Continuum (free) field from lattice wavelet renormalization

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Constructing continuum quantum field from lattice

- A classical field is a function on the spacetime $\mathbb{R}^{d+1}$ (or a section in a principal bundle...)
- A quantum field is an operator-valued distribution $\Phi(x)$: for a test function f, $\Phi(f)$ gives an (unbounded) operator.
- Quantization of fields requires renormalization: If $\Phi(x)$ a quantum field (operator-valued distribution), $\Phi(x)^2$ does not make sense in general.
- Constructive QFT: regularize $\Phi(x)$ by restricting x to lattices, then $\Phi(x)$ is an operator. Then take the continuum limit.
- Osterwalder-Schrader axioms $\Rightarrow$ Wightman axioms $\Rightarrow$ Haag-Kastler axioms...
- Can one regularize quantum fields directly on the Minkowski space by lattices? **Yes**, at least for the massive free field at time zero.

Chiral conformal net

Definition

A **conformal net** on S^1 is $(\mathcal{A}, U, \Omega)$, where $\mathcal{A}$ is a map from the set of intervals in S^1 into the set of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$.
- Diffeomorphism covariance (continuous)
- Positive energy
- Vacuum

If there is already a quantum conformal field Φ, then one can take $A(I) = \{e^{i\Phi(f)} : \mathrm{supp}\, f \subset I\}''$.

Can one construct such a net from lattices?

Jones' no-go theorem

(in general one can use the planar algebra, but we consider the simplest tensor case)

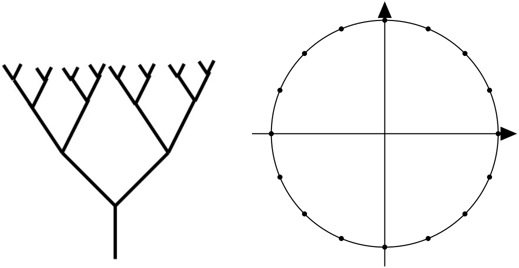


Figure: A binary tree (taken from arXiv:1607.08769 by V. Jones, CMP '17.)

Fix a vector space V, and an isometry $V \to V \otimes V$, and consider the inductive limit of $V^{\otimes 2^n} \to V^{\otimes 2^{n+1}}$ (the block spin renormalization). Think of operators acting on one component V at level n as the observables in an interval of length $\frac{2\pi}{2^n}$.

Jones' no-go theorem

In this way, the larger interval corresponds to a large algebra (isotony), and disjoint intervals commute (locality) (and covariance with respect to the Thompson group T).

But the continuity of rotations fails. "Small" rotations do not correspond to "small" unitary transformations in the inductive limit.

So, this method does not give (directly) the continuum field theory. What is wrong with it?

What can we hope from the lattices?

Instead of considering intervals, let us look at the boundary points. We have algebras on lattice points, satisfyng locality.
Suppose that we could construct the continuum field theory from the lattices, *not* in the sense of the Osterwalder-Schrader + Wightman reconstruction, but in a more direct sense.

- We can consider algebras on lattices.
- The algebras on disjoint lattice points should commute (**locality**).
- The coarser lattice algebras should embed in the finer lattice, in some way (**isotony**).
- These algebras should embed in and generate the algebras of the continuum field.
- The algebras on a finer lattice points do **not necessarily consist tensor component** of a lattice point in a coarser lattice.

Guessing the scaling limit of lattice fields

If a continuum limit could be constructed from the lattice algebras, then

- the algebras on different points should commute.
- the algebra on one point should decompose into the algebra of more points in the finer scale.
- each element in the point should be some observable in the continuum.

More concretely, if we should obtain the free field in the end, **the observables on the lattice points should be (some combinations of) the free field.**

In fact, in principle there are various ways to embedd a lattice algebra into a finer lattice algebra.

Guessing the scaling limit of lattice free fields

We consider the spacelike dimension $d \geq 1$ and take a sequence of gradually refined lattices.

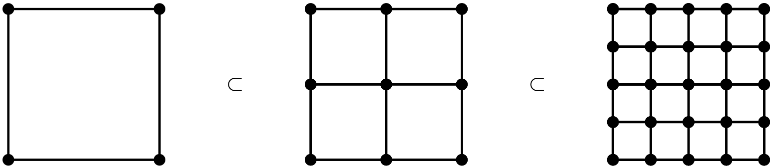


Figure: Refinement of lattices for $d = 2$.

