Low dimensional Quantum Field Theory and Operator Algebras

Yoh Tanimoto

Ph.D thesis in Mathematics
Supervisor: Roberto Longo

Università di Roma "Tor Vergata", Dipartimento di Matematica,
October 2011

Overview of the thesis

This Ph.D. thesis collects a summary of fundamental notions, preceding researches and the results obtained in the author's doctoral study on low-dimensional Quantum Field Theory (QFT), especially on Conformal QFT (CFT). There are various interests in QFT from physical and mathematical points of view. In particular, low-dimensional CFT is an intersection of methods from different disciplines in physics and mathematics. In this thesis, the author concentrates on thermal states and scattering theory of low-dimensional QFT with operator-algebraic and representation-theoretic techniques.

Although some of results are purely mathematical, the author's study has its root in Algebraic approach to QFT. In AQFT, a model of QFT is realized as a net of von Neumann algebras. Many examples in low-dimensional spacetime can be constructed from certain representations of infinite dimensional Lie group or Lie algebras. The constructed nets are analysed through the theory of von Neumann algebras, particularly the subfactor theory and the Tomita-Takesaki modular theory. Then several problems in physics, e.g. the structure of charge, classification of thermal states or determination of scattering amplitude can be stated in terms of nets of von Neumann algebras.

A certain class of representations of the group $\mathrm{Diff}(S^1)$ of the orientiation preserving diffeomorphisms of S^1 is used to construct nets of von Neumann algebras on S^1. The circle S^1 is considered to be the one-point compactification of the real line $\mathbb{R}$, the one-dimensional spacetime, hence it is important to consider the stabilizer subgroup B_0 of the "point of infinity" in $\mathrm{Diff}(S^1)$. The group $\mathbb{R}$ naturally acts on B_0 and one can consider positive energy representations. In Chapter 2, the author investigates both the algebraic property and the representation theory of the group B_0. The first and second cohomology groups and the ideal structure are completely determined. A natural question is whether there are representations of B_0 which do not extend to $\mathrm{Diff}(S^1)$. This turns out to be affirmative and a family of such positive-energy representations is constructed.

A futher study on representations of infinite dimensional Lie algebra is carried out in Chapter 3. Here the Lie algebra of the smooth maps from $\mathbb{R}$ into a simple Lie algebra with compact support is considered. The author studies positive-energy (projective) representation with an invariant vector. Such a representation is called a ground state representation, and related with a physical state with zero temperature. The second cohomology group is shown to be isomorphic to $\mathbb{C}$ and it is shown that there is a one-to-one correspondence between ground state representations and positive integers in the second cohomology group. The relation between representations of the loop algebras and the corresponding nets is discussed.

In Chapter 4, the finite temperature states are investigated. A thermal state on a net of von Neumann algebras is realized as a state on the quasilocal C^*-algebra satisfying the KMS condition. It is possible to consider representations of nets of von Neumann algebras. There is a family of nets which are characterinzed by the finiteness of the equivalence classes of representations and an additional technical condition (completely rational nets). It is proved that any completely rational net admit only one thermal state at each temperature. In contrast, for several non completely rational nets, all the thermal states are classified. For some other nets, a continuous family of thermal states is constructed. Each such state is connected with a representation of the net.

In Chapter 5, the scattering theory of two-dimensional massless QFT is investigated. As for CFT, the usual notion of the scattering matrix of particles turns out to be always trivial. Furthermore, a new method to construct (not necessarily local) nets of von Neumann algebras is proposed and in fact this construction exhausts the class of two-dimensional massless QFT which allows a complete interpretation as particles. The interaction and locality property of these newly constructed exmaples are examined.

Each Chapter except the Introduction contains results of independent studies, hence can be read separately. At the end of each chapter, important open problems and future directions are discussed.

Acknowledgments/Ringraziamenti

È il mio grande piacere ringraziare il mio supervisore, il Professore Roberto Longo, che mi ha dato l'opportunità di mettermi a questa ricerca. Le collaborazioni con lui erano sempre impressionanti: ha delle idee chiare e prospettive profonde, anche se non sembravano technicamente facili all'inizio, che mi guidavano sempre nella direzione giusta. Lui mi ha dato anche tante possibilità di interagire con degli altri, che hanno risultato in vari componenti di questa tesi.

I am greatful to professor Yasuyuki Kawahigashi, who recommended me to come to Rome and helped me in various ways. Also the supports by Daniele Guido and Karl-Henning Rehren are highly appreciated.

I am deeply indebted to my collaborators, Marcel Bischoff, Paolo Camassa, Wojciech Dybalski and Mihály Weiner. Without them, many of the results of the present thesis would have been impossible. Furthermore, it was an encouraging experience to share interests and to exchange techniques with them, which considerably expanded my view on the topics, which I seldom had before.

I would like to thank my friends, who helped me in various stages of this work, and who and I have been stimulating each other: Daniela Cadamuro, Roberto Conti, Kenny De Commer, Robin Hillier, Gabriela Octaviana Jianu, Xiaohong Lan, Masato Mimura, Micaela Rizzi and Makoto Yamashita.

Finally I wish to thank my family for their understanding and encouragement.

Contents

Chapter 1

Introduction

Quantum field theory (QFT) is a physical theory which treats particles with production and annihilation. Since its birth, QFT has been successful in prediction of high energy physics in a very high precision. The central theory is called the standard model and it is considered as a definitive theory in its range of application.

On the other hand, its mathematical foundation is still unsettled and is an active field of research. In many field of physics, there is a precise corresopondence between physical concepts and mathematical objects. Just for example, I mention quantum mechanics. In quantum mechanics, the space of states is represented by a Hilbert space, physical observables correspond to self-adjoint operators, the time-evolusion of the system is given by a one-parameter unitary group and the statistical prediction of experiments is described in terms of the spectral measure of the self-adjoint operator corresponding to the physical observable to be measured. In QFT, although it has a great number of practical success, a precise mathematical formulation is still missing.

There have been mathematical approaches to axiomatize QFT. Namely, one defines quantum field as some mathematical object, formulate certain physical requirements as axioms and investigate their consequences. In this case, the difficulty appears as the lack of examples of such axioms. The present status of QFT is summarized as follows: There is no interacting example of QFT in four-spacetime dimension.

The situation is different in two spacetime dimensions. There is a well-accepted set of axioms (with certain variations) and a wide variety of examples. Hence it is possible to study such examples in a mathematically sound way and there have been obtained several structural consequences as well as classification results of certain classes of models.

There is a subclass of two-dimensional models which, in a certain sense, decompose further into a pair of one-dimensional models and each one-dimensional component acquires a higher spacetime symmetry. Such a two-dimensional model is called a chiral conformal theory and the higher symmetry is the conformal symmetry. Conversely, from a pair of one-dimensional models it is possible to construct a two-dimensional model, first simply by coupling two components, then even interactin models by "twisting" the simple construction. This is one of the main results in this thesis.

Hence the study of two-dimensional models is split into two parts. One is to study one-dimensional components. The other is to study how to couple these components. The present thesis is organized according to this splitting. In the rest of this Introduction, I present the fundamental notions and give an overview of the problems treated in each chapter. In Section 1.1, I explain the algebraic approach to QFT used throughout this thesis. Then in Section 1.2 I present the theory of infinite dimensional Lie groups and algebras used in connection with QFT. In Chapter 2 and 3, we study in particular the group of diffeomorphisms of the circle and the loop groups. In Section 1.3, I review the theory of thermal states in QFT. In Chapter 4, the thermal states in several models are discussed. It turns out that the variety of thermal states is related with the representation theory in preceding Chapters. Then in Chapter 5, I address the issue of scattering theory in two-dimensional spacetime. A general framework is summarized in Section 1.4.

Each Chapter may use different notations and should be read independently.

Publication status of the results

Most of the results I obtained, some in collaboration, are contained in articles submitted to, accepted to or published in various journals. Chapter 2 is based on [87] published in International Journal of Mathematics. The materials in Chapter 3 come from [85] published in Annals Henri Poincaré. Joint works with Paolo Camassa, Roberto Longo and Mihály Weiner on KMS states on conformal nets [20] (accepted to Communications in Mathematical Physics) [21] (submitted) are explained in Chapter 4. Studies on scattering theory in Chapter 5 resulted in several works, namely, collaborations with Wojciech Dybalski [34] (accepted to Communications in Mathematical Physics) [33] (submitted) and single-authored papers [86] (accepted to Communications in Mathematical Physics) [84] (submitted).

1.1 Algebraic approach to Quantum Field Theory

1.1.1 One-dimensional nets of observables

Here we exhibit the mathematical setting which we use to describe physical systems on one-dimensional spacetime S^1. Let $\mathcal{I}$ be the set of all open, connected, non-dense, non-empty subsets of S^1. We call elements of $\mathcal{I}$ **intervals** in S^1. For an interval I, we denote by I' the interior of the complement $S^1 \setminus I$. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on S^1 by the linear fractional transformations.

A **(local) Möbius covariant net** is an assignment $\mathcal{A}$ to each interval of a von Neumann algebra $\mathcal{A}(I)$ on a fixed separable Hilbert space $\mathcal{H}$ with the following conditions:

(1) **Isotony.** If $I_1 \subset I_2$, then $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

(2) **Locality.** If $I_1 \cap I_2 = \emptyset$, then $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$.

(3) **Möbius covariance.** There exists a strongly continuous unitary representation U of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ such that for any interval I it holds that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \text{ for } g \in \mathrm{PSL}(2,\mathbb{R}).$$

(4) **Positivity of energy.** The generator of the one-parameter subgroup of rotations in the representation U is positive.

(5) **Existence of vacuum.** There is a unique (up to a phase) unit vector Ω in $\mathcal{H}$ which is invariant under the action of U, and cyclic for $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

It is well-known that, from these conditions, the following properties automatically follow (see, for example, [41]):

(6) **Additivity.** If $I = \bigcup_i I_i$, then $\mathcal{A}(I) = \bigvee_i \mathcal{A}(I_i)$.

(7) **Reeh-Schlieder property.** The vector Ω is cyclic and separating for each $\mathcal{A}(I)$.

(8) **Haag duality.** For any interval I it holds that $\mathcal{A}(I)' = \mathcal{A}(I')$.

(9) **Bisognano-Wichmann property.** The Tomita-Takesaki operator Δ_I of $\mathcal{A}(I)$ with respect to Ω satisfies the following:

$$U(\delta^I(2\pi t)) = \Delta_I^{-it},$$

where δ^I is the one-parameter group in $\mathrm{PSL}(2,\mathbb{R})$ which preserves the interval I (which we call "the dilation associated to I": in the real line picture $\delta^I : x \mapsto e^s x$ if $I \equiv \mathbb{R}^+$: see Section 1.1.2).

(10) **Factoriality.** Each local algebra $\mathcal{A}(I)$ is a type III_1-factor (unless $\mathcal{H}$ is one dimensional).

The Bisognano-Wichmann property is of particular importance in several contexts of this thesis. This property means that the vacuum state $\omega(\cdot) = \langle\Omega, \cdot\Omega\rangle$ is a KMS state for $\mathcal{A}(I)$ with respect to δ^I (at inverse temperature 2π), see below. This will be exploited to construct a standard KMS state with respect to the spacetime translation in Section 1.3.2.

We say that $\mathcal{A}$ is **strongly additive** if it holds that $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$, where I_1 and I_2 are intervals obtained by removing an interior point of I.

Let $\mathcal{A}$ be a Möbius covariant net on S^1. If a unitary operator V commutes with the translation unitaries $T(t)$ and it holds that $V\mathcal{A}(\mathbb{R}_+)V^* \subset \mathcal{A}(\mathbb{R}_+)$, then we say that V implements a **Longo-Witten endomorphism** of $\mathcal{A}$. In particular V preserves Ω up to a scalar since Ω is the unique invariant vector under $T(t)$. Such endomorphisms have been studied first in [64] and they found a large family of endomorphisms for the $U(1)$-current net, its extensions and the free fermion net.

1.1.2 The restriction of a net to the real line

Although Möbius covariant nets are defined on the circle S^1, it is natural from a physical point of view to consider a theory on the real line $\mathbb{R}$. We identify $\mathbb{R}$ with the punctured circle $S^1 \setminus \{-1\}$ by the Cayley transform:

$$t = i\frac{1+z}{1-z} \Longleftrightarrow z = \frac{t-i}{t+i}, t \in \mathbb{R}, z \in S^1 \subset \mathbb{C}.$$

The point $-1 \in S^1$ is referred to as "the point at infinity" ∞ when considered in the real-line picture.

We recall that the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ is generated by the following three one-parameter groups, namely rotations, translations and dilations [63]:

$$\begin{aligned} \rho_s(z) &= e^{is}z, \text{ for } z \in S^1 \subset \mathbb{C} \\ \tau_s(t) &= t+s, \text{ for } t \in \mathbb{R} \\ \delta_s(t) &= e^s t, \text{ for } t \in \mathbb{R}, \end{aligned}$$

where rotations are defined in the circle picture, on the other hand translations and dilations are defined in the real line picture. Of these, translations and dilations do not move the point at infinity.

According to this identification, we also restrict a conformal net $\mathcal{A}$ to the real line. Namely, we consider all the finite-length open intervals $I \Subset \mathbb{R} = S^1 \setminus \{-1\}$ under the identification. We still have an isotonic and local net of von Neumann algebras corresponding to intervals in $\mathbb{R}$, which is covariant under translation, dilation and diffeomorphisms of S^1 which preserve -1. It is known that the positivity of energy (the generator of rotations) is equivalent to the positivity of the generator of translations [93], and the vacuum vector Ω is invariant under translations and dilations. We denote this restriction to the real line by $\mathcal{A}|_\mathbb{R}$.

The terminology of representations easily translates to the real-line picture. Namely, a representation of $\mathcal{A}|_\mathbb{R}$ is a consistent family $\{\pi_I\}_{I\Subset\mathbb{R}}$ of representations of $\{\mathcal{A}(I)\}_{I\Subset\mathbb{R}}$, and an endomorphism (respectively an automorphism) is a representation on the same Hilbert space which maps $\mathcal{A}(I)$ into (respectively onto) itself. Note that the family of bounded (connected) intervals is directed. We shall denote by $\mathfrak{A}_\mathcal{A}$ the associated quasi-local algebra, that is the C^*-algebra

$$\mathfrak{A}_\mathcal{A} := \overline{\cup_{I\Subset\mathbb{R}}\mathcal{A}(I)}$$

where the closure is meant in the operator norm topology. By the directedness, any representation (resp. endomorphism, automorphism) of $\mathcal{A}|_\mathbb{R}$ extends to a representation (resp. endomorphism, automorphism) of the C^*-algebra $\mathfrak{A}_\mathcal{A}$. Translations and dilations take bounded intervals $I \Subset \mathbb{R}$ to bounded intervals, hence these transformations give rise to automorphisms of $\mathfrak{A}_\mathcal{A}$.

1.1.3 Subnets and extensions

Let $\mathcal{B}$ be a Möbius covariant net on $\mathcal{H}$. Another assignment $\mathcal{A}$ of von Neumann algebras $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ on $\mathcal{H}$ is called a **subnet** of $\mathcal{B}$ if it satisfies isotony, Möbius covariance with respect to the same U for $\mathcal{B}$ and it holds that $\mathcal{A}(I) \subset \mathcal{B}(I)$ for every interval $I \in \mathcal{I}$. If $\mathcal{A}(I)' \cap \mathcal{B}(I) = \mathbb{C}\mathbb{1}$ for an interval I (hence for any interval, by the covariance and the transitivity of the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathcal{I}$), we say that the inclusion of nets $\mathcal{A} \subset \mathcal{B}$ is **irreducible**.

