analytic continuation in all variables in the cube.

- $(A\Omega)_n \in \mathrm{Dom}(\Delta_n^{\frac{1}{2}}) = \mathrm{Dom}(\Delta_1^{\frac{1}{2}\otimes n})$ so it is analytic on the diagonal.
- By $\Delta^{\frac{1}{2}} A\Omega = JA^*\Omega$, $(U(\frac{a}{2})A\Omega)_n$, it is analytic on the lower cube.

$\Longrightarrow$ Estimate of $(U(\frac{a}{2})A\Omega)_n$ around $\left(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2}\right)$ by $\|A\|$

$\Longrightarrow$ nuclearity for minimal distance (Alazzawi-Lechner '17).



Towards modular nuclearity

- Represent the analytic continuiation of $A\Omega$ as a Cauchy integral.
  - $A\Omega$ is analytic in the hypercube.
  - The value $(A\Omega)_n(\theta_1 - \frac{\pi i}{2}, \cdots, \theta_n - \frac{\pi i}{2})$ can be Cauchy integral on the boundary.
  - The integral kernel is L^2, so it is a nuclear operator.
  - The esimate can be improved if $S(0) = -1$ by mapping to the fermionic Fock space.
  - The esimate can be improved if the distance between two wedges is large (Lechner '08, Alazzawi-Lechner '15).
- $\Rightarrow$ modular nuclearity $\Rightarrow$ split property $\Rightarrow$ large relative commutant (observables in double cones) $\Rightarrow$ Haag-Kastler net.

Summary

- input: two-particle factorizing S-matrix with **poles**
- **new observables** $\widetilde{\phi}(\xi) = \phi(\xi) + \chi(\xi)$
- strong commutativity + modular nuclearity $\Rightarrow$ interacting net

Open problems

- ☑ Bullough-Dodd (scalar)
- ☐ $Z(N)$-Ising, Sine-Gordon, Gross-Neveu, Toda field theories...
- Equivalence with other constructions (exponential interaction by Hoegh-Krohn): what about other examples?
  - sinh-Gordon (Hoegh-Krohn vs Lechner)
  - Federbush (Ruijsenaars vs T.)
  - sine-Gordon ((Fröhlich-)Park(-Seiler) / Bahns-Rejzner vs ??)
- Relations with CFT (scaling limit, integrable perturbation...)
- quantum group symmetry?