Free products in AQFT

Yoh Tanimoto

(joint with R. Longo and Y. Ueda)
University of Rome "Tor Vergata"
Supported by Rita Levi Montalcini grant of MIUR

July 6th 2018, Timișoara

Studying von Neumann algebras and inclusions

von Neumann algebra $\mathcal{M}$: subalgebra of $\mathcal{B}(\mathcal{H})$, containing $\mathbb{1}$ and closed in weak operator topology.

- classification, construction from group/quantum group, subalgebras, group actions...

subfactor $\mathcal{N} \subset \mathcal{M}$: inclusion of von Neumann algebras (with trivial centers).

- classification, invariants, representation theory...
- $\mathcal{N}' \cap \mathcal{M}$ is typically "small" and there is a conditional expectation $E : \mathcal{M} \to \mathcal{N}$.

Half-sided modular inclusion

There are interesting inclusions $\mathcal{N} \subset \mathcal{M}$ where $\mathcal{N}' \cap \mathcal{M}$ where there is no conditional expectation from $\mathcal{M}$ to $\mathcal{N}$.
Result: **new examples from free product**, applications to **quantum field theory**.

Tomita-Takesaki modular theory

- $\mathcal{M}$: von Neumann algebra on $\mathcal{H}$.
- Ω: vector in $\mathcal{H}$, cyclic ($\mathcal{M}\Omega$ is dense in $\mathcal{H}$) and separating ($x\Omega \neq 0$ for $x \neq 0$ for $\mathcal{M}$).
- $\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{H}) : [x, y] = 0 \text{ for all } y \in \mathcal{M}\}$, the commutant of $\mathcal{M}$.
- S: closure of $x\Omega \longmapsto x^*\Omega$. $S = J\Delta^{\frac{1}{2}}$.

Theorem (Tomita)

$\Delta^{it}\mathcal{M}\Delta^{-it} \subset \mathcal{M}$, $J\mathcal{M}J = \mathcal{M}'$.

$\sigma^t(x) = \Delta^{it}x\Delta^{-it}$ is called the **modular automorphism** of $\mathcal{M}$ with respect to Ω, and plays a crucial role in the study of (type III) von Neumann algebras.

In conformal field theory, where $\mathcal{M} = \mathcal{A}(\mathbb{R})$ is a local algebra and Ω is the vacuum vector, σ^t is the group of **spacetime dilations** (the Bisognano-Wichmann property).

Half-sided modular inclusions

- $\mathcal{N} \subset \mathcal{M}$: von Neumann algebras,
- Ω: cyclic and separating for $\mathcal{M}, \mathcal{N}$
- $\Delta_{\mathcal{M}}^{it}$: the modular group of $\mathcal{M}$ with respect to Ω.

$\mathcal{N} \subset \mathcal{M}$ is a **half-sided modular inclusion** if $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}(\mathcal{N}) \subset \mathcal{N}$ for $t \leq 0$.
(If $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{it}(\mathcal{N}) \subset \mathcal{N}$ for all $t \in \mathbb{R}$, there is conditional expectation $E : \mathcal{M} \to \mathcal{N}$ (Takesaki) $\Rightarrow \mathcal{M} = \mathcal{N}$. **We do not want it**)

Theorem (Wiesbrock '93, Araki-Zsido '05)

If $\mathcal{N} \subset \mathcal{M}$ is a half-sided modular inclusion with respect to Ω, then

$$P := \frac{1}{2\pi}(\log \Delta_{\mathcal{N}} - \log \Delta_{\mathcal{M}})$$

is self-adjoint and positive, where $\Delta_{\mathcal{N}}$ is the modular operator of $\mathcal{N}$. If we set $U(s) = e^{isP}$, then it holds that $\mathrm{Ad}\,\Delta^{it}(U(s)) = U(e^{-2\pi t}s)$ (the $ax + b$ group) and $\mathrm{Ad}\,U(1)(\mathcal{M}) = \mathcal{N}$.

Free products and AQFT

Algebraic quantum field theory

standard half-sided modular inclusions $\Leftrightarrow$ (strongly additive) QFT on S^1.
Construct new examples?