Let us denote by $\mathcal{H}_\mathcal{A}$ the subspace of $\mathcal{H}$ generated by $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ from Ω, and by $P_\mathcal{A}$ the orthogonal projection onto $\mathcal{H}_\mathcal{A}$. Then it is easy to see that $P_\mathcal{A}$ commutes with all $\mathcal{A}(I)$ and U. The assignment $\{\mathcal{A}(I)|_{\mathcal{H}_\mathcal{A}}\}_{I\in\mathcal{I}}$ with the representation $U|_{\mathcal{H}_\mathcal{A}}$ of $\mathrm{PSL}(2,\mathbb{R})$ and the vacuum Ω is a Möbius covariant net on $\mathcal{H}_\mathcal{A}$. Conversely, if a Möbius covariant net $\mathcal{A}_0$ is unitarily equivalent to such a restriction $\mathcal{A}|_{\mathcal{H}_\mathcal{A}}$ of a subnet $\mathcal{A}$ of $\mathcal{B}$, then $\mathcal{B}$ is called an **extension** of $\mathcal{A}_0$. We write simply $\mathcal{A}_0 \subset \mathcal{B}$ if no confusion arises.

When we have an inclusion of nets $\mathcal{A} \subset \mathcal{B}$, for each interval I there is a canonical conditional expectation $E_I : \mathcal{A}(I) \to \mathcal{B}(I)$ which preserves the vacuum state ω thanks to the Reeh-Schlieder property and Takesaki's theorem [82, Theorem IX.4.2]. We define the **index** of the inclusion $\mathcal{A} \subset \mathcal{B}$ as the index $[\mathcal{B}(I), \mathcal{A}(I)]$ with respect to this conditional expectation [56], which does not depend on I (again by covariance, or even without covariance [60]). If the index is finite, the inclusion is irreducible.

1.1.4 Diffeomorphism covariance and Virasoro nets

In the present thesis we will consider a class of nets with a much larger group of symmetry, which still contains many interesting examples. Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving diffeomorphisms of the circle S^1. This group naturally contains $\mathrm{PSL}(2,\mathbb{R})$.

A Möbius covariant net $\mathcal{A}$ is said to be a **conformal net** if the representation U extends to a projective unitary representation of $\mathrm{Diff}(S^1)$ such that for any interval I and $x \in \mathcal{A}(I)$ it holds that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \text{ for } g \in \mathrm{Diff}(S^1),$$
$$U(g)xU(g)^* = x, \text{ if } \mathrm{supp}(g) \subset I',$$

where $\mathrm{supp}(g) \subset I'$ means that g acts identically on I. In this case we say that $\mathcal{A}$ is **diffeomorphism covariant**.

From the second equation above we see that $U(g) \in \mathcal{A}(I)$ if $\mathrm{supp}(g) \subset I$ by Haag duality. If we define

$$\mathrm{Vir}(I) = \{U(g) : \mathrm{supp}(g) \subset I\}'',$$

one can show that Vir is a subnet of $\mathcal{A}$. Such a net is called a **Virasoro net**. Let us consider its restriction to the space $\mathcal{H}_{\mathrm{Vir}}$. The representation U of $\mathrm{Diff}(S^1)$ restricts to $\mathcal{H}_{\mathrm{Vir}}$ as well, and this restriction is irreducible by the Haag duality. In addition, the restriction of U to $\mathrm{PSL}(2,\mathbb{R})$ admits an invariant vector Ω and the rotation still has positive energy. Such representations have been completely classified by positive numbers c, the **central**

charge, see for example [24, Appendix A]. It is known that even to the full representation U on $\mathcal{H}$ we can assign the central charge c. Since the representation U which makes $\mathcal{A}$ diffeomorphism covariant is unique [26], the value of c is an invariant of $\mathcal{A}$. We say that the net $\mathcal{A}$ has the central charge c (see also Section 1.5.3).

1.1.5 Complete rationality

We now define the class of conformal nets to which our main result applies. Let us consider the following conditions on a net $\mathcal{A}$. For intervals I_1, I_2, we shall write $I_1 \Subset I_2$ if the closure of I_1 is contained in the interior of I_2.

(a) **Split property.** For intervals $I_1 \Subset I_2$ there exists a type I factor F such that $\mathcal{A}(I_1) \subset F \subset \mathcal{A}(I_2)$.

(b) **Strong additivity.** For intervals I, I_1, I_2 such that $I_1 \cup I_2 \subset I$, $I_1 \cap I_2 = \emptyset$, and $I \setminus (I_1 \cup I_2)$ consists of one point, it holds that $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$.

(c) **Finiteness of μ-index.** For disjoint intervals I_1, I_2, I_3, I_4 in a clockwise (or counter-clockwise) order with a dense union in S^1, the Jones index of the inclusion $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ is finite (it does not depend on the choice of intervals [54] and we call it the **μ-index** of $\mathcal{A}$).

A conformal net $\mathcal{A}$ is said to be **completely rational** if it satisfies the three conditions above. If $\mathcal{A}$ is diffeomorphism covariant, the strong additivity condition (b) follows from the other two (a) and (c) [65].

An important class of completely rational nets is given by the conformal nets with $c < 1$, which have been completely classified [52]. Among other examples of completely rational nets (with $c \geq 1$) are $SU(N)_k$ loop group nets (Section 1.5.3, [41, 97]). It is known that complete rationality passes to finite index extensions and finite index subnets [62]. The importance of complete rationality is revealed in representation theory of nets (see Section 1.1.6).

1.1.6 Representations and sectors of conformal nets

Let $\mathcal{A}$ be a conformal net on S^1. A **representation** π of $\mathcal{A}$ is a family of (normal) representations π_I of algebras $\mathcal{A}(I)$ on a common Hilbert space $\mathcal{H}_\pi$ with the consistency condition

$$\pi_J|_{\mathcal{A}(I)} = \pi_I, \text{ for } I \subset J.$$

A representation π satisfying $\{\cup_I \pi_I(\mathcal{A}(I))\}' = \mathbb{C}\mathbb{1}$ is called **irreducible**. Two representations π, π' are **unitarily equivalent** iff there is a unitary operator W such that $\mathrm{Ad}(W) \circ \pi_I = \pi'_I$ for every interval I. A unitary equivalence class of an irreducible representations is called a **sector**. It is known that any completely rational net admits only finitely many sectors [54].

A representation may be given also on the original (vacuum-)Hilbert space. Such a representation ρ which preserves each local algebra $\mathcal{A}(I)$ is called an **endomorphism** of $\mathcal{A}$. Note that this notion of endomorphisms differs from the terminology of localized endomorphisms of DHR representation theory, in which not all local algebras are preserved. If each representation of the local algebra is surjective, it is called an **automorphism**. An automorphism which preserves the vacuum state is called an **inner symmetry**. Any inner symmetry is implemented by a unitary operator and it is in the same sector as the vacuum representation.

1.1.7 Half-sided modular inclusions

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras. If there is a cyclic and separating vector Ω for $\mathcal{N}, \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N}'$, then the inclusion $\mathcal{N} \subset \mathcal{M}$ is said to be **standard** in the sense of [31]. If $\sigma_t^{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for $\tau \in \mathbb{R}_\pm$ where $\sigma_t^{\mathcal{M}}$ is the modular automorphism of $\mathcal{M}$ with respect to Ω, then it is called a $\pm$**half-sided modular inclusion**.

Theorem 1.1.1 ([95, 2])**.** *If $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is a $+$(respectively $-$)half-sided modular inclusion, then there is a Möbius covariant net $\mathcal{A}$ on S^1 such that $\mathcal{A}(\mathbb{R}_-) = \mathcal{M}$ and $\mathcal{A}(\mathbb{R}_- - 1) = \mathcal{N}$ (respectively $\mathcal{A}(\mathbb{R}_+) = \mathcal{M}$ and $\mathcal{A}(\mathbb{R}_+ + 1) = \mathcal{N}$).*

If a unitary representation V of $\mathbb{R}$ with positive spectrum satisfies $V(t)\Omega = \Omega$ for $t \in \mathbb{R}$, $\mathrm{Ad}V(t)(\mathcal{M}) \subset \mathcal{M}$ for $t \leq 0$ (respectively $t \geq 0$) and $\mathrm{Ad}V(-1)(\mathcal{M}) = \mathcal{N}$ (respectively $\mathrm{Ad}V(1)(\mathcal{M}) = \mathcal{N}$), then V is the representation of the translation group of the Möbius covariant net constructed above.

1.1.8 Two-dimensional nets of observables

The two-dimensional Minkowski space $\mathbb{R}^2$ is represented as a product of two lightlines $\mathbb{R}^2 = \mathrm{L}_+ \times \mathrm{L}_-$, where $\mathrm{L}_\pm := \{(t_0, t_1) \in \mathbb{R}^2 : t_0 \pm t_1 = 0\}$ are the positive and the negative lightlines. The fundamental group of spacetime symmetry is the (proper orthochronous) Poincaré group $\mathcal{P}_+^\uparrow$, which is generated by translations and Lorentz boost.

Let $\mathcal{O}$ be the family of open bounded regions in $\mathbb{R}^2$. A **(local) Poincaré covariant net** $\mathcal{A}$ assigns to $O \in \mathcal{O}$ a von Neumann algebra $\mathcal{A}(O)$ on a common separable Hilbert space $\mathcal{H}$ satisfying the following conditions:

(1) **Isotony.** If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

(2) **Locality.** If O_1 and O_2 are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$.

(3) **Additivity.** If $O = \bigcup_i O_i$, then $\mathcal{A}(O) = \bigvee_i \mathcal{A}(O_i)$.

(4) **Poincaré covariance.** There exists a strongly continuous unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$ such that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \mathcal{P}_+^\uparrow.$$

(5) **Positivity of energy.** The joint spectrum of the translation subgroup in $\mathcal{P}_+^\uparrow$ in the representation U is contained in the forward lightcone $V_+ := \{(p_0, p_1) \in \mathbb{R}^2 : p_0 + p_1 \geq 0, p_0 - p_1 \geq 0\}$.

(6) **Existence of the vacuum.** There is a unique (up to a phase) unit vector Ω in $\mathcal{H}$ which is invariant under the action of U, and cyclic for $\bigvee_{O \in \mathcal{O}} \mathcal{A}(O)$.

From these assumptions, the following properties automatically follow [3].

(7) **Reeh-Schlieder property.** The vector Ω is cyclic and separating for each $\mathcal{A}(O)$.

(8) **Irreducibility.** The von Neumann algebra $\bigvee_{O \in \mathcal{O}} \mathcal{A}(O)$ is equal to $B(\mathcal{H})$.

It is sometimes appropriate to extend the definition of a net also to unbounded regions. Let $\mathcal{A}$ be a Poincaré covariant net. For an arbitrary open region O, we define $\mathcal{A}(O) := \bigvee_{D \subset O} \mathcal{A}(D)$, where D runs over all bounded regions included in O (this definition coincides with the original net if O is bounded). Among important unbounded regions are **wedges**. The standard left and right wedges are defined as follows:

$$\begin{aligned} W_{\mathrm{L}} &:= \{(t_0, t_1) : t_0 > t_1, t_0 < -t_1\} \\ W_{\mathrm{R}} &:= \{(t_0, t_1) : t_0 < t_1, t_0 > -t_1\} \end{aligned}$$

The regions W_{L} and W_{R} are invariant under Lorentz boosts. The causal complement of W_{L} is W_{R} (and vice versa). All the regions obtained by translations of standard wedges are still called left- and right- wedges, respectively. Moreover, any double cone is obtained as the intersection of a left wedge and a right wedge. Let O' denote the causal complement of O. It holds that $W_{\mathrm{L}}' = W_{\mathrm{R}}$, and if $D = (W_{\mathrm{L}} + a) \cap (W_{\mathrm{R}} + b)$ is a double cone, $a, b \in \mathbb{R}^2$, then $D' = (W_{\mathrm{R}} + a) \cup (W_{\mathrm{L}} + b)$. It is easy to see that Ω is still cyclic and separating for $\mathcal{A}(W_{\mathrm{L}})$ and $\mathcal{A}(W_{\mathrm{R}})$.

Let us introduce some additional assumptions on the structure of nets.

- **Haag duality.** If O is a wedge or a double cone, then it holds that $\mathcal{A}(O)' = \mathcal{A}(O')$.
- **Bisognano-Wichmann property.** The modular group Δ^{it} of $\mathcal{A}(W_{\mathrm{R}})$ with respect to Ω is equal to $U(\Lambda(-2\pi t))$, where $\Lambda(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ denotes the Lorentz boost.

Duality for wedges (namely $\mathcal{A}(W_{\mathrm{L}})' = \mathcal{A}(W_{\mathrm{R}})$) follows from the Bisognano-Wichmann property (see Proposition 1.1.4). If $\mathcal{A}$ is Möbius covariant, then the Bisognano-Wichmann property is automatic [10], and Haag duality is equivalent to strong additivity [77]. Apart from Möbius nets, these properties are common even in massive interacting models [59]. Furthermore, starting with $\mathcal{A}(W_{\mathrm{L}})$, it is possible to construct a net which satisfies both properties [6, 59]. Hence we believe that these additional assumptions are natural and at some points of the present thesis we assume that the net $\mathcal{A}$ satisfies them.

We identify the circle S^1 as the one-point compactification of the real line $\mathbb{R}$ by the Cayley transform:

$$t = i\frac{z-1}{z+1} \Longleftrightarrow z = -\frac{t-i}{t+i}, \quad t \in \mathbb{R}, \quad z \in S^1 \subset \mathbb{C}.$$

The Möbius group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ by the linear fractional transformations, hence it acts on $\mathbb{R}$ locally (see [10] for local actions). Then the group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ acts locally on $\mathbb{R}^2$, where $\overline{\mathrm{PSL}(2,\mathbb{R})}$ is the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. Note that the group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ contains translations, Lorentz boosts and dilations, so in particuar it includes the Poincaré group $\mathcal{P}_+^\uparrow$. We refer to [53] for details.

Let $\mathcal{A}$ be a Poincaré covariant net. If the representation U of $\mathcal{P}_+^\uparrow$ (associated to the net $\mathcal{A}$) extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ such that for any open region O there is a small neighborhood $\mathcal{U}$ of the unit element in $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ such that $gO \subset \mathcal{M}$ and it holds that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})},$$

then we say that $\mathcal{A}$ is a **Möbius covariant net**.