On each lattice point there should be an algebra of observables.
When a lattice is included in a finer lattice, this algebra should be mapped to somewhere. But where?

The Weyl algebra

Let $\mathfrak{h}$ be a Hilbert space, and we call it the **one-particle space**. $\mathfrak{h}$ can be considered as a symplectic space with the symplectic form $\mathrm{Im}\,\langle\cdot,\cdot\rangle$.
A family of operators parametrized by $\xi \in \mathfrak{h}$ is said to satisfy the **canonical commutation relations** (CCR) if

$$W(\xi)W(\eta) = e^{-\frac{i}{2}\mathrm{Im}\,\langle\xi,\eta\rangle}W(\xi+\eta), \qquad \xi,\eta\in\mathfrak{h}.$$

One can consider the C^*-algebra generated by these relations, called the Weyl algebra. It is a simple C^*-algebra.
A representation of this Weyl algebra is given on the Fock space $\mathcal{F}(\mathfrak{h}) = \oplus_{n=0}^{\infty} P_n\mathfrak{h}^{\otimes n}$, where P_n is the symmetrization, and $n=0$ is spanned by Ω, and its action is given by

$$W(\xi)\Omega = e^{-\frac{1}{2}\|\xi\|^2}e^{\xi}, \qquad \text{where } \xi\in\mathfrak{h} \text{ and}$$
$$e^{\xi} = 1\oplus\xi\oplus\frac{1}{2!}\xi^{\otimes 2}\oplus\cdots\oplus\frac{1}{n!}\xi^{\otimes n}\oplus\cdots.$$

The Weyl algebra

It is convenient to take unbounded generators of the Weyl algebra: When $\mathfrak{h}$ can be written as a complexification of a real Hilbert space $\mathfrak{h} = \mathfrak{k} \oplus i\mathfrak{k}$, and $\mathfrak{h} \ni \xi = p_\xi + iq_\xi$, we write the generators as

$$W(\xi) = \exp(i\Phi(p_\xi) + i\Pi(iq_\xi))$$

Then $\Phi(p_\xi)$ and $\Phi(p_\eta)$ commute, $\Pi(iq_\xi)$ and $\Pi(iq_\eta)$ commute, while $[\Phi(p_\xi), \Pi(iq_\eta)] = i\langle p_\xi, q_\eta\rangle$ (the unbounded form of CCR).
Φ is called the **field**, while Π is called the **momentum**.

The continuum massive free field

We depart from the goal: we wish to construct the continuum field theory in **a finite volume at time zero**. Fix the "dispersion relation" $\gamma(k) = \sqrt{k^2 + m^2}$.

The one-particle space $\mathfrak{h}_{\mathrm{ct}}$ of the continuum theory is the completion of $C^\infty(\mathbb{T}^d, \mathbb{C})$, with the norm

$$\|\xi\|^2 = \sum_{k \in \mathbb{Z}^d} \left| \gamma^{-\frac{1}{2}}(k) \hat{q}_\xi(k) + i \gamma^{\frac{1}{2}}(k) \hat{p}_\xi(k) \right|^2,$$

where $\xi = p_\xi + i q_\xi$ is the real and imaginary parts, and $\hat{p}_\xi, \hat{q}_\xi$ are their Fourier transforms.

The one-particle space $\mathfrak{h}_{\mathrm{ct}}$ is the direct sum of real Sobolev spaces $H_{\mathbb{R}}^{-\frac{1}{2}}(\mathbb{T}^d) \oplus H_{\mathbb{R}}^{\frac{1}{2}}(\mathbb{T}^d)$.

On the Fock space $\mathcal{F}(\mathfrak{h}_{\mathrm{ct}})$ we have the field Φ_{ct} and the momentum Π_{ct}. Note that only functions with **some regularity** can be applied to Π_{ct}.

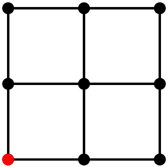
Guessing the scaling limit of lattice free fields

We consider a finite lattice Λ_N with the periodic boundary condition, that contains 2^{dN} points and embedded in the unit torus $\mathbb{T}^d$.