Free product of von Neumann algebras

$\{\mathcal{M}_k, \Omega_k\} \Rightarrow \mathcal{M}$, $\mathcal{M}$ contains copies of $\mathcal{M}_k$ and they are freely independent.
$\{\mathcal{N}_k \subset \mathcal{M}_k, \Omega_k\} \Rightarrow \mathcal{N} \subset \mathcal{M}$. What can one say about $\mathcal{N}' \cap \mathcal{M}$?

Results

- Half-sided inclusions $(\mathcal{N} \subset \mathcal{M}, \Omega)$ with $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}\mathbb{1}$.
- There is $\{\mathcal{N}_k \subset \mathcal{M}_k, \Omega_k\}$ whose free products satisfy $\mathcal{N}' \cap \mathcal{M} \neq \mathbb{C}\mathbb{1}$.
- Some candidates for new QFT.

Haag-Kastler net on S^1

Definition

A **Möbius covariant net** on S^1 is $(\mathcal{A}, U, \Omega)$, where $\{\mathcal{A}(I)\}$ is a family of von Neumann algebras parametrized by intervals in S^1 such that

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$.
- Möbius covariance: U is a unitary representation of $\mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{Ad}\, U(g)(\mathcal{A}(I)) = \mathcal{A}(gI)$.
- Positive energy: the restriction of U to rotations has the positive generator L_0.
- Vacuum: there is a unique (up to a scalar) unit vector Ω such that $U(g)\Omega = \Omega$ for $g \in \mathrm{PSL}(2, \mathbb{R})$ and cyclic for $\mathcal{A}(I)$.

Many examples coming from **quantum field theory**.

Half-sided modular inclusions and Möbius covariant nets on S^1

Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a half-sided modular inclusion. It is **standard** if Ω is cyclic for $\mathcal{M} \cap \mathcal{N}'$.

Theorem (Guido-Longo-Wiesbrock '98)

*There is a **one-to-one correspondence** between*

- *standard half-sided modular inclusions*
- *Möbius covariant nets on $S^1 = \mathbb{R} \cup \{\infty\}$ which are "strongly additive" ($\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$, where $I \setminus \{p\} = I_1 \cup I_2$).*

Correspondence: $\mathcal{M} = \mathcal{A}(\mathbb{R}_+), \mathcal{N} = \mathcal{A}(\mathbb{R}_+ + 1), \mathcal{M} \cap \mathcal{N}' = \mathcal{A}(0,1)$.

Some problems:

- Can one construct half-sided modular inclusions "algebraically"?
- Is there half-sided modular inclusions which satisfy $\mathcal{M} \cap \mathcal{N}' = \mathbb{C}\mathbb{1}$?
- Find handy criteria which give standardness.

Free product of von Neumann algebras

- K: index set. $\kappa \in K$.
- $\mathcal{M}_\kappa$: von Neumann algebras on $\mathcal{H}_\kappa$
- Ω_k: cyclic and separating for $\mathcal{M}_\kappa$
- $\mathcal{H}_\kappa^\circ := \mathcal{H}_\kappa \ominus \Omega_\kappa$

Free product von Neumann algebra (Voiculescu '85)

- $\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{\kappa_j \neq \kappa_{j+1} \\ 1\leq j\leq n-1}} \mathcal{H}_{\kappa_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\kappa_n}^\circ$ with $\mathcal{H}_\kappa^\circ := \mathcal{H}_\kappa \ominus \mathbb{C}\Omega_\kappa$.
- $\mathcal{H} \cong \mathcal{H}_\kappa \otimes \mathcal{K}_\kappa$
- $\mathcal{M}_\kappa$ acts on $\mathcal{H}$ by this identification: λ_κ
- $\mathcal{M}(= \bigstar_\kappa \mathcal{M}_\kappa) := \{\lambda_\kappa(\mathcal{M}_\kappa)\}''$

$\mathcal{M}$ is often a factor, e.g. if one of $\mathcal{M}_\kappa$ is diffuse (Ueda '11).