If the net $\mathcal{A}$ is Möbius covariant, then it extends to a net on the Einstein cylinder $\mathcal{E} := \mathbb{R} \times S^1$ [53]. On $\mathcal{E}$ one can define a natural causal structure which extends the one on $\mathcal{M}$ (see [66]). We take a coordinate system on $\mathcal{E}$ used in [77]: Let $\mathbb{R} \times \mathbb{R}$ be the universal cover of $S^1 \times S^1$. The cylinder $\mathcal{E}$ is obtained from $\mathbb{R} \times \mathbb{R}$ by identifying points (a,b) and $(a+2\pi, b-2\pi) \in \mathbb{R} \times \mathbb{R}$. Any double cone of the form $(a, a+2\pi) \times (b, b+2\pi) \subset \mathbb{R} \times \mathbb{R}$ represents a copy of the Minkowski space. The causal complement of a double cone $(a,c) \times (b,d)$, where $0 < c-a < 2\pi, 0 < d-b < 2\pi$, is $(c, a+2\pi) \times (d-2\pi, b)$ or equivalently $(c-2\pi, a) \times (d, b+2\pi)$. If O is a double cone, we denote the causal complement by O'. For an interval $I = (a,b)$, we denote by I^+ the interval $(b, a+2\pi) \subset \mathbb{R}$ and by I^- the interval $(b-2\pi, a) \subset \mathbb{R}$.

Furthermore, it is well-known that, from Möbius covariance, the following properties automatically follow (see [10]):

- **Haag duality in $\mathcal{E}$.** For a double cone O in $\mathcal{E}$ it holds that $\mathcal{A}(O)' = \mathcal{A}(O')$, where O' is defined in $\mathcal{E}$ as above.

- **Bisognano-Wichmann property in** $\mathcal{E}$ For a double cone O in $\mathcal{E}$, the modular automorphism group Δ_O^{it} of $\mathcal{A}(O)$ with respect to the vacuum state $\omega := \langle \Omega, \cdot \Omega \rangle$ equals to $U(\Lambda_t^O)$ where Λ_t^O is a one-parameter group in $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ which preserves O (see [10] for concrete expressions).

We denote by $\mathrm{Diff}(\mathbb{R})$ the group of diffeomorphisms of S^1 which preserve the point $\{-1\}$. If we identify $S^1 \setminus \{-1\}$ with $\mathbb{R}$, this can be considered as a group of diffeomorphisms of $\mathbb{R}$ [1]. The Minkowski space $\mathbb{R}^2$ can be identified with a double cone in $\mathcal{E}$. The group

[1]Note that not all diffeomorphisms of $\mathbb{R}$ extend to diffeomorphisms of S^1, hence the group $\mathrm{Diff}(\mathbb{R})$ is *not* the group of all the diffeomorphisms of $\mathbb{R}$. However, this notation is common in the context of conformal field theory.

$\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ acts on $\mathbb{R}^2$ and this action extends to $\mathcal{E}$ by periodicity. The group generated by this action of $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ and the action of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ (which acts on $\mathcal{E}$ through quotient by the relation $(r_{2\pi}, r_{-2\pi}) = (\mathrm{id}, \mathrm{id})$ [53]) is denoted by $\mathrm{Conf}(\mathcal{E})$. Explicitly, $\mathrm{Conf}(\mathcal{E})$ is isomorphic to the quotient group of $\overline{\mathrm{Diff}(S^1)} \times \overline{\mathrm{Diff}(S^1)}$ by the normal subgroup generated by $(r_{2\pi}, r_{-2\pi})$, where $\overline{\mathrm{Diff}(S^1)}$ is the universal covering group of $\mathrm{Diff}(S^1)$ (note that $r_{2\pi}$ is an element in the center of $\overline{\mathrm{Diff}(S^1)}$).

A Möbius covariant net is said to be **conformal** if the representation U further extends to a projective representation of $\mathrm{Conf}(\mathcal{E})$ such that

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO), \text{ for } g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R}),$$

and if it holds that

$$U(g)xU(g)^* = x$$

for $x \in \mathcal{A}(O)$, where O is a double cone and $g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ has a support disjoint from O.

Proposition 1.1.2. *If the net $\mathcal{A}$ is conformal, the intersection $\bigcap_J \mathcal{A}(I \times J)$ contains representatives of diffeomorphisms of the form $g_+ \times \mathrm{id}$ where $\mathrm{supp}(g_+) \subset I$,*

Proof. If g is a diffeomorphism of the form $g_+ \times \mathrm{id}$ and $\mathrm{supp}(g_+) \subset I$, then $U(g)$ commutes with $\mathcal{A}(I^+ \times J)$ for arbitrary J, thus Proposition follows by the Haag duality on $\mathcal{E}$. $\square$

If it holds that $\mathcal{A}(O_1) \vee \mathcal{A}(O_2) = \mathcal{A}(O)$ where O_1 and O_2 are the two components of the causal complement (in O) of an interior point of a double cone O, we say that $\mathcal{A}$ is **strongly additive**. This implies the **chiral additivity** [77], namely that $\mathcal{A}(I \times J_1) \vee \mathcal{A}(I \times J_2) = \mathcal{A}(I \times J)$ if J_1 and J_2 are obtained from J by removing an interior point.

Let $\mathcal{A}_\pm$ be two Möbius covariant nets on S^1 defined on the Hilbert spaces $\mathcal{H}_\pm$ with the vacuum vectors $\Omega_\pm$ and the representations of $U_\pm$ (see Section 1.1.1). We define a two-dimensional net $\mathcal{A}$ as follows: Let $\mathrm{L}_\pm := \{(t_0, t_1) \in \mathbb{R}^2 : t_0 \pm t_1 = 0\}$ be two lightrays. For a double cone O of the form $I \times J$ where I and J $I \subset \mathrm{L}_+$ and $J \subset \mathrm{L}_-$, we set $\mathcal{A}(O) = \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$. For a general open region $O \subset \mathbb{R}^2$, we set $\mathcal{A}(O) := \bigvee_{I \times J} \mathcal{A}(I \times J)$ where the union is taken among intervals such that $I \times J \subset O$. If we take the vacuum vector as $\Omega := \Omega_+ \otimes \Omega_-$ and define the representation U of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ by $U(g_+ \times g_-) := U_+(g_1) \times U_-(g_2)$, it is easy to see that all the conditions for Möbius covariant net follow from the corresponding properties of nets on S^1. We say that such $\mathcal{A}$ is **chiral**. If $\mathcal{A}_\pm$ are conformal, then the representation U naturally extends to a projective representation of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. Hence $\mathcal{A}$ is a two-dimensional conformal net.

We recall that $\mathcal{A}(O)$ is interpreted as the algebra of observables measured in a spacetime region O. A typical example of a conformal net is constructed in the following way: If we have a local conformal field Ψ, namely an operator-valued distribution, then we define $\mathcal{A}(O)$ as the von Neumann algebra generated by $e^{i\Psi(f)}$ where the support of f is included in O. But our framework does not assume the existence of any field. Indeed, there are examples of conformal nets (on S^1; See Section 1.1.1) for which no field description is at hand [52]. Thus the algebraic approach is more general than the conventional one.

1.1.9 Conditional expectations in nets of observables

The Bisognano-Wichmann property asserts a relation between the dynamics of the net and the Tomita-Takesaki modular theory. In the modular theory, one of the fundamental tools is conditional expectation. We briefly recall here its definition and discuss some immediate consequences. A **condtional expectation** from a von Neumann algebra $\mathcal{M}$ onto a subalgebra $\mathcal{N}$ is a linear map $E : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:

- $E(x) = x$ for $x \in \mathcal{N}$.
- $E(xyz) = xE(y)z$ for $x, z \in \mathcal{N}$, $y \in \mathcal{M}$.
- $E(x)^*E(x) \leq E(x^*x)$ for $x \in \mathcal{M}$.

Let us recall the fundamental theorem of Takesaki [82, Theorem IX.4.2].

Theorem 1.1.3. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras and φ be a faithful normal state on $\mathcal{M}$. Then the following are equivalent:*

- *$\mathcal{N}$ is invariant under the modular automorphism group σ_t^φ.*
- *There is a normal conditional expectation E from $\mathcal{M}$ onto $\mathcal{N}$ such that $\varphi = \varphi \circ E$.*

Furthermore, if the above conditions hold, then the conditional expectation E is implemented by a projection in the following sense: We consider the GNS representation π_φ and Φ be the GNS vector. Let $P_\mathcal{N}$ be the projection onto the subspace $\overline{\mathcal{N}\Phi}$. Then it holds that $E(x)\Phi = P_\mathcal{N} x\Phi$. In particular, $\mathcal{N} = \mathcal{M}$ if and only if $P_\mathcal{N} = \mathbb{1}$ (hence $E = \mathrm{id}$). The modular automorphism group of $\varphi|_\mathcal{N}$ is equal to $\sigma^\varphi|_\mathcal{N}$.

A Poincaré covariant net $\mathcal{A}$ is said to be **wedge dual** if it holds that $\mathcal{A}(W_\mathrm{L})' = \mathcal{A}(W_\mathrm{L}')(= \mathcal{A}(W_\mathrm{R}))$ (see Section 1.1.8 for W_L and W_R). With the help of conditional expectation, it is easy to deduce that Bisognano-Wichmann property (defined in Section 1.1.8) implies wedge duality, although this implication has been essentially known [6, 79].

Proposition 1.1.4. *If a Poincaré covariant net $\mathcal{A}$ satisfies Bisognano-Wichmann property, then it is wedge dual.*

Proof. The modular automorphism group σ_t^Ω of $\mathcal{A}(W_\mathrm{L})'$ is implemented by Δ_Ω^{-it}, which is equal to $U(\Lambda(-2\pi t))$ by Bisognano-Wichmann property. It is obvious that $\mathcal{A}(W_\mathrm{R}) \subset \mathcal{A}(W_\mathrm{L})'$ and $\mathcal{A}(W_\mathrm{R})$ is invariant under $\mathrm{Ad}U(\Lambda(-2\pi t)) = \mathrm{Ad}\Delta_\Omega^{-it}$. Hence by Takesaki's Theorem 1.1.3, there is a conditional expectation E from $\mathcal{A}(W_\mathrm{L})'$ onto $\mathcal{A}(W_\mathrm{R})$ which preserves $\omega = \langle \Omega, \cdot \Omega \rangle$ and it is implemented by the projection onto the subspace $\overline{\mathcal{A}(W_\mathrm{R})\Omega}$. But by Reeh-Schlieder property it is the whole space $\mathcal{H}$, hence E is the identity map and we obtain $\mathcal{A}(W_\mathrm{L})' = \mathcal{A}(W_\mathrm{R})$. □

For a given net $\mathcal{A}$, we can associate the **dual net** $\mathcal{A}^{\mathrm{d}}$ ([3, Section 1.14]), defined by

$$\mathcal{A}^{\mathrm{d}}(O_0) = \bigcap_{O \perp O_0} \mathcal{A}(O)',$$

where $O \perp O_0$ means that O and O_0 are causally disjoint. $\mathcal{A}^{\mathrm{d}}$ does not necessarily satisfy locality nor additivity. Additivity is usually necessary only in proving Reeh-Schlieder property, so we do not discuss here. We have the following.

Proposition 1.1.5 ([3]). *If a Poincaré covariant net $\mathcal{A}$ is wedge dual, then $\mathcal{A}^{\mathrm{d}}$ is local and Haag dual.*

Thus, if we consider the dual net $\mathcal{A}^{\mathrm{d}}$ as a natural extension, Haag duality for a net with Bisognano-Wichmann property is not a strong requirement.

1.2 Infinite dimensional Lie groups and algebras

1.2.1 2-cocycles and projective representations

A 2-cocycle on a complex Lie algebra $\mathfrak{h}$ is a bilinear form $\omega : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ which satisfies

$$\omega(\xi, \eta) = -\omega(\eta, \xi), \tag{1.1}$$
$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0. \tag{1.2}$$

For a given cocycle ω, we can define a new Lie algebra $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}$ with the following operation,

$$[(\xi, a_1), (\eta, a_2)] = ([\xi, \eta], \omega(\xi, \eta)),$$

and we call it the central extension of $\mathfrak{h}$ by the cocycle ω. It is customary to express this algebra using a formal central element C and to define the commutation relation

$$[\xi + a_1 C, \eta + a_2 C] = [\xi, \eta] + \omega(\xi, \eta)C.$$

A projective unitary representation π with a 2-cocycle ω of $\mathfrak{h}$ assigns to any element ξ of $\mathfrak{h}$ a linear operator $\pi(\xi)$ on a (inner product) linear space V such that

$$\pi([\xi, \eta]) = \pi(\xi)\pi(\eta) - \pi(\eta)\pi(\xi) + \omega(\xi, \eta).$$

Such a projective representation π can be naturally considered as a (true, non projective) representation $\widetilde{\pi}$ of $\widetilde{\mathfrak{h}}$ by the formula $\widetilde{\pi}(\xi + aC) = \pi(x) + a \cdot \mathbb{1}$.

1.2.2 The loop algebras

We define an infinite-dimensional Lie algebra called loop algebra of the Lie algebra $\mathfrak{g}$ of a compact Lie group G:

$$L\mathfrak{g} := \{\xi : S^1 \to \mathfrak{g}, \text{smooth}\},$$
$$[\xi_1, \xi_2](z) := [\xi_1(z), \xi_2(z)], z \in S^1.$$

$L\mathfrak{g}$ has the natural topology by the uniform convergence of each derivative and the differential structure.

For the loop algebra $L\mathfrak{g}$, the complexification $(L\mathfrak{g})_\mathbb{C}$ can be naturally defined and be identified with $L\mathfrak{g}_\mathbb{C}$. It obtains a structure of $*$-Lie algebra by defining $\xi^*(z) := (\xi(z))^*$, where in the right hand side * means the $*$-operation with respect to the compact form.

Instead of analysing the loop algebras directly, it is customary to consider the polynomial loops $\xi(z) = \sum_k \xi_k z^k$, where $\xi_k \in \mathfrak{g}_\mathbb{C}$ and only finitely many terms appear in the sum. Let us denote the polynomial subalgebra by $\widetilde{\mathfrak{g}_\mathbb{C}}$. It is easy to see that $\widetilde{\mathfrak{g}_\mathbb{C}} \cong \mathfrak{g}_\mathbb{C} \otimes_\mathbb{C} \mathbb{C}[t, t^{-1}]$ with the bracket $[x \otimes t^k, y \otimes t^l] = [x, y] \otimes t^{k+l}$.

The G-invariant 2-cocycle on the algebra $L\mathfrak{g}_\mathbb{C}$ (see Section 1.2) is unique up to a scalar [75, Proposition 4.2.4].

Theorem 1.2.1. *Any G-invariant 2-cocycle on $L\mathfrak{g}_\mathbb{C}$ is proportional to the following one.*

$$\omega(\xi, \eta) = \frac{1}{2\pi i} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle d\theta,$$

where $\langle \cdot, \cdot \rangle$ is the unique invariant symmetric form on $\mathfrak{g}_\mathbb{C}$.

At the Lie algebra level, central extensions by proportional cocycles are isomorphic, hence we identify them and denote the equivalence class by $\widehat{\mathfrak{g}_\mathbb{C}}$.

It is not convenient to treat general representations and decomposition into irreducible representations. Instead, in the following we consider a special class of irreducible representations.

First of all, $\mathfrak{g}_\mathbb{C}$ has the triangular decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let $\{H_i\}$ be the basis of $\mathfrak{h}$ with respect to the root decomposition and $\omega_i \in \mathfrak{h}^*$ such that $\omega_i(H_j) = \delta_{i,j}$, and $\widetilde{\alpha}$ be the highest root.