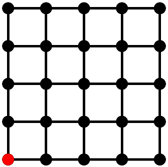
Suppose that on each lattice point we have the Weyl algebra: With the one-particle space $\mathfrak{h} = \mathbb{C}$, the Weyl algebra acts on the Fock space $\mathcal{F}(\mathfrak{h}) = \oplus_n P_n \mathfrak{h}^{\otimes n}$, where P_n is the symmetrization. The algebra on the lattice Λ_N is then the Weyl algebra based on $\mathfrak{h}_N = \ell^2(\Lambda_N)$.
If there is a reasonable continuum limit, then this should be embedded in the algebra of the continuum free field. It can be formulated at **time zero** torus $\mathbb{T}^d$ as the second quantization of $L^2(\mathfrak{h}_{\mathrm{ct}})$, on the Fock space $\mathcal{F}(\mathfrak{h}_{\mathrm{ct}})$.
We make the Ansatz that there should be a nice embedding of $\ell^2(\Lambda_N)$ into $\ell^2(\Lambda_{N+1})$.
But there are various ways to embed a lattice to a finer lattice...

Pointlike scaling (failed)

The simplest possibility is the simple inclusion $\Lambda_N \subset \Lambda_{N+1}$:



$\subset$

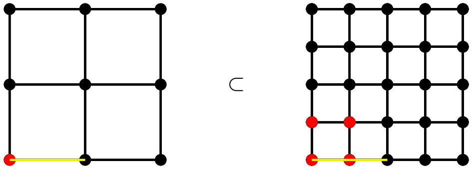


Correspondingly, we can consider the inclusion $\ell^2(\Lambda_N) \subset \ell^2(\Lambda_{N+1})$. However, in this way, the algebra on one point of $\ell^2(\Lambda_N)$ must commute with algebra of any other point. If there were the continuum limit, the algebra would be localized in one point in the continuum. On the other hand, in the Haag-Kastler axioms, there are no operators localized in one point.

$\Longrightarrow$ this does not work (this may work if one considers only correlation functions).

Block spin scaling (failed)

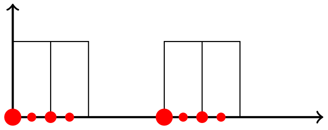
Instead, we can consider one point in a lattice as the characteristic function of the upper-right square:



This means we identify the field on one point in the lattice as the field smeared by the characteristic function.

Block spin scaling (failed)

In this way, different points on one lattice are mapped to disjoint sets of points on finer lattices. This is the same feature as that in Jones' decomposition of intervals into finer intervals.
In terms of the lattice algebra, we consider the map from $\ell^2(\Lambda_N) \to \ell^2(\Lambda_{N+1})$ where a constant function supported in one point is mapped to a constant function (with the halved value) supported in the same point and its adjacent point(s) (the dual of this mapping is known as the **block spin renormalization**).



Assume that this led to a continuum field. Then we would have to apply characteristic functions to the momentum Π_{ct}, but this is not defined: it is defined only on the Sobolev space $H^{\frac{1}{2}}_{\mathbb{R}}(\mathbb{T}^d)$.

Wavelet scaling: a generalization

But maybe some ideas can be extracted, with $d = 1$.
In the block spin case, we have

- Characteristic functions of intervals with diadic end points.
- By dilating such a characteristic function, we obtain again a characteristic function with diadic end points.
- Such characteristic functions are decomposed into finer characteristic functions.
- These characteristic functions span the space $L^2(\mathbb{T}^d)$. Furthermore,

  $$V_0 \subset \cdots \subset V_N \subset V_{N+1} \subset \cdots \subset L^2(\mathbb{T}^d),$$

  where V_N is spanned by characteristic functions with end points $p/2^N$.

This structure is known as the **multi resolution analysis**: by choosing a **scaling function** ϕ, in this case a characteristic function of the unit interval, and take the subspace generated by dilations of ϕ.