Some results on free product

A generic element $x \in \mathcal{M} = \bigstar_\kappa \mathcal{M}_\kappa$ can be written as $\sum x_1^\circ \star x_2^\circ \star \cdots \star x_n^\circ$, where $x_\kappa^\circ \in \mathcal{H}_{\kappa_k}, \kappa_k \neq \kappa_{k+1}$ such that $\langle \Omega_\kappa, x_{\kappa_k}^\circ \Omega_\kappa \rangle = 0$.
On a vector $\xi_1 \cdots \xi_n$, $\xi_\kappa \in \mathcal{H}_{\kappa_k}^\circ$, $x_\kappa^\circ \in \mathcal{M}_\kappa^\circ$ acts as

$$x_\kappa^\circ \cdot \xi_1 \otimes \cdots \otimes \xi_n = \begin{cases} x_\kappa^\circ \Omega_\kappa \otimes \xi_1 \otimes \cdots \otimes \xi_n & \text{if } \kappa \neq \kappa_1 \\ (x_\kappa^\circ \xi_1 - \langle \Omega_{\kappa_1}, x_\kappa^\circ \xi_1 \rangle \Omega_{\kappa_1}) \otimes \xi_2 \otimes \cdots \otimes \xi_n & \text{if } \kappa = \kappa_1 \\ + \langle \Omega_{\kappa_1}, x_\kappa^\circ \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_n & \end{cases}$$

Structure of the commutant (Voiculescu '85)

$\mathcal{M}'$ is generated by $x_1^{\circ\prime} \star \cdots \star x_n^{\circ\prime}$, $x_k^{\circ\prime} \in \mathcal{M}'_{\kappa_k}$, and they act from the right.

Modular automorphisms (Dykema '94, Barnett '95)

Let σ_κ^t be the modular automorphism of $\mathcal{M}_\kappa, \Omega_\kappa$. The modular automorphism σ^t of $\mathcal{M}, \Omega$ acts as

$$\sigma^t(x_1 \star \cdots \star x_n) = \sigma_\kappa^t(x_1) \star \cdots \star \sigma_{\kappa_n}^t(x_n).$$

Free product of half-sided modular inclusions

Lemma

Let $\{(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ *be a family of half-sided modular inclusions. The inclusion of free product von Neumann algebras*

$$\mathcal{N} = \bigstar_\kappa \mathcal{N}_\kappa \subset \mathcal{M} = \bigstar_\kappa \mathcal{M}_\kappa, \quad \Omega$$

is a half-sided modular inclusion.

Proof: For $x_1 \star \cdots \star x_n \in \mathcal{M}$,

$$\sigma^t(x_1 \star \cdots \star x_n) = \sigma^t_{\kappa_1}(x_1) \star \cdots \star \sigma^t_{\kappa_n}(x_n) \in \mathcal{N}.$$

Question: What about the relative commutant $\mathcal{N}' \cap \mathcal{M}$?
Answer: it depends on $\{\mathcal{N}_\kappa \subset \mathcal{M}_\kappa\}$ and also on $|K|$.

Half-sided modular inclusions with trivial relative commutant

Theorem (Longo-T.-Ueda, arXiv:1706.06070, to appear in Ann. Inst. Fourier)

Let $\{(\mathcal{N}_\kappa \subset \mathcal{M}_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ *be* **infinite copies** *of a same half-sided modular inclusion* $(\mathcal{N}_0 \subset \mathcal{M}_0, \Omega_0)$.
For the half-sided modular inclusion $(\mathcal{N} = \bigstar_\kappa \mathcal{N}_\kappa \subset \mathcal{M} = \bigstar_\kappa \mathcal{M}_\kappa,\ \Omega)$, *it holds that* $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}\mathbb{1}$.

Notes on the proof:

- The same result holds for general inclusions of von Neumann algebras
- For a subset $K_1 \subset K$ and the corresponding subalgebra $\mathcal{M}_{K_1} = \bigstar_{K_1} \mathcal{M}_\kappa$, there is a conditional expectation $E_{K_1} : \mathcal{M} \to \mathcal{M}_{K_1}$. This holds if K is infinite.
- Estimate $x^\circ \Omega, x^\circ \in \mathcal{N}' \cap \mathcal{M}$ using analytic elements $\lambda_{\kappa'}(b), \kappa' \in K \setminus K_1$.