Following this decomposition of $\mathfrak{g}_\mathbb{C}$, we can decompose $\widehat{\mathfrak{g}_\mathbb{C}}$ as follows.

$$\widehat{\mathfrak{g}_\mathbb{C}} = \left(\mathfrak{g}_\mathbb{C} \otimes_\mathbb{C} (\mathbb{C}[t] \ominus \mathbb{C}) \oplus \mathfrak{n}^+\right) \oplus (\mathfrak{h} \oplus \mathbb{C}C) \oplus \left(\mathfrak{g}_\mathbb{C} \otimes_\mathbb{C} (\mathbb{C}[t^{-1}] \ominus \mathbb{C}) \oplus \mathfrak{n}^-\right).$$

This is the triangular decomposition of $\widehat{\mathfrak{g}_\mathbb{C}}$. Put $\hat{\mathfrak{h}} := (\mathfrak{h} \oplus \mathbb{C}C)$. We define weights on $\hat{\mathfrak{h}}$:

$$\gamma(C) = 1, \quad \gamma(H_i) = 0,$$
$$\widetilde{\omega}_i = \omega_i + (\omega_i, \widetilde{\alpha})\gamma,$$
$$\widetilde{\omega}_0 = \frac{1}{2}(\widetilde{\alpha}, \widetilde{\alpha})\gamma.$$

A representation of $\widehat{\mathfrak{g}_{\mathbb{C}}}$ is called a lowest weight representation with weight $\lambda \in \hat{\mathfrak{h}}^*$ if there is a cyclic vector v_0 such that

$$\begin{aligned} hv_0 &= \lambda(h)v_0 \text{ for } h \in \hat{\mathfrak{h}}, \\ x_+v_0 &= 0 \text{ for } x_+ \in \mathfrak{g}_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbb{C}[t] \ominus \mathbb{C}) \oplus \mathfrak{n}^+ \end{aligned}$$

We have the following result [42].

Theorem 1.2.2 (Garland)**.** *A lowest weight representation of $\widehat{\mathfrak{g}_{\mathbb{C}}}$ with weight λ admits a positive-definite contravariant form if and only if λ is dominant integral, namely, $\lambda(H_i) \in \mathbb{N}$.*

Furthermore, in this case the representation is unitary, and admits the action of S^1 as rotation. Moreover, all such representations integrate to positive-energy projective representations of the loop group LG (see Section 1.5.2). There is a one-to-one correspondence between dominant integral condition above and (1.3). Group representations are classified by the level and the weight satisfying (1.3), so there is a one-to-one correspondence between lowest weight irreducible representations of $\widehat{\mathfrak{g}_{\mathbb{C}}}$ and positive-energy projective unitary irreducible representations of LG.

1.2.3 The Virasoro algebra

The Witt algebra (we denote it Witt. See also [81]) is the Lie algebra generated by L_n for $n \in \mathbb{Z}$ with the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n}.$$

The Witt algebra has a central extension with a central element C, unique up to isomorphisms, with the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m,-n}.$$

In other words, this corresponds to a 2-cocycle $\omega(L_m, L_n) = \frac{1}{12}m(m^2-1)\delta_{m,-n}$. This algebra is called the Virasoro algebra Vir. On Witt and Vir we can define a *-operation by

$$(L_n)^* = L_{-n}, C^* = C.$$

The Witt algebra is a subalgebra of the Lie algebra $\mathrm{Vect}(S^1)$ of smooth complex functions on the circle S^1 with the following commutation relations:

$$[f, g] = fg' - f'g,$$

and the correspondence $L_n \mapsto ie^{in\theta}$. Its real part is the Lie algebra of the group of diffeomorphisms of S^1[70]. This algebra is equipped with the smooth topology, namely, a net of functions f_n converges to f if and only if the k-th derivatives $f_n^{(k)}$ converge to $f^{(k)}$ uniformly on S^1 for all $k \geq 0$. The central extension above extends continuously to this algebra. As the group $\mathrm{Diff}(S^1)$ is a manifold modelled on $\mathrm{Vect}(S^1)$, it is equipped with the induced topology of the smooth topology of $\mathrm{Vect}(S^1)$.

1.3 Thermal states in conformal QFT

1.3.1 KMS states on chiral nets: general remarks

In what follows we shall use the "real-line" picture. A linear functional $\psi : \mathfrak{A}_{\mathcal{A}} \to \mathbb{C}$ such that its *local restriction* $\psi|_{\mathcal{A}(I)}$ is normal for every bounded open interval $I \Subset \mathbb{R}$ is said to be **locally normal** on $\mathfrak{A}_{\mathcal{A}}$. Let now ψ be a locally normal state on $\mathfrak{A}_{\mathcal{A}}$ and consider the associated GNS representation π_ψ of $\mathfrak{A}_{\mathcal{A}}$ on the Hilbert space $\mathcal{H}_\psi$ with GNS vector Ψ. By construction, the vector Ψ is cyclic for the algebra $\pi_\psi(\mathfrak{A}_{\mathcal{A}})$ and $\langle \Psi, \pi_\psi(x)\Psi\rangle = \psi(x)$ for every $x \in \mathfrak{A}_{\mathcal{A}}$.

Lemma 1.3.1. *$\mathcal{H}_\pi$ is separable.*

Proof. Let $I \Subset \mathbb{R}$ be a bounded interval. The restriction of $\pi_\psi|_{\mathcal{A}(I)}$ to the Hilbert space $\overline{\pi_\psi(\mathcal{A}(I))\Psi}$ may be viewed as the GNS representation of $\mathcal{A}(I)$ coming from the state $\psi|_{\mathcal{A}(I)}$. It follows that $\overline{\pi_\psi(\mathcal{A}(I))\Psi}$ is separable, since (property (10) in Section 1.1.1) the local algebra $\mathcal{A}(I)$ is a type III_1 factor given on a separable Hilbert space.

Let now $I_n := (-n, n) \in \mathbb{R}$ and $\mathcal{H}_{\psi,n} := \overline{\pi_\psi(\mathcal{A}(I_n))\Psi}$ for every $n \in \mathbb{N}$. Then, on one hand, $\mathcal{H}_{\psi,n}$ is separable for every $n \in \mathbb{N}$; on the other hand, using that every finite length interval I is contained in *some* interval I_n, it follows easily that $\cup_n \mathcal{H}_{\psi,n}$ is dense in $\mathcal{H}_\psi$. Thus $\mathcal{H}_\psi$ is separable, as it is the closure of the union of a countable number of separable Hilbert spaces. □

Corollary 1.3.2. *The restriction of π_ψ to any local algebra $\mathcal{A}(I)$ ($I \Subset \mathbb{R}$) is normal; thus $\mathcal{A}_\psi(I) := \pi_\psi(\mathcal{A}(I))$ is a von Neumann algebra on $\mathcal{H}_\psi$, and $\pi_\psi|_{\mathcal{A}(I)} : \mathcal{A}(I) \to \mathcal{A}_\psi(I)$ is actually a unitarily implementable isomorphism between type III_1 factors.*

Proof. The listed facts follow from the last lemma since $\mathcal{A}(I)$ is a type III_1 factor given on a separable Hilbert space. □

A translation of the real line takes every bounded interval into a bounded interval. Thus the adjoint action of the strongly continuous one-parameter group of unitaries $t \mapsto U(\tau_t)$ associated to translations, which is originally given for the chiral net $\mathcal{A}$, may be viewed as a one-parameter group of $*$-automorphisms of $\mathfrak{A}_{\mathcal{A}}$. Similarly, we may consider dilations, too, as a one-parameter group $t \mapsto \mathrm{Ad}U(\delta_t)$ of $*$-automorphisms of $\mathfrak{A}_{\mathcal{A}}$. We have that

$$\mathrm{Ad}U(\tau_t)(\mathcal{A}(I)) = \mathcal{A}(t+I), ; \qquad \mathrm{Ad}U(\delta_t)(\mathcal{A}(I)) = \mathcal{A}(e^t I)$$

and we have the group relations

$$\delta_s \circ \tau_t = \tau_{e^s t} \circ \delta_s.$$

Let α_t be a one-parameter automorphism group of the C^*-algebra $\mathfrak{A}_{\mathcal{A}}$. A **$\beta$-KMS state** φ on $\mathfrak{A}_{\mathcal{A}}$ with respect to α_t is a state with the following condition: for any $x, y \in \mathfrak{A}_{\mathcal{A}}$ there is an analytic function f on the strip $0 < \Im z < \beta$, bounded and continuous on the closure of the strip, such that

$$f(t) = \varphi(x\alpha_t(y)), \quad f(t + i\beta) = \varphi(\alpha_t(y)x).$$

In Chapter 4 we will be interested in states on $\mathfrak{A}_\mathcal{A}$ satisfying the β-KMS condition w.r.t. the one-parameter group $t \mapsto \mathrm{Ad}U(\tau_t)$. As a direct consequence of the last recalled group-relations, φ is such a β-KMS state if and only if $\varphi \circ \mathrm{Ad}U(\delta_t)$ is a KMS state with *inverse temperature* β/e^t. Thus it is enough to study KMS states at the fixed inverse temperature $\beta = 1$, which we shall simply call a **KMS state**.

A KMS state φ of $\mathfrak{A}_\mathcal{A}$ w.r.t. $t \mapsto \mathrm{Ad}U(\tau_t)$ is in particular an invariant state for $t \mapsto \mathrm{Ad}U(\tau_t)$. Thus, considering the GNS representation π_φ associated to φ on the Hilbert space $\mathcal{H}_\varphi$ with GNS vector Φ, we have that there exists a unique one-parameter group of unitaries $t \mapsto V_\varphi(t)$ of $\mathcal{H}_\varphi$ such that

$$V_\varphi(t)\pi_\varphi(x)\Phi = \pi_\varphi(\mathrm{Ad}U(\tau_t(x)))\Phi$$

for all $t \in \mathbb{R}$ and $x \in \mathfrak{A}_\mathcal{A}$. It is well-known that Φ is automatically cyclic and separating for the von Neumann algebra $\pi_\varphi(\mathfrak{A}_\mathcal{A})''$ [83], and that the associated modular group $t \mapsto \Delta^{it}$ actually coincides with $t \mapsto V_\varphi(t)$.

By the general result [83, Theorem 1], a KMS state is automatically locally normal. Moreover, by [83, Theorem 4.5] every KMS state can be decomposed into *primary* KMS states. We recall that a KMS state φ is **primary** iff it cannot be written as a nontrivial convex combination of other KMS states and that it is equivalent with the property that $\pi_\varphi(\mathfrak{A}_\mathcal{A})''$ is a factor.

We also recall the KMS version of the well-known Reeh-Schlieder property. Its proof relies on standard arguments, see e.g. [61, Prop. 3.1].

Lemma 1.3.3. *Let φ be a KMS state on $\mathfrak{A}_\mathcal{A}$ w.r.t. the one-parameter group $t \mapsto \mathrm{Ad}U(\tau_t)$, and let π_φ be the associated GNS representation with GNS vector Φ. Then Φ is cyclic and separating for $\pi_\varphi(\mathcal{A}(I))$ for every bounded (nonempty, open) interval $I \Subset \mathbb{R}$.*

1.3.2 The geometric KMS state

Here we show that every local, diffeomorphism covariant net $\mathcal{A}$ admits at least one KMS state, indeed this state has a geometric origin. The construction of this geometric KMS state φ_{geo} is essential for our results, hence we include it in the present Chatpter.

The geometric KMS state is constructed using two properties: Bisognano-Wichmann property (valid also in higher dimensions), which implies that the vacuum state is a KMS state for the C^*-algebra $\mathcal{A}(\mathbb{R}_+)$ w.r.t. dilations; diffeomorphism covariance, by which it is (locally) possible to find a map from $\mathbb{R}$ to $\mathbb{R}_+$ that sends translations to dilations. Such a map would (globally) be the exponential, which is not a diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, but for any given interval we can find a diffeomorphism which coincides with the exponential map on that interval.

Proposition 1.3.4. *For any conformal net* $\mathcal{A}$*, there is a canonical injective endomorphism* Exp *of the* C^**-algebra* $\mathfrak{A}_\mathcal{A} \equiv \overline{\bigcup_{I \Subset \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}$ *such that*

(1) $\mathrm{Exp}\,(\mathcal{A}\,(I)) = \mathcal{A}\left(e^{2\pi I}\right)$

(2) $\mathrm{Exp} \circ \mathrm{Ad}U(\tau_t) = \mathrm{Ad}U(\delta_{2\pi t}) \circ \mathrm{Exp}$,

(3) Exp *is a* C^**-algebra isomorphism of* $\mathfrak{A}_\mathcal{A}$ *with* $\mathfrak{A}(\mathbb{R}_+) \equiv \overline{\bigcup_{I \Subset \mathbb{R}_+} \mathcal{A}(I)}^{\|\cdot\|}$.

Proof. For any $I \Subset \mathbb{R}$, choose a map $\eta_I \in C^\infty(\mathbb{R}, \mathbb{R})$ such that: $\eta_I(t) = e^{2\pi t}$, $\forall t \in I$; outside an interval $J \Subset \mathbb{R}$ (J has to contain both I and $e^{2\pi I}$) η_I is the identity map $\eta_I(t) = t$; $\eta_I^{-1} \in C^\infty(\mathbb{R}, \mathbb{R})$. Then η_I is a diffeomorphism and has a unitary representative $U(\eta_I)$ such that $\mathrm{Ad}U(\eta_I)(\mathcal{A}(J)) = \mathcal{A}(\eta_I J)$ and in particular $\mathrm{Ad}U(\eta_I)(\mathcal{A}(I)) = \mathcal{A}\left(e^{2\pi I}\right)$. Set $\mathrm{Exp}|_{\mathcal{A}(I)} = \mathrm{Ad}U(\eta_I)$, this is a well-defined endomorphism of $\cup_{I\Subset\mathbb{R}}\mathcal{A}(I)$ (since $\mathrm{Ad}U(\eta_I)|_{\mathcal{A}(I)} = \mathrm{Ad}U(\eta_J)|_{\mathcal{A}(I)}$ whenever $I \subset J$) which can be extended to the norm closure $\mathfrak{A}_\mathcal{A}$ satisfying (1) and (3). Condition (2) follows from the corresponding relation for maps of $\mathbb{R}$, $\eta_I \circ \tau_t = \delta_{2\pi t} \circ \eta_I$, and the fact that, on every local algebra $\mathcal{A}(I)$,

$$\begin{aligned}\mathrm{Exp} \circ \mathrm{Ad}U(\tau_t) &= \mathrm{Ad}U(\eta_I) \circ \mathrm{Ad}U(\tau_t) = \mathrm{Ad}U(\eta_I \circ \tau_t) = \\ &= \mathrm{Ad}U(\delta_{2\pi t} \circ \eta_I) = \mathrm{Ad}U(\delta_{2\pi t}) \circ \mathrm{Ad}U(\eta_I) = \mathrm{Ad}U(\delta_{2\pi t}) \circ \mathrm{Exp}.\end{aligned}$$

□

Theorem 1.3.5. *For any conformal net* $\mathcal{A}$*, the state* $\varphi_{\mathrm{geo}} := \omega \circ \mathrm{Exp}$ *is a primary KMS state w.r.t. translations.*

Proof. By definition, the GNS representation of φ_{geo} is (unitarily equivalent to) the composition of the vacuum (identity) representation with Exp: $(\mathrm{Exp}, \mathcal{H}_\Omega, \Omega)$. Thus $\pi_{\varphi_{\mathrm{geo}}}(\mathfrak{A}_\mathcal{A})'' = \mathcal{A}(\mathbb{R}_+)$ which is a factor: φ_{geo} is a primary state.