Wavelet scaling: a generalization

Let $\phi_{N,k}(x) = \phi(2^N x - k)$. A **scaling function** ϕ should satisfy:

- $\{\phi_{N,k}\}_{k\in\Lambda_N}$ for a fixed N is an orthonormal system.
- $\phi = \phi_{0,0}$ can be written as a linear combination of $\phi_{1,k}$.
- $\{\phi_{N,k}\}_{N\in\mathbb{N}_0, k\in\Lambda_N}$ span $L^2(\mathbb{T})$.

The case where $\phi_{0,0} = \chi_{[0,\frac{1}{2}]}$ is called the **Haar wavelets** (wavelets consist an orthonormal system in each V_N).

But there are other scaling functions. For any given $K \in \mathbb{N}$, there is the scaling function ${}_K\phi$ that generates **Daubechies wavelets**. ${}_K\phi \in H^{\frac{1}{2}}_{\mathbb{R}}(\mathbb{T})$ (actually in $H^{0.839}_{\mathbb{R}}(\mathbb{T})$).

The scaling function ${}_K\phi$ is supported in $[0, 2K-1]$. By scaling and normalizing appropriately, we obtain a scaling function compactly supported in $\mathbb{T}$.

Daubechies wavelets

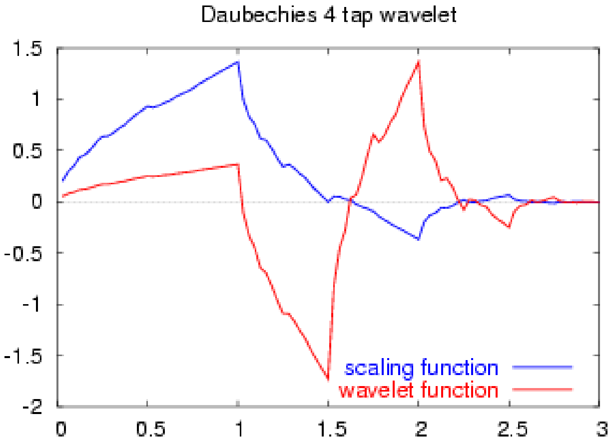


Figure: Daubechies' scaling function with $K = 2$ $(D = 4)$ (taken from commons.wikimedia.org/wiki/File:Daubechies4-functions.png created by LutzL)

Wavelet scaling

Let ϕ be a (Daubechies) scaling function.

$$\phi = \phi_{0,0} = \sum_k h_k \phi_{1,k}$$

We use these coefficients to define the scaling map $\ell^2(\Lambda_N) \to \ell^2(\Lambda_{N+1})$:

$$R^N_{N+1}\delta^{(N)}(x) = 2^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} h_n \delta^{(N+1)}_{n2^{-N-1}}(x),$$

and linearity and translation invariance. This R^N_{N+1} defines a symplectic map, and preserves the real/imaginary subspaces. In particular, if ξ and η have disjoint supports, then $\mathrm{Im}\,\langle \xi, \eta\rangle = 0$ and $\mathrm{Im}\,\langle R^N_{N+1}\xi, R^N_{N+1}\eta\rangle = 0$, even if $R^N_{N+1}\xi$ and $R^N_{N+1}\eta$ **do not have necessarily disjoint supports**.

Wavelet scaling

This amounts to identify the lattice fields $\Phi_N(k)$ and $\Pi_N(k)$ with $\Phi_{\mathrm{cn}}(\phi_{N,k}), \Pi_{\mathrm{cn}}(\phi_{N,k})$. As we take a Daubechies scaling function that is sufficiently regular, these operators make sense.
By promoting the maps R^N_{N+1} to the algebra morphisms, we obtain

Theorem

There is the embedding

$$W_N(\Lambda_N) \subset W_N(\Lambda_{N+1}) \subset \cdots \subset W_{\mathrm{ct}}(\mathbb{T}),$$

where local subalgebras are mapped to local subalgebras (with slightly extended support).

Extending this to $d > 1$ is straightforward by using the tensor product of scaling functions.

States and representations

We have assumed the existence of the continuum algebra. The case where the continuum limit is known is not the most interesting one, but we should construct it from lattices.

We should be able to do the following.