Nuclearity conditions and split property

$(\{\mathcal{A}(I)\}, U, \Omega)$: Möbius covariant net on S^1, **not necessarily local**.
$U(\rho_t) = e^{itL_0}$, where ρ_t is a rotation of S^1.

Theorem (Buchholz-D'Antoni-Longo '07)

If $e^{-\beta L_0}$ is trace class for some $\beta > 0$, then for certain inclusion $I \subset \tilde{I}$, $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is split, namely, there is an intermediate type I factor ($\cong \mathcal{B}(\mathcal{K})$).

Proof: The trace class condition implies that the map

$$\mathcal{A}(I) \ni x \longmapsto \langle J_{\mathcal{A}(\tilde{I}),\Omega} x\Omega, \cdot\,\Omega\rangle \in \mathcal{A}(\tilde{I})_*$$

is nuclear. Decompose it into normal maps $\sum \varphi_j \psi_j$, so that $xy \to \langle J_{\mathcal{A}(\tilde{I}),\Omega} x\Omega, y\Omega\rangle = \sum \varphi_j(x)\psi_j(y)$ is a normal state on $\mathcal{N} \vee \mathcal{M}'$.
$\mathcal{A}(I)$ are type III_1 factors (Gabbiani-Fröhlich '93). If $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ is split, then the relative commutant $\mathcal{A}(I)' \cap \mathcal{A}(\tilde{I})$ is of type III, in particular, cannot be trivial.

Free product inclusion with nontrivial relative commutant

$(\{\mathcal{A}_\kappa(I)\}, U_\kappa, \Omega_\kappa), \kappa = 1, 2$: Möbius covariant nets on S^1 with trace class property for all β (many such examples). In concrete examples, $\dim\ker(n - L_0) \approx e^{\alpha n}, 0 < \alpha < 1$.

Theorem (Longo-T.-Ueda)

$(\mathcal{A}(I) = \mathcal{A}_1(I) \star \mathcal{A}_2(I), \bigstar U, \Omega)$ *is a Möbius covariant net* **without locality**. $e^{-\beta L_0}$ *is trace class for some* $\beta > 0$.
Consequently, $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$ *is split and has nontrivial relative commutant.*

Proof: $\bigstar U$ can be written explicitly and $\mathrm{Tr}\, e^{-\beta L_0}$ can be estimated by geometric series. Take β so that $\mathrm{Tr}\, e^{-\beta L_{j,0}} < 1$.

This gives a first example of inclusion of free product von Neumann algebras with nontrivial relative commutant.

Possible construction of new Haag-Kastler net

Take two Möbius covariant nets and half-sided modular inclusions $(\mathcal{A}_j(\mathbb{R}_+ + 1) \subset \mathcal{A}_j(\mathbb{R}_+), \Omega_j), j = 1, 2$.

Problem: finite free product

Take the free product $\mathcal{N} = \mathcal{A}_1(\mathbb{R}_+ + 1) \star \mathcal{A}_2(\mathbb{R}_+ + 1) \subset \mathcal{M} = \mathcal{A}_1(\mathbb{R}_+) \star \mathcal{A}_2(\mathbb{R}_+), \Omega$. It is a half-sided modular inclusion. Is the relative commutant nontrivial?

Good criterion for relative commutant is needed. The trace class condition is useful for inclusions $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$, where $\bar{I} \subset \tilde{I}$, but not for $\mathbb{R} + 1 \subset \mathbb{R}$.

Similar problem in two-dimensions, **modular nuclearity**? (used in 2d Haag-Kastler net, Lechner '08, T. '14, etc.): For $(\mathcal{N} \subset \mathcal{M}, \Omega)$, if

$$\mathcal{N} \ni x \mapsto \Delta^{\frac{1}{4}}_{\mathcal{M},\Omega} x\Omega$$

is nuclear, then $\mathcal{N} \subset \mathcal{M}$ is split.

Summary and outlook

- Half-sided modular inclusion with **trivial** relative commutant
- Inclusion of free product von Neumann algebras with **nontrivial** relative commutant

- Possibly new Haag-Kastler nets
- More techniques to determine relative commutant