The vector Ω is cyclic and separating for $\mathcal{A}(\mathbb{R}_+)$ and by the Bisognano-Wichmann property the modular group is the group $t \mapsto U(\delta_{2\pi t})$ of (rescaled) dilations (dilations associated to the interval $\mathbb{R}_+ \subset S^1$, i.e. the "true" dilations), therefore $\mathrm{Ad}\Delta_\Omega^{it} \circ \mathrm{Exp} = \mathrm{Ad}U(\delta_{2\pi t}) \circ \mathrm{Exp} = \mathrm{Exp} \circ \mathrm{Ad}U(\tau_t)$.

Hence, as the modular group w.r.t. Ω is the translation group for the represented net $\mathbb{R} \Supset I \mapsto \mathrm{Exp}\,(\mathcal{A}\,(I))$, the vector state Ω is a KMS state w.r.t. translations. □

Remark 1.3.6. Consider the case where $\mathcal{A}$ is strongly additive. Then, in the vacuum representation of $\mathcal{A}$, we have $\mathcal{A}\,(e^{2\pi a}, \infty) \cap \mathcal{A}\left(e^{2\pi b}, \infty\right)' = \mathcal{A}\left(e^{2\pi a}, e^{2\pi b}\right)$, therefore, by construction, $\mathcal{A}_{\mathrm{geo}}(a, \infty) \cap \mathcal{A}_{\mathrm{geo}}(b, \infty)' = \mathcal{A}_{\mathrm{geo}}(a, b)$, for any $a < b < \infty$, where $\mathcal{A}_{\mathrm{geo}} = \mathcal{A}_{\varphi_{\mathrm{geo}}}$ is defined as in eq. (4.3) here below.

By the same arguments used in the proof of Theorem 1.3.5, we have the following.

Proposition 1.3.7. *There is a one-to-one map between the sets of*

- *KMS states on* $\mathfrak{A}(\mathbb{R}_+) \equiv \overline{\bigcup_{I \Subset \mathbb{R}_+} \mathcal{A}(I)}^{\|\cdot\|}$ *with respect to dilations*

- *KMS states on* $\mathfrak{A}_{\mathcal{A}} \equiv \overline{\bigcup_{I \Subset \mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}$ *with respect to translations.*

The correspondence is given by $\varphi \mapsto \varphi \circ \mathrm{Exp}$.

By definition, the *geometric* KMS state φ_{geo} of $\mathfrak{A}_{\mathcal{A}}$ is the KMS state corresponding to the vacuum state on $\mathfrak{A}(\mathbb{R}_+)$ according to the above proposition: $\varphi_{\mathrm{geo}} \equiv \omega \circ \mathrm{Exp}$.

1.4 Scattering theory of massless models in two dimensions

1.4.1 Scattering theory of waves

Here we summarize the scattering theory of massless two-dimensional models established in [11]. This theory is stated in terms of Poincaré covariant nets of observables.

Let us denote by $T(a) := U(\tau(a))$ the representative of spacetime translation $\tau(a)$ by $a \in \mathbb{R}^2$. Furthermore, we denote the lightlike translations by $T_\pm(t) := T((t, \pm t))$. Let $\mathbf{P}$ denote the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by (one-dimensional) translations and dilations. Note that $\mathbf{P}$ is simply connected, hence it can be considered as a subgroup of $\overline{\mathrm{PSL}(2,\mathbb{R})}$. As will be seen in the following, the representation U of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ restricted to $\mathbf{P} \times \mathbf{P}$ has typically a big multiplicity in Möbius covariant theories. The subspaces $\mathcal{H}_+ = \{\xi \in \mathcal{H} : T_+(t)\xi = \xi \text{ for all } t\}$ and $\mathcal{H}_- = \{\xi \in \mathcal{H} : T_-(t)\xi = \xi \text{ for all } t\}$ are referred to as the spaces of waves with positive and negative momentum, respectively. Let $P_\pm$ be the orthogonal projections onto $\mathcal{H}_\pm$, respectively.

Let x be a local operator, i.e., an element of $\mathcal{A}(O)$ for some O. We set $x(a) := T(a)xT(a)^*$ for $a \in \mathbb{R}^2$ and consider a family of operators parametrized by $\mathcal{T}$:

$$x_\pm(h_{\mathcal{T}}) := \int dt\, h_{\mathcal{T}}(t) x((t, \pm t)),$$

where $h_{\mathcal{T}}(t) = |\mathcal{T}|^{-\varepsilon} h(|\mathcal{T}|^{-\varepsilon}(t - \mathcal{T}))$, $0 < \varepsilon < 1$ is a constant, $\mathcal{T} \in \mathbb{R}$ and h is a nonnegative symmetric smooth function on $\mathbb{R}$ such that $\int dt\, h(t) = 1$.

Lemma 1.4.1 ([11] Lemma 1,2,3). *Let x be a local operator. Then the limit $\Phi^{\mathrm{in}}_\pm(x) := \operatorname*{s-lim}_{\mathcal{T} \to -\infty} x_\pm(h_{\mathcal{T}})$ exists and it holds that*

- $\Phi^{\mathrm{in}}_\pm(x)\Omega = P_\pm x \Omega$.
- $\Phi^{\mathrm{in}}_\pm(x)\mathcal{H}_\pm \subset \mathcal{H}_\pm$.
- $\mathrm{Ad}U(g)\Phi^{\mathrm{in}}_\pm(x) = \Phi^{\mathrm{in}}_\pm(\mathrm{Ad}U(g)(x))$, *where* $g \in \mathcal{P}^\uparrow_+$.

Furthermore, the limit $\Phi^{\mathrm{in}}_\pm(x)$ depends only on $P_\pm x\Omega$, respectively. We call these limit operators the "incoming asymptotic fields". It holds that $[\Phi^{\mathrm{in}}_+(x), \Phi^{\mathrm{in}}_-(y)] = 0$ for arbitrary local x and y.

Similarly one defines the "outgoing asymptotic fields" by $\Phi^{\mathrm{out}}_\pm(x) := \operatorname*{s-lim}_{\mathcal{T} \to \infty} x_\pm(h_{\mathcal{T}})$

Remark 1.4.2. As the asymptotic field is defined as the limit of local operators, it still has certain local properties. For example, let O_+ and O_0 be two regions such that O_+ stays in the future of O_0 and $x \in O_+, y \in O_0$. Then it holds that $[\Phi^{\mathrm{in}}_\pm(x), y] = 0$, since for a negative $\mathcal{T}$ with sufficiently large absolute value, $x_\pm(h_\mathcal{T})$ lies in the spacelike complement of y. Similar observations apply also to $\Phi^{\mathrm{out}}_\pm$.

Lemma 1.4.1 captures the dispersionless kinematics of elementary excitations in two-dimensional massless theories: since $\Phi^{\mathrm{in}}_\pm(x)\mathcal{H}_\pm \subset \mathcal{H}_\pm$, by composing two waves travelling to the right we obtain again a wave travelling to the right. Thus waves are, in general, composite objects, associated with reducible representations of the Poincaré group. Moreover, it follows that collision states of waves may contain at most two excitations: One wave with positive momentum and the other with negative momentum.

Let us now construct these collision states: For $\xi_\pm \in \mathcal{H}_\pm$, there are sequences of local operators $\{x_{\pm,n}\}$ such that $\operatorname*{s-lim}_{n\to\infty} P_\pm x_{\pm,n}\Omega = \xi_\pm$ and Using these sequences let us define collision states following [11] (see also [34]):

$$\begin{aligned}
\xi_+ \overset{\mathrm{in}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi^{\mathrm{in}}_+(x_{+,n})\Phi^{\mathrm{in}}_-(x_{-,n})\Omega \\
\xi_+ \overset{\mathrm{out}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi^{\mathrm{out}}_+(x_{+,n})\Phi^{\mathrm{out}}_-(x_{-,n})\Omega
\end{aligned}$$

We interpret $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$) as the incoming (respectively outgoing) state which describes two non-interacting waves ξ_+ and ξ_-. These asymptotic states have the following natural properties.

Lemma 1.4.3 ([11] Lemma 4). *For the collision states $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ and $\eta_+ \overset{\mathrm{in}}{\times} \eta_-$ it holds that*

1. $\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle = \langle \xi_+, \eta_+ \rangle \cdot \langle \xi_-, \eta_- \rangle$.
2. $U(g)(\xi_+ \overset{\mathrm{in}}{\times} \xi_-) = (U(g)\xi_+) \overset{\mathrm{in}}{\times} (U(g)\xi_-)$ *for all* $g \in \mathcal{P}^\uparrow_+$.

And analogous formulae hold for outgoing collision states.

Furthermore, we define the spaces of collision states: Namely, we let $\mathcal{H}^{\mathrm{in}}$ (respectively $\mathcal{H}^{\mathrm{out}}$) be the subspace generated by $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$). From the Lemma above, we see that the following map

$$S : \xi_+ \overset{\mathrm{out}}{\times} \xi_- \longmapsto \xi_+ \overset{\mathrm{in}}{\times} \xi_-$$

is an isometry. The operator $S : \mathcal{H}^{\mathrm{out}} \to \mathcal{H}^{\mathrm{in}}$ is called the **scattering operator** or the **S-matrix**. We say the waves in $\mathcal{A}$ are **interacting** if S is not a constant multiple of the identity operator on $\mathcal{H}^{\mathrm{out}}$. The purpose of these Sections is to show that $S = \mathbb{1}$ on $\mathcal{H}^{\mathrm{out}}$ for Möbius covariant nets and to determine $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}}$ in terms of chiral observables (see Section 5.2). As a corollary one observes that a Möbius covariant net is chiral if and only if it is **asymptotically complete**, i.e. $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}} = \mathcal{H}$. Moreover, we show in Subsection 5.3.1 that if a net is Poincaré covariant and asymptotically complete, then it is noninteracting if and only if it is a chiral Möbius covariant net (see Section 1.1.8).

1.5 Examples of nets on S^1

1.5.1 The $U(1)$-current net and Longo-Witten endomorphisms

We briefly summarize here some facts about the net called the $U(1)$-current net, or the (chiral part of) free massless bosonic field. On this model, there has been found a family of Longo-Witten endomorphisms [64]. We will construct a wedge-local net for each of these endomorphisms. This model has been studied with the algebraic approach since the fundamental paper [16]. We refer to [63] for the notations and the facts in the following.

A fundamental ingredient is an irreducible unitary representation of the Möbius group with the lowest weight 1: Namely, we take the representation of $\mathrm{PSL}(2,\mathbb{R})$ of which the smallest eigenvalue of the rotation subgroup is 1. We call the Hilbert space $\mathcal{H}^1$. We take a specific realization of this representation. Namely, let $C^\infty(S^1,\mathbb{R})$ be the space of real-valued smooth functions on S^1. This space admits a seminorm

$$\|f\| := \sum_{k\geq 0} 2k|\hat{f}_k|^2,$$

where $\hat{f}_k$ is the k-th Fourier component of f, and a complex structure

$$(\mathcal{I}f)_k = -i\mathrm{sign}(k)\hat{f}_k.$$

Then, by taking the quotient space by the null space with respect to the seminorm, we obtain the complex Hilbert space $\mathcal{H}^1$. We say $C^\infty(S^1,\mathbb{R}) \subset \mathcal{H}^1$. On this space, there acts $\mathrm{PSL}(2,\mathbb{R})$ by naturally extending the action on $C^\infty(S^1,\mathbb{R})$.

Let us denote $\mathcal{H}^n := \mathcal{H}^{\otimes n}$ for a nonnegative integer n. On this space, acts the symmetric group S_n. Let P_n be the projection onto the invariant subspace with respect to this action. We put $\mathcal{H}^n_s := P_n\mathcal{H}^n$ and the **symmetric Fock space**

$$\mathcal{H}^\Sigma_s := \bigoplus_n \mathcal{H}^n_s,$$

and this will be the Hilbert space of the $U(1)$-current net on S^1. For $\xi \in \mathcal{H}^1$, we denote by e^ξ a vector of the form $\sum_n \frac{1}{n!}\xi^{\otimes n} = 1 \oplus \xi \oplus \left(\frac{1}{2}\xi\otimes\xi\right) \oplus \cdots$. Such vectors form a total set in $\mathcal{H}^\Sigma_s$. The **Weyl operator** of ξ is defined by $W(\xi)e^\eta = e^{-\frac{1}{2}\langle\xi,\xi\rangle - \langle\xi,\eta\rangle}e^{\xi+\eta}$.

The Hilbert space $\mathcal{H}^\Sigma_s$ is naturally included in the **unsymmetrized Fock space**:

$$\mathcal{H}^\Sigma := \bigoplus_n (\mathcal{H}^1)^{\otimes n} = \mathbb{C} \oplus \mathcal{H}^1 \oplus \left(\mathcal{H}^1 \otimes \mathcal{H}^1\right) \oplus \cdots$$

We denote by P_s the projection onto $\mathcal{H}^\Sigma_s$. For an operator X_1 on the one particle space$\mathcal{H}^1$, we define the **second quantization** of X_1 on $\mathcal{H}^\Sigma_s$ by

$$\Gamma(X_1) := \bigoplus_n (X_1)^{\otimes n} = 1 \oplus X_1 \oplus (X_1 \otimes X_1) \oplus \cdots$$

Obviously, $\Gamma(X_1)$ restricts to the symmetric Fock space $\mathcal{H}_s^\Sigma$. We still write this restriction by $\Gamma(X_1)$ if no confusion arises. For a unitary operator $V_1 \in B(\mathcal{H}^1)$ and $\xi \in \mathcal{H}^1$, it holds that $\Gamma(V_1)e^\xi = e^{V_1\xi}$ and $\mathrm{Ad}\Gamma(V_1)(W(\xi)) = W(V_1\xi)$. On the one particle space $\mathcal{H}^1$, there acts the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ irreducibly by U_1. Then $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathcal{H}^\Sigma$ and on $\mathcal{H}_s^\Sigma$ and by $\Gamma(U_1(g))$, $g \in \mathrm{PSL}(2,\mathbb{R})$. The representation of the translation subgroup in $\mathcal{H}^1$ is denoted by $T_1(t) = e^{itP_1}$ with the generator P_1.

The $U(1)$**-current net** $\mathcal{A}_{U(1)}$ is defined as follows:

$$\mathcal{A}_{U(1)}(I) := \{W(f) : f \in C^\infty(S^1,\mathbb{R}) \subset \mathcal{H}^1, \mathrm{supp}(f) \subset I\}''.$$

The vector $1 \in \mathbb{C} = \mathcal{H}^0 \subset \mathcal{H}_s^\Sigma$ serves as the vacuum vector and $\Gamma(U_1(\cdot))$ implements the Möbius symmetry. We denote by T_s^Σ the representation of one-dimensional translation of $\mathcal{A}_{U(1)}$.

For this model, a large family of endomorphisms has been found by Longo and Witten.