- To find the morphisms $\alpha^N_{N+1} : W_N \to W_{N+1}$
- On each lattice, there is a natural **Hamiltonian**.
- Consider the lowest energy state $\omega_0^{(N)}$ of it.
- By taking the dual of α^N_{N+1} and others, $\omega_0^{(M)}$ can be restricted to W_N if $M > N$.
- These restrictions should tend to the continuum vacuum state as $M \to \infty$.

$$
\begin{tikzcd}
 & & & & \vdots & \vdots \\
 & & \omega_0^{(N+2)} \arrow[r] \arrow[d] & \cdots & \omega_\infty^{(N+2)} & W_{N+2} \\
 & \omega_0^{(N+1)} \arrow[r] \arrow[d] & \omega_1^{(N+1)} \arrow[r] \arrow[d] & \cdots & \omega_\infty^{(N+1)} & W_{N+1} \arrow[u, "\alpha_{N+2}^{N+1}"] \\
\omega_0^{(N)} \arrow[r] & \omega_1^{(N)} \arrow[r] & \omega_2^{(N)} \arrow[r] & \cdots & \omega_\infty^{(N)} & W_N \arrow[u, "\alpha_{N+1}^{N}"]
\end{tikzcd}
$$

Figure: Wilson's triangle of renormalization in terms of algebras and states at subsequent scales.

The free vacuum states

The free Hamiltonian on the lattice Λ_N is, with $\mu_N > \sqrt{2d}$, given by

$$H_0^{(N)} = \frac{1}{2}2^{-dN}\left(\sum_{x\in\Lambda_N}\Pi_{N|x}^2 + \mu_N^2 2^{2N}\Phi_{N|x}^2 - 2\sum_{\substack{x,y\in\Lambda_N \\ \text{adjacent}}} 2^{2N}\Phi_{N|x}\Phi_{N|y}\right).$$

This can be diagonalized by the Fourier transform. Its ground state is given by

$$\omega_{\mu_N}(W_N(\xi)) = e^{-\frac{1}{4}\left(\left|\left|\gamma_{\mu_N}^{-1/2}\hat{q}\right|\right|^2_{\mathfrak{h}_{N,L}} + \left|\left|\gamma_{\mu_N}^{1/2}\hat{p}\right|\right|^2_{\mathfrak{h}_{N,L}}\right)},$$

where $\xi = 2^{-N\frac{d+1}{2}}q + i2^{-N\frac{d-1}{2}}p$.
For $M > N$, the state ω_{μ_M} is restricted through the map $\ell^2(\Lambda_N) \to \ell^2(\Lambda_M)$ defined by the scaling function.

The free vacuum states

Recall that $\xi \in \ell^2(\Lambda_N)$ can be embedded in $\mathfrak{h}_{\mathrm{ct}}$, by identifying $\delta(x-k)$ with $\phi_{N,k}(x)$. Let us call this map R.

Theorem

Let μ_N be such that $\mu_N 2^{-N} \to m > 0$. Then, $\omega_{\mu_N}(W_N(\xi))$ converges in the weak topology to $\omega_{m,\mathrm{ct}}(W_{\mathrm{ct}}(R\xi))$, where $\omega_{m,\mathrm{ct}}$ is the free vacuum state with mass m.*

Theorem

Let us use the Daubechies scaling function ${}_K\phi$ with $K \geq 6$. Then the algebra generated by the lattice observables is dense in the algebra in the continuum field in $\mathbb{T}^d$.

This entails that we obtain the **time-zero algebra** of the continuum field theory from the lattice algebras.

Some remarks

- It is also possible to prove that the dynamics generated by the lattice Hamiltonian converges to the continuum Hamiltonian. So, in this sense, all the elements of the continuum field theory can be obtained from lattices.
- On the other hand, proving the Haag-Kastler axioms is difficult. We know it in this case just because the free field satisfies the Haag-Kastler axioms and we embedded the lattice theories in it.
- Of particular interest is the Lieb-Robinson bound. Does it imply **relativistic** locality?
- Until now we have considered the unit torus $\mathbb{T}^d$. It is possible to change the volume, and the local algebras are just unitarily equivalent.

Outlook

- Interacting QFT?
  - Change the Hamiltonian, change the states.
  - Locality from Lieb-Robinson bound?
- Gauge theories?
- Free (massless) fermion, CFT?
- Entropy on lattice algebras?
- De Sitter space?