Theorem 1.5.1 ([64], Theorem 3.6)**.** *Let φ be an inner symmetric function on the upper-half plane $\mathbb{S}_\infty \subset \mathbb{C}$: Namely, φ is a bounded analytic function of $\mathbb{S}_\infty$ with the boundary value $|\varphi(p)| = 1$ and $\varphi(-p) = \overline{\varphi(p)}$ for $p \in \mathbb{R}$. Then $\Gamma(\varphi(P_1))$ commutes with T_s^Σ (in particular $\Gamma(\varphi(P_1))\Omega = \Omega$) and $\mathrm{Ad}\Gamma(\varphi(P_1))$ preserves $\mathcal{A}_{U(1)}(\mathbb{R}_+)$. In other words, $\Gamma(\varphi(P_1))$ implements a Longo-Witten endomorphism of $\mathcal{A}_{U(1)}$.*

1.5.2 The loop group nets

In this section, we collect notations and basic results on loop groups and loop algebras. G represents a simple and simply connected Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. Traditionally, infinite-dimensional Lie algebras have been a fundamental tool in conformal field theory. The representation theory of infinite-dimensional algebras inevitably contains unbounded operators on infinite dimensional vector spaces, and standard notions as irreducibility and decomposition of representations could be difficult or practically not appropriate to define in total generality. On the other hand, at the group level the representation theory of compact group gives us a strong device for such reduction.

Loop groups

Let us indroduce the loop group LG of G:

$$LG := \{g : S^1 \to G, \text{smooth}\},$$
$$g_1 \cdot g_2(z) := g_1(z) \cdot g_2(z), z \in S^1.$$

It is also possible to define a natural topology on the group LG, and there is a smooth map from the neighbourhood of the unit element of LG to the neighbourhood of 0 in $L\mathfrak{g}$. The group operation corresponds to the bracket [70] [75].

A 2-cocycle on a group H with values in $\mathbb{T}$ is a map $\gamma : H \times H \to \mathbb{T}$ which satisfies (see [81, Chapter 3])

$$\gamma(e,e) = 1, \quad \gamma(f,g)\gamma(fg,h) = \gamma(f,gh)\gamma(g,h),$$

where e is the unit element of H. If there is $\beta : H \to \mathbb{T}$ such that $\gamma(f,g) = \frac{\beta(f)\beta(g)}{\beta(fg)}$, then γ is said to be coboundary. The set of 2-cocycles forms a group by defining the product with pointwise multiplication. If one cocycle is a multiple of a coboundary with another cocycle, these two cocycles are said to be equivalent.

A group $\widetilde{LG}$ is called a central extension of LG by $\mathbb{T}$ if there is an exact sequence

$$0 \to \mathbb{T} \to \widetilde{LG} \to LG \to 0.$$

A central extension is said to be split if $\widetilde{LG} \cong \mathbb{T} \times LG$. There is a one-to-one correspondence between central extensions of LG and equivalence classes of 2-cocycles [81].

The following is fundamental [75, Chapter 4].

Theorem 1.5.2 (Pressley and Segal). *If G is simple and simply connected, then there exists a family of central extensions of LG which are parametrized by positive integers, and all such extensions come from the central extensions of $L\mathfrak{g}$ (see below).*

The 2-cocycles of the group appear when we consider projective representations.

Definition 1.5.3. *Let $\mathcal{H}$ be a Hilbert space. A map $\pi : LG \to U(\mathcal{H})$ is a projective unitary representation if there is a 2-cocycle γ of G such that*

$$\pi(g_1)\pi(g_2) = \gamma(g_1, g_2)\pi(g_1 \cdot g_2).$$

If we have a projective representation of LG, by definition it also specifies a 2-cocycle of LG, and this cocycle determines the class of central extensions. We call this class the **level** of the representation. We can naturally think that the given projective representation of LG as a "true" representation, not projective, of the central extension.

Note that the circle S^1 acts on LG by rotation:

$$g_\theta(z) := g(e^{-i\theta}z).$$

Definition 1.5.4. *A projective unitary representation π of LG is said to have positive energy if there is a unitary representation U of S^1 on the same Hilbert space with positive spectrum such that*

$$U(\theta)\pi(g)U(\theta)^* = \pi(g_\theta).$$

Remark 1.5.5. Let us define the action of rotation on the space of 2-cocycles by $\gamma_\theta(g_1, g_2) = \gamma((g_1)_{-\theta}, (g_2)_{-\theta})$. Then any positive energy representation has a 2-cocycle which is invariant under this action of translation.

The loop group LG contains constant loops. The set of constant loops forms a subgroup isomorphic to G. A constant loop is of course invariant under rotation, hence it commutes with the action of rotation. As seen below, the restriction of the central extension to this subgroup of constant loops splits, hence we may assume that any projective representation of LG is associated with a true representation of G. By the positivity of energy, U has a lowest eigenvalue. Since the subgroup of constant loops commutes with U, G acts on this eigenspace. When the full representation is irreducible, this restriction should be irreducible.

Now we can state the classification result [75, Chapter 9]

Theorem 1.5.6 (Pressley and Segal). *Any smooth positive-energy projective unitary representation of LG is completely reducible. Smooth positive-energy projective unitary irreducible representations of LG can be classified by the level h and the lowest weight λ of G on the lowest eigenspace of U. Such a representation is possible if and only if*

$$-\frac{1}{2}h\|h_\alpha\|^2 \leq \lambda(h_\alpha) \leq 0 \tag{1.3}$$

for each positive root α and h_α is the coroot.

All such representations are diffeomorphism covariant: Namely, there is a projective unitary representation U of $\mathrm{Diff}(S^1)$ such that $U(\gamma)\pi(g)U(\gamma)^* = \pi(g \circ \gamma^{-1})$, where the composition $g \circ \gamma^{-1}$ is again an element of LG.

In particular, for each level h there are finitely many representations. In terms of CFT, a representation with $\lambda = 0$ corresponds to the vacuum representation. Each representation of LG with a different level corresponds to a different theory, and different weights corresponds to different sectors. Finiteness of representations is the source of rationality of these loop group models.

The loop group nets

For a vacuum representation π ($\lambda = 0$) at level h of the loop group LG, we define

$$\mathcal{A}_{G,h}(I) := \{\pi(\gamma) : \mathrm{supp}\gamma \subset I\}''$$

and we call it the **loop group net** of G at level h. All such nets are conformal thanks to the diffeomorphism covariance of the group representation. It is known that for $G = SU(N)$, $\mathcal{A}_{G,h}$ is completely rational [97].

1.5.3 The Virasoro nets

There are examples of nets generated by the diffeomorphism symmetry itself (see also Section 1.1.4). More precisely, we consider the positive-energy (projective) representations of the group $\mathrm{Diff}(S^1)$: Namely, π is said to be a positive-energy representation if π is a projective representation of $\mathrm{Diff}(S^1)$ and the one-parameter group of rotation has a positive generator. Since we consider projective representations, π determines a class of cocycle. This is called the central charge of π and it is parametrized by a real number c. The (central extension of the) group $\mathrm{Diff}(S^1)$ has a subgroup S^1 of rotations and by positivity of energy the subgroup has the lowest eigenvalue $h \geq 0$. It is known for which values of c and h there exist irreducible, unitary, positive-energy, projective representations of $\mathrm{Diff}(S^1)$ [49]. All such representations are classified by c and h. If π has $h = 0$, it is called a vacuum representation.

For a vacuum representation π, we construct a net as follows.

$$\mathrm{Vir}_c(I) := \{\pi(g) : \mathrm{supp}(g) \subset I\}''.$$

This is called the Virasoro net with the central charge c. Virasoro nets have a very different nature for $c < 1$ and $c \geq 1$. Indeed, for $c < 1$, it is known that Vir_c is completely rational [52]. On the other hand, for $c \geq 1$, Vir_c admits infinitely many sectors and for $c > 1$, it is not even strongly additive [18].

The importance of the Virasoro nets lies in the fact that any conformal (diffeomorphism covariant) net contains a Virasoro net as a subnet. This fact has been exploited in the classification of the nets with the central charge $c < 1$ [52]. On the other hand, Vir_c with $c \geq 1$ can be embedded in the $U(1)$-current net in a translation-covariant way, and we use this fact to construct infintely many KMS states on Vir_c in Chapter 4.

The group $\mathrm{Diff}(S^1)$ is an infinite-dimensional Lie group, and its Lie algebra is $\mathrm{Vect}(S^1)$, the space of all the smooth vector fields on S^1 [70]. The algebra $\mathrm{Vect}(S^1)$ includes a subalgebra of finite trigonometric series called the Vitt algebra. We will study a subalgebra of Vitt algebra in Chapter 2.

Chapter 2

The stabilizer subgroup of one point in $\mathrm{Diff}(S^1)$

Chapter Introduction

In this Chapter we study a certain subalgebra of the Virasoro algebra. The Virasoro algebra is a fundamental object in conformal quantum field theory.

The symmetry group of the chiral component of a conformal field theory in 1+1 dimension is B_0, the group of all orientation-preserving diffeomorphisms of the real line which are smooth at the point at infinity (for example, see [81]). Instead of working on $\mathbb{R}$, it is customary to consider a chiral model on the compactified line S^1 with the symmetry group $\mathrm{Diff}(S^1)$ as we saw in Section 1.1.1. In a quantum theory, we are interested in its projective representations.

With positivity of the energy, which is a physical requirement, the representation theory of the central extension of $\mathrm{Diff}(S^1)$ has been well studied [81]. In any irreducible unitary projective representation of $\mathrm{Diff}(S^1)$, the central element acts as a scalar c. The (central extension of the) group $\mathrm{Diff}(S^1)$ has a subgroup S^1 of rotations and by positivity of energy the subgroup has the lowest eigenvalue $h \geq 0$. It is known for which values of c and h there exist irreducible, unitary, positive-energy, projective representations of $\mathrm{Diff}(S^1)$. All such representations are classified by c and h.

The Lie algebra of $\mathrm{Diff}(S^1)$ is the algebra of all the smooth vector fields on S^1 [70]. It is sometimes convenient to study its polynomial subalgebra, the Witt algebra. The Witt algebra has a unique central extension [81] called the Virasoro algebra Vir. In a similar way as above, we can define lowest energy representations of Vir with parameters c, h and it is known when these representations are unitary [49]. On the other hand, for any positive energy, unitary lowest weight representation of Vir there is a corresponding projective representation of $\mathrm{Diff}(S^1)$ [43].

In a physical context, conformal field theory in 1+1 dimensional Minkowski space has the chiral components on two lightlines (see Section 5.4). Thus it is mathematically useful to study the subgroup B_0 of stabilizers of one point ("the point at infinity") of $\mathrm{Diff}(S^1)$.

We can construct nets of von Neumann algebras on $\mathbb{R}$ from representations of B_0, and nets on $\mathbb{R}^2$ by tensor product. The theory of local quantum physics are extensively studied with techniques of von Neumann algebras [46, 7, 51]. In the case of nets on S^1, the nets generated by $\mathrm{Diff}(S^1)$ play a key role in the classification of diffeomorphism covariant nets [52]. This gives a strong motivation for studying the representation theory of B_0, since for nets on $\mathbb{R}$ the group B_0 should play a similar role to that of $\mathrm{Diff}(S^1)$ for nets on S^1.

Some properties of the restrictions of representations of $\mathrm{Diff}(S^1)$ to B_0 have been studied. For example, the restriction to B_0 of every irreducible unitary positive energy representation of $\mathrm{Diff}(S^1)$ is irreducible [94]. Different values of c, h may correspond to equivalent representations [94]. Unfortunately little is known about representations which are not restrictions. In this Chapter we address this problem.

The positivity of the energy for $\mathrm{Diff}(S^1)$ is usually defined as the boundedness from below of the generator of the group of rotation (since we consider projective representations, the generator of a one-parameter subgroup is defined only up to an addition of a real scalar multiple of the identity). It is well known that this is equivalent to the boundedness from below of the generator of the group of translations (see [63]). The latter definition is the one having its origin in physics. Concerning the group B_0, as it does not include the group of rotations, the positivity of energy is defined by boundedness from below of the generator of the group of translations.

In section 2.2, we determine the first and second cohomologies of the Lie algebra $\mathcal{K}_0$ of the group B_0. The first cohomology corresponds to one dimensional representations and the second cohomology corresponds to central extensions. It will be shown that the only possible central extension is the natural inclusion into the Virasoro algebra. On the other hand the first cohomology is one dimensional and does not extend to Vir.

In section 2.3, we determine the ideal structure of $\mathcal{K}_0$ and calculate their commutator subalgebras. It will be shown that all of these ideals can be defined by the vanishing of certain derivatives at the point at infinity.

In section 2.4, we determine the automorphism group of the central extension $\mathcal{K}$ of $\mathcal{K}_0$. This group turns out to be very small but contains some elements not extending to automorphisms of the Virasoro algebra.

In section 2.5, we construct several representations of $\mathcal{K}$. Each of these representations has an analogue of a lowest weight vector and has the universal property. Thanks to the result of Feigin and Fuks [36], we can completely determine which of these representations are irreducible.

In section 2.6, we investigate the endomorphism semigroup of $\mathcal{K}$. Compositions of these endomorphisms with known unitary representations give rise some strange kinds of representations. Corresponding representations of the group B_0 are studied in section 2.7.

2.1 Preliminaries

We consider a subspace $\mathcal{K}_0$ of the Witt algebra (see Section 1.2.3) spanned by $K_n = L_n - L_0$ for $n \neq 0$. By a straightforward calculation this subspace is indeed a *-subalgebra with the following commutation relations:

$$[K_m, K_n] = \begin{cases} (m-n)K_{m+n} - mK_m + nK_n & (m \neq -n) \\ -mK_m - mK_{-m} & (m = -n) \end{cases} .$$

We denote $\mathrm{Vect}(S^1)_0 \subset \mathrm{Vect}(S^1)$ the subalgebra of smooth functions which vanish on $\theta = 0$. This is the Lie algebra of the group B_0 of all the diffeomorphisms of S^1 which stabilize $\theta = 0$. The algebra $\mathcal{K}_0$ is a *-subalgebra of $\mathrm{Vect}(S^1)_0$.

We will show that $\mathcal{K}_0$ has a unique (up to isomorphisms) central extension which is a subalgebra of Vir. The central extension is denoted by $\mathcal{K}$ and has the following commutation relations:

$$[K_m, K_n] = \begin{cases} (m-n)K_{m+n} - mK_m + nK_n & (m \neq -n) \\ -mK_m - mK_{-m} + \frac{C}{12}m(m^2-1) & (m = -n) \end{cases} . \tag{2.1}$$

2.2 First and Second cohomologies of $\mathcal{K}_0$

We will discuss the following cohomology groups of $\mathcal{K}_0$ [81]:

$$\begin{aligned} H^1(\mathcal{K}_0, \mathbb{C}) &:= \{\phi : \mathcal{K}_0 \to \mathbb{C} |\ \phi \text{ is linear and vanishes on } [\mathcal{K}_0, \mathcal{K}_0].\} \\ Z^2(\mathcal{K}_0, \mathbb{C}) &:= \{\omega : \mathcal{K}_0 \times \mathcal{K}_0 \to \mathbb{C} |\ \omega \text{ is bilinear and} \\ &\quad \text{for } a, b, c \in \mathcal{K}_0 \text{ satisfies } \omega(a,b) = -\omega(b,a), \\ &\quad \omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b) = 0\} \\ B^2(\mathcal{K}_0, \mathbb{C}) &:= \{\omega : \mathcal{K}_0 \times \mathcal{K}_0 \to \mathbb{C} |\ \text{there is } \mu \text{ s.t } \omega(a,b) = \mu([a,b]).\} \\ H^2(\mathcal{K}_0, \mathbb{C}) &:= Z^2/B^2. \end{aligned}$$

Elements in the (additive) group H^1 correspond to one dimensional representations of $\mathcal{K}_0$. The group H^2 corresponds to the set of all central extensions of $\mathcal{K}_0$. We call H^1 and H^2 the first and the second cohomology groups of $\mathcal{K}_0$, respectively.

Lemma 2.2.1. $[\mathcal{K}_0, \mathcal{K}_0]$ *has codimension one in* $\mathcal{K}_0$.

Proof. Let us define a linear functional ϕ on $\mathcal{K}_0$ by the following:

$$\phi(K_n) = n.$$

As K_n's form a basis of $\mathcal{K}_0$, this defines a linear functional. By the commutation relation above, we have

$$\phi([K_m, K_n]) \begin{cases} & \text{(for the case } m \neq -n) \\ = & (m-n)\phi(K_{m+n}) - m\phi(K_m) + n\phi(K_n) \\ = & (m-n)(m+n) - m^2 + n^2 \\ = & 0 \\ & \\ & \text{(for the case } m = -n) \\ = & -m\phi(K_m) - m\phi(K_{-m}) \\ = & -m^2 - m(-m) \\ = & 0 \end{cases} .$$

Hence this vanishes on the commutator. The linear functional ϕ is nontrivial and the commutator subalgebra $[\mathcal{K}_0, \mathcal{K}_0]$ is in the nontrivial kernel of ϕ. In particular, $[\mathcal{K}_0, \mathcal{K}_0]$ is not equal to $\mathcal{K}_0$.

To see that the commutator subalgebra of $\mathcal{K}_0$ has codimension one, we will show that all the element of $\mathcal{K}_0$ can be obtained as the linear combination of K_1 and elements of $[\mathcal{K}_0, \mathcal{K}_0]$. Let us note that

$$\begin{array}{rcl} [K_1, K_{-1}] & = & -K_1 - K_{-1} \\ [K_2, K_{-1}] & = & 3K_1 - 2K_2 - K_{-1} \\ [K_{-2}, K_1] & = & -3K_{-1} + 2K_{-2} + K_1. \end{array}$$

So K_{-1}, K_2, K_{-2} can be obtained. For other elements in the basis, we only need to see

$$\begin{array}{rcl} [K_n, K_1] & = & (n-1)K_{n+1} - nK_n + K_1 \\ [K_{-n}, K_{-1}] & = & -(n-1)K_{-n-1} + nK_{-n} - K_{-1}, \end{array}$$

and to use mathematical induction. □

Corollary 2.2.2. *$H^1(\mathcal{K}_0, \mathbb{C})$ is one dimensional. In particular, there is a unique (up to scalar) one dimensional representation of $\mathcal{K}_0$.*

Next we will determine the second cohomology group of $\mathcal{K}_0$.

Lemma 2.2.3. *The following set forms a basis of the commutator subalgebra of $\mathcal{K}_0$.*

$$[K_n, K_1], [K_{-n}, K_{-1}] \text{ for } n > 1, [K_{-2}, K_1], [K_2, K_{-1}], [K_1, K_{-1}].$$

Proof. As we have seen, the commutator subalgebra is the kernel of the functional of lemma 2.2.1. The last three elements in the set are linearly independent and contained in the subspace spanned by K_{-2}, K_{-1}, K_1 and K_2. The elements $[K_n, K_1]$ (respectively the elements $[K_{-n}, K_{-1}]$,) contain K_{n+1} terms (respectively $K_{-(n+1)}$ terms,) hence they are independent and form the basis of the commutator subalgebra. □

Theorem 2.2.4. *$H^2(\mathcal{K}_0, \mathbb{C})$ is one dimensional.*

Proof. Take an element ω of $Z^2(\mathcal{K}_0, \mathbb{C})$. Let $\omega_{m,n} := \omega(K_m, K_n)$ for $m, n \in \mathbb{Z} \setminus \{0\}$ be complex numbers. From the definition of $Z^2(\mathcal{K}_0, \mathbb{C})$, the following holds:

$$\begin{array}{rcl} & & \omega_{m,n} = -\omega_{n,m} \\ 0 & = & \omega(K_l, [K_m, K_n]) + \omega(K_n, [K_l, K_m]) + \omega(K_m, [K_n, K_l]) \\ & = & (m-n)\omega_{l,m+n} - m\omega_{l,m} + n\omega_{l,n} \\ & & +(l-m)\omega_{n,l+m} - l\omega_{n,l} + m\omega_{n,m} \\ & & +(n-l)\omega_{m,n+l} - n\omega_{m,n} + l\omega_{m,l}, \end{array} \tag{2.2}$$

where this holds also for the cases $l + m = 0, m + n = 0$, or $n + l = 0$ if we define $w_{k,0} = w_{0,k} = 0$ for $k \in \mathbb{Z}$.

Let α be a linear functional on the commutator subalgebra defined by

$$\begin{aligned}
\alpha([K_n, K_1]) &= \omega_{n,1} \text{ for } n > 1 \\
\alpha([K_{-n}, K_{-1}]) &= \omega_{-n,-1} \text{ for } n > 1 \\
\alpha([K_{-2}, K_1]) &= \omega_{-2,1} \\
\alpha([K_2, K_{-1}]) &= \omega_{2,-1} \\
\alpha([K_1, K_{-1}]) &= \omega_{1,-1}.
\end{aligned}$$

This definition is legitimate by lemma 2.2.3.

If we define $\omega'_{m,n} = \omega_{m,n} - \alpha([K_m, K_n])$, there is a corresponding element ω' in $Z^2(\mathcal{K}_0, \mathbb{C})$ and belongs to the same class in $Z^2/B^2(\mathcal{K}_0)$. To keep the brief notation, we assume from the beginning the following:

$$\omega_{n,1} = \omega_{-n,-1} = \omega_{-2,1} = \omega_{2,-1} = \omega_{1,-1} = 0 \text{ for } n > 1$$

and we will show that $\omega_{m,n} = 0$ if $m \neq -n$.

Now we set $l = 2, m = 1, n = -1$ in (2.2) to get:

$$0 = 2\omega_{2,0} - \omega_{2,1} - \omega_{2,-1} + \omega_{-1,3} - 2\omega_{-1,2} + \omega_{-1,1} - 3\omega_{1,1} + \omega_{1,-1} + 2\omega_{1,2}.$$

From this we see that $\omega_{-1,3}$ vanishes because by assumption all the other terms are zero. Similarly if we let $l = -2, m = 1, n = 1$, we have $\omega_{1,-3} = 0$.

Furthermore, setting $l > 1, m = 1, n = -1$ we get

$$0 = 2\omega_{l,0} - \omega_{l,1} - \omega_{l,-1} + (l-1)\omega_{-1,l+1} - l\omega_{-1,l} + \omega_{-1,1} - (l+1)\omega_{1,l-1} + \omega_{1,-1} + l\omega_{1,l}.$$

This implies $\omega_{-1,l+1} = 0$ by induction for $l > 1$. Similarly, letting $l < -1, m = 1, n = -1$ we see $\omega_{1,l-1} = 0$ for $l < -1$.

Next we use formula (2.2) substituting $l = 1, n = -m$ to get

$$\begin{aligned}
0 = 2m\omega_{1,0} - m\omega_{1,m} - m\omega_{1,-m} + (1-m)\omega_{-m,m+1} - \omega_{-m,1} + m\omega_{-m,m} \\
+ (-m-1)\omega_{m,1-m} + m\omega_{m,-m} + \omega_{m,1}.
\end{aligned}$$

Since $\omega_{1,m} = \omega_{-1,m} = 0$, as we have seen above, and by the antisymmetry $\omega_{-m,m} = -\omega_{m,-m}$, we have

$$(1-m)\omega_{-m,1+m} + (-m-1)\omega_{m,1-m} = 0.$$

By assumption, we have $\omega_{-1,2} = 0$. By induction on m, we observe $\omega_{-m,m+1} = 0$. Similarly it holds $\omega_{-m,m-1} = 0$.

Finally we fix $k \in \mathbb{N}$ and let $l = 1, n = k - m$ to get

$$\begin{aligned}
0 = (2m-k)\omega_{1,k} - m\omega_{1,m} + (k-m)\omega_{1,k-m} + (1-m)\omega_{k-m,m+1} - \omega_{k-m,1} \\
+ m\omega_{k-m,m} + (k-m-1)\omega_{m,k-m+1} - (k-m)\omega_{m,k-m} + \omega_{m,1}.
\end{aligned}$$

By assumption, as before, the preceding equation becomes the following:

$$\begin{aligned} 0 &= (1-m)\omega_{k-m,m+1} + k\omega_{k-m,m} + (k-m-1)\omega_{m,k-m+1} \\ &= (1-m)\omega_{(k+1)-(m+1),m+1} + k\omega_{k-m,m} + (k-m-1)\omega_{m,(k+1)-m} \end{aligned} \tag{2.3}$$

If we let $k=1$, the second term vanishes by the observation above and we see

$$(1-m)\omega_{1-m,m+1} - m\omega_{m,2-m} = 0$$

Again by induction on m, we see $\omega_{2-m,m}$ vanishes for all m. Then by induction on k and using (2.3), we can conclude $\omega_{k-m,m}$ vanishes for all $k \in \mathbb{N}, m \in \mathbb{Z}$. Similar argument applies for $k<0$.

Summarizing, if we have an element in $Z^2(\mathcal{K}_0, \mathbb{C})$, we may assume that all the off-diagonal parts vanish. Letting $l = -m-n$ in (2.2), we see that there is a possibility of one (and only) dimensional second cohomology as in the case of Virasoro algebra (see [81]). □

This theorem shows that there is a unique central extension (up to isomorphism) of $\mathcal{K}_0$. We denote the central extension by $\mathcal{K}$. Fixing a cocycle $\omega \in Z^2(\mathcal{K}_0, \mathbb{C}) \setminus B^2(\mathcal{K}_0, \mathbb{C})$ the algebra $\mathcal{K}$ is formally defined as $\mathcal{K}_0 \oplus \mathbb{C}$ with the commutation relations

$$[(x,a),(y,b)] := ([x,y], \omega(x,y)) \text{ for } x,y \in \mathcal{K}_0, a,b \in \mathbb{C}.$$

Equivalently, in this thesis and in literature, using a formal central element C, one writes:

$$[x+aC, y+bC] = [x,y] + \omega(x,y)C.$$

Proposition 2.2.5. *Let us fix a real number λ. On $\mathcal{K}$, there is a *-automorphism Λ defined by $\Lambda(K_n) = K_n + in\lambda C$ and $\Lambda(C) \mapsto C$.*

Proof. It is clear that this preserves the *-operation. Since the change by this mapping is just an addition of a scalar multiple of the central element, this does not change the commutator. On the other hand, as seen in lemma 2.2.1, the map $K_n \mapsto n$ vanishes on the commutator subalgebra, hence the linear map in question preserves the commutators. □

Proposition 2.2.6. *The *-automorphism in Proposition 2.2.5 does not extend to the Virasoro algebra unless $\lambda = 0$.*

Proof. Assume the contrary, namely that Λ extends to Vir. Since $\mathcal{K}$ has codimension one in the Virasoro algebra, we only have to determine where L_0 is mapped. The algebra Vir is the linear span of K_n's, C and L_0, hence $\Lambda(L_0)$ takes the following form.

$$\Lambda(L_0) = \sum_{n\neq 0} a_n K_n + a_0 L_0 + bC,$$

where a_n's and b are complex numbers and a_n's vanish except for finitely many n.

On the other hand, in Vir, we have

$$[K_n, L_0] = [L_n - L_0, L_0] = nL_n = nK_n + nL_0.$$

Since in the sum of $\Lambda(L_0)$ only finitely many terms appear, let N be the largest integer with which a_N does not vanish. If $N > 1$, recalling $[K_1, L_0] = K_1 + L_0$, we have

$$\begin{aligned} \Lambda([K_1, L_0]) &= [K_1 + i\lambda C, \Lambda(L_0)] \\ &= \Lambda(K_1) + \Lambda(L_0), \end{aligned}$$

which is impossible because the second expression contains K_{N+1} term but the last expression does not. Hence N must be less than 2. By the same argument replacing K_1 by K_2, we have that N must be less than 1. Similarly replacing K_1 by K_{-1} or K_{-2}, it can be shown that $\Lambda(L_0)$ must be of the form

$$\Lambda(L_0) = a_0 L_0 + bC.$$

We need to note that a_0 and b must be real as Λ is a *-automorphism.

Now let us calculate again

$$\begin{aligned} [\Lambda(K_1), \Lambda(L_0)] &= [K_1 + i\lambda C, a_0 L_0 + b \cdot C] \\ &= a_0 K_1 + a_0 L_0, \end{aligned}$$

by assumption this must be equal to

$$\begin{aligned} \Lambda([K_1, L_0]) &= \Lambda(K_1 + L_0) \\ &= K_1 + a_0 L_0 + (b + i\lambda)C, \end{aligned}$$

which is impossible since b is real, except the case $\lambda = 0$ (and in this case $b = 0, a = 1$). $\square$

Remark 2.2.7. When we make compositions of these automorphisms with a representation of $\mathcal{K}$, we might obtain inequivalent representations of $\mathcal{K}$. However these representations integrate to equivalent projective unitary representations of the group B_0, since with these automorphisms the changes of self-adjoint elements in $\mathcal{K}$ are only scalars and the changes of their exponentials are only phases, therefore equivalent as projective representations of B_0.

2.3 Derived subalgebras and groups

2.3.1 A sequence of ideals in $\mathcal{K}_0$

We will investigate the derived subalgebras of $\mathcal{K}_0$. The derived subalgebra (or the commutator subalgebra) of a Lie algebra is, by definition, the subalgebra generated by all the commutators of the given Lie algebra.

The easiest and most important property of the commutator subalgebra is that it is an ideal. This is clear from the definition. If a Lie algebra is simple, then the commutator subalgebra must coincide with the Lie algebra itself. This is the case for the Virasoro algebra.

On the other hand, the algebra $\mathcal{K}_0$ and its unique nontrivial central extension $\mathcal{K}$ are not simple. This can be seen from lemma 2.2.1: the commutator subalgebra (which we denote by $\mathcal{K}_0^{(1)}$) has codimension 1 in $\mathcal{K}_0$ and it is the kernel of a homomorphism of the Lie algebra.

Let us denote $\mathrm{Vect}(S^1)_0$ the subalgebra of $\mathrm{Vect}(S^1)$ whose element vanish at $\theta = 0$. We remind that the commutator on $\mathrm{Vect}(S^1)$ is the following.

$$[f, g] = fg' - f'g. \tag{2.4}$$

Now it is easy to see that $\mathrm{Vect}(S^1)_0$ is a subalgebra. Let us recall that we embed $\mathcal{K}_0$ in $\mathrm{Vect}(S^1)_0$ by the correspondence $K_n \mapsto i(\exp(in\cdot) - 1)$. We clarify the meaning of the homomorphism ϕ by considering the larger algebra $\mathrm{Vect}(S^1)_0$.

Lemma 2.3.1. *The homomorphism $\phi : K_n \mapsto -n$ on $\mathcal{K}_0$ continuously extends to* $\mathrm{Vect}(S^1)_0$ *and the result is*

$$\begin{array}{rcl} \phi : \mathrm{Vect}(S^1)_0 & \to & \mathbb{R} \\ f & \mapsto & f'(0). \end{array}$$

Proof. It is easy to see that ϕ and the derivative on 0 coincide. The latter is clearly continuous on $\mathrm{Vect}(S^1)_0$ in its smooth topology.

To see that the extension is still a homomorphism of $\mathrm{Vect}(S^1)_0$, we only have to calculate the derivative of $[f, g]$ on $\theta = 0$:

$$\begin{aligned} \frac{d}{dt}[f, g](0) &= \left.\frac{d}{dt}\left(fg' - f'g\right)\right|_{t=0} \\ &= \left(f'g' + fg'' - f''g - f'g'\right)(0) \\ &= \left(f''g - fg''\right)(0) \\ &= 0, \end{aligned}$$

since f and g are elements of $\mathrm{Vect}(S^1)_0$. □

We set $\phi_1 := \phi$ and we define similarly,

$$\begin{array}{rcl} \phi_k : \mathrm{Vect}(S^1)_0 & \to & \mathbb{R} \\ f & \mapsto & f^{(k)}(0), \end{array}$$

where $f^{(k)}$ is the k-th derivative of the function f. Again these maps are continuous in the topology of smooth vectors.

We show the following.

Lemma 2.3.2. *Let f and g be in* $\mathrm{Vect}(S^1)_0$. *Suppose $\phi_m(f) = \phi_m(g) = 0$ for $m = 1, \cdots k$. Then $\phi_m([f, g]) = \phi_m(fg' - f'g) = 0$ for $m = 1, \cdots 2k + 1$.*

Proof. First we recall the general Leibniz rule:

$$(F \cdot G)^{(k)}(\theta) = \sum_{m=0}^{k} {}_kC_m F^{(m)}(\theta) G^{(k-m)}(\theta),$$

where ${}_kC_m$ denotes the choose function $\frac{k!}{m!(k-m)!}$ Then, in each term of the m-th derivatives of $[f, g] = fg' - f'g$ where $m \leq 2k$, there appears a factor which is a derivative f or g of order $m \leq k$ and the term vanishes by assumption. To consider the $(2k + 1)$-th derivative, the only nonvanishing terms are

$$\begin{aligned} [f, g]^{(2k+1)}(\theta) &= {}_{2k+1}C_{k+1} f^{(k+1)} g^{(k+1)} - {}_{2k+1}C_k f^{(k+1)} g^{(k+1)} \\ &= 0. \end{aligned}$$

□

Proposition 2.3.3. *The subspace* $\mathrm{Vect}(S^1)_k = \{f \in \mathrm{Vect}(S^1)_0 : \phi_1(f) = \cdots = \phi_k(f) = 0\}$ *is an ideal of* $\mathrm{Vect}(S^1)_0$ *and it holds that*

$$[\mathrm{Vect}(S^1)_k, \mathrm{Vect}(S^1)_k] \subset \mathrm{Vect}(S^1)_{2k+1}$$

Proof. The latter part follows directly from lemma 2.3.2. To show that $\mathrm{Vect}(S^1)_k$ is an ideal, we only have to take $f \in \mathrm{Vect}(S^1)_0$ and $g \in \mathrm{Vect}(S^1)_k$ and to calculate derivatives of $[f, g]$. By the Leibniz rule above, for $m \leq k$, in each term of the m-th derivative of $[f, g]$ there is a factor which is a derivative of g of order $m \leq k$ or a derivative f and they must vanish at $\theta = 0$ by assumption. □

Note that if we restrict ϕ_m to $\mathcal{K}_0$, it acts like $\phi_m(K_k) = i(ik)^m$. Defining $\mathcal{K}_k = \{x \in \mathcal{K}_0 : \phi_1(x) = \cdots \phi_k(x) = 0\}$, we can see similarly that $\{\mathcal{K}_k\}$ are ideals of $\mathcal{K}_0$ and that $[\mathcal{K}_k, \mathcal{K}_k] \subset \mathcal{K}_{2k+1}$.

2.3.2 Basis for $\mathcal{K}_k$

Our next task is to determine the derived subalgebras of $\{\mathcal{K}_k\}$. For this purpose, it is appropriate to take a new basis for each $\mathcal{K}_k$.

The following observation is easy.

Lemma 2.3.4. *If V is the vector space spanned by a countable basis $\{B_n\}_{n\in\mathbb{Z}}$, then $\{B_n - B_{n+1}\}_{n\in\mathbb{Z}}$ is a linearly independent set and the vector space spanned by them has codimension 1 in V.*

We set recursively,

$$\begin{aligned} M_n^0 &:= L_n - L_{n+1} \\ M_n^1 &:= M_n^0 - M_{n+1}^0 \\ M_n^{k+1} &:= M_n^k - M_{n+1}^k, \end{aligned}$$

where $\{L_n\}$ is the basis of the Witt algebra. By lemma 2.3.4, we have a sequence of subspaces of Witt. We will see that they coincide with $\{\mathcal{K}_n\}$. For this purpose we need the combinatorial formula in lemma 2.3.7.

Remark 2.3.5. We use the convention that a polynomial of degree -1 is 0.

Lemma 2.3.6. *If $k \geq 0$ and if $p(x)$ is a polynomial of x of degree k, then $p(x) - p(x+1)$ is a polynomial of degree $k-1$.*

Proof. We just have to consider the terms of the highest and the second highest degrees. □

We fix a natural number k. Let us define a sequence of polynomials recursively by

$$\begin{aligned} p_k(x) &= x^k, \\ p_{m-1}(x) &= p_m(x) - p_m(x+1) \text{ for } 0 \leq m \leq k. \end{aligned}$$

Lemma 2.3.7. *We have the explicit formulae for $-1 \leq m \leq k$.*

$$p_m(x) = \sum_{l=0}^{k-m} (x+l)^k (-1)^l {}_{k-m}C_l.$$

Proof. We show this lemma by induction. If $m = k$, $p_m(x) = x^k$ and the lemma holds.

Let us assume that the formula holds for m. We use the well-known combinatorial fact that if $1 < j \ \leq i$ then ${}_iC_{j-1} + {}_iC_j = {}_{i+1}C_j$. Now let us calculate

$$\begin{aligned}
p_{m-1}(x) &= p_m(x) - p_m(x+1) \\
&= \sum_{l=0}^{k-m}(x+l)^k(-1)^l{}_{k-m}C_l - \sum_{l=0}^{k-m}(x+1+l)^k(-1)^l{}_{k-m}C_l \\
&= \sum_{l=0}^{k-m}(x+l)^k(-1)^l{}_{k-m}C_l - \sum_{l'=1}^{k-m+1}(x+l')^k(-1)^{l'-1}{}_{k-m}C_{l'-1} \\
&= \sum_{l=0}^{k-m}(x+l)^k(-1)^l{}_{k-m}C_l + \sum_{l=1}^{k-m+1}(x+l)^k(-1)^l{}_{k-m}C_{l-1} \\
&= (x+k-m+1)^k(-1)^{k-m+1} \\
&\quad + \sum_{l=1}^{k-m}(x+l)^k(-1)^l\left({}_{k-m}C_{l-1} + {}_{k-m}C_l\right) + x^k \\
&= (x+k-m+1)^k(-1)^{k-m+1} + \sum_{l=1}^{k-m}(x+l)^k(-1)^l{}_{k-m+1}C_l + x^k \\
&= \sum_{l=0}^{k-m+1}(x+l)^k(-1)^l{}_{k-m}C_l.
\end{aligned}$$

$\square$

Proposition 2.3.8. *For $k \geq 0$, as a polynomial of x, it holds*

$$\sum_{l=0}^{k+1}(x+l)^k(-1)^l{}_{k+1}C_l = 0.$$

Proof. If we put $m = -1$ in lemma 2.3.7, we get the left hand side of this formula. On the other hand, by definition of p_{-1} and by lemma 2.3.6, it must be a polynomial of degree -1, in other words, it vanishes. $\square$

We want to apply this formula to the calculation of the functionals ϕ_k. For this purpose we need formulae for $\{M_n^k\}$ (which are defined at the beginning of this subsection) in terms $\{L_n\}$.

Proposition 2.3.9. *It holds that*

$$M_n^k = \sum_{l=0}^{k+1}(-1)^l{}_{k+1}C_l L_{n+l}$$

Proof. Again we show this by induction. If $k = 0$, then $M_n^0 = L_n - L_{n+1}$ and this case is proved.

Assume it holds $M_n^k = \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l L_{n+l}$. Again using the combinatorial formula ${}_iC_{j-1} + {}_iC_j = {}_{i+1}C_j$, let us calculate

$$\begin{aligned}
M_n^{k+1} &= M_n^k - M_{n+1}^k \\
&= \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l L_{n+l} - \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l L_{n+1+l} \\
&= \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l L_{n+l} - \sum_{l'=1}^{k+2}(-1)^{l'-1} {}_{k+1}C_{l'-1} L_{n+l'} \\
&= \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l L_{n+l} + \sum_{l=1}^{k+2}(-1)^l {}_{k+1}C_{l-1} L_{n+l} \\
&= (-1)^{k+2} L_{n+k+2} + \sum_{l=1}^{k+1}(-1)^l \left({}_{k+1}C_l + {}_{k+1}C_{l-1}\right) L_{n+l} + L_0 \\
&= \sum_{l=0}^{k+2}(-1)^l {}_{k+2}C_l L_{n+l}.
\end{aligned}$$

And this is what we had to prove. □

Corollary 2.3.10. *For fixed $k \geq 0$, $\{M_n^k | n \in \mathbb{Z}\}$ is a basis of $\mathcal{K}_k$.*

Proof. We can extend ϕ_k to the Witt algebra by $\phi_k(L_n) = i(in)^k$ (for $k = 0$, $\phi_0(L_n) = i$ by definition).

Then, it is immediate that we have the following.

$$\begin{aligned}
\mathcal{K}_0 &= \{x \in \text{Witt} : \phi_0(x) = 0\} \\
\mathcal{K}_k &= \{x \in \text{Witt} : \phi_0(x) = \phi_1(x) = \cdots = \phi_k(x) = 0\}.
\end{aligned}$$

Clearly $\{\phi_k\}$ are independent and each $\mathcal{K}_{k+1}$ has codimension 1 in $\mathcal{K}_k$.

We will prove the corollary by induction. The set $\{M_n^0\}$ spans a subspace of Witt with codimension 1 by lemma 2.3.4 and it is immediate to see that $\phi_0(M_n^0) = 0$. On the other hand $\mathcal{K}_0$ is the kernel of ϕ_0 and has codimension one in Witt. Hence they must coincide.

Assume that $\{M_n^{k-1}\}$ is the basis of $\mathcal{K}_{k-1}$. Then it is obvious that $M_n^k = M_n^{k-1} - M_{n+1}^{k-1} \in \mathcal{K}_{k-1}$. Now, by proposition 2.3.9 and proposition 2.3.7, we see easily that for $n \in \mathbb{Z}$

$$\begin{aligned}
\phi_k(M_n^k) &= \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l \phi_k(L_{n+l}) \\
&= \sum_{l=0}^{k+1}(-1)^l {}_{k+1}C_l (n+l)^k \\
&= 0.
\end{aligned}$$

This means that $M_n^k \in \mathcal{K}_k$.

The linear span of $\{M_n^k\}_{n\in\mathbb{Z}}$ must have codimension 1 by lemma 2.3.4 in $\mathcal{K}_{k-1}$, therefore it must coincide with $\mathcal{K}_k$, since $\mathcal{K}_k$ has codimension 1 in $\mathcal{K}_{k-1}$ by definition. □

2.3.3 Commutator subalgebras of $\mathcal{K}_k$

Now we can completely determine all the commutator subalgebras of $\mathcal{K}_k$. The key fact is that we can easily calculate the commutator in the basis we have obtained in the previous section.

Proposition 2.3.11. *Let $k \geq 0$ and $m, n \in \mathbb{Z}$. It holds that*

$$[M_m^k, M_n^k] = (m-n)M_{m+n}^{2k+1}$$

Proof. We prove the proposition by induction. The case for $k = 0$ is shown as follows.

$$\begin{aligned}
[M_m^0, M_n^0] &= [L_m - L_{m+1}, L_n - L_{n+1}] \\
&= (m-n)L_{m+n} - (m+1-n)L_{m+1+n} \\
&\quad -(m-n-1)L_{m+n+1} + (m-n)L_{m+1+n+1} \\
&= (m-n)\left(L_{m+n} - L_{m+n+1}\right) - (m-n)\left(L_{m+n+1} - L_{m+n+2}\right) \\
&= (m-n)(M_{m+n}^0 - M_{m+n+1}^0) \\
&= (m-n)M_{m+n}^1
\end{aligned}$$

Let us assume that the formula holds for k. We calculate

$$\begin{aligned}
[M_m^{k+1}, M_n^{k+1}] &= [M_m^k - M_{m+1}^k, M_n^k - M_{n+1}^k] \\
&= (m-n)M_{m+n}^{2k+1} - (m+1-n)M_{m+1+n}^{2k+1} \\
&\quad -(m-n-1)M_{m+n+1}^{2k+1} + (m-n)M_{m+1+n+1}^{2k+1} \\
&= (m-n)\left(\left(M_{m+n}^{2k+1} - M_{m+n+1}^{2k+1}\right) - \left(M_{m+n+1}^{2k+1} - M_{m+n+2}^{2k+1}\right)\right) \\
&= (m-n)(M_{m+n}^{2k+2} - M_{m+n+1}^{2k+2}) \\
&= (m-n)M_{m+n}^{2k+3}.
\end{aligned}$$

This completes the induction.

□

Remark 2.3.12. The Witt algebra can be treated as $\mathcal{K}_{-1}$ in this context, in the sense that the formula of the proposition holds for $k = -1$.

Theorem 2.3.13. *It holds that $\mathcal{K}_{2k+1} = \mathcal{K}_k^{(1)}$, where $\mathcal{K}_k^{(1)}$ is the derived subalgebra of $\mathcal{K}_k$.*

Proof. It is clear from corollary 2.3.10 and proposition 2.3.11 that the derived subalgebra of $\mathcal{K}_k$ is included in $\mathcal{K}_{2k+1}$ and the commutators of elements in the basis of $\mathcal{K}_k$ exhaust the basis of $\mathcal{K}_{2k+1}$. □

2.3.4 The ideal structure of $\mathcal{K}_0$

The basis obtained in the previous subsection is suitable to determine all the ideals of $\mathcal{K}_0$. In fact, we will see that any ideal of $\mathcal{K}_0$ must coincide with one of $\{\mathcal{K}_k\}$ or $\ker\phi_1 \cap \ker\phi_3$.

Lemma 2.3.14. *If $\mathcal{I}$ is a nontrivial ideal of $\mathcal{K}_0$, then it includes $\mathcal{K}_k$ for some k.*

Proof. Let x be a nontrivial element of $\mathcal{I}$. It has an expansion $x = \sum_{j=1}^N a_j M_{n_j}^0$ and we may assume $a_j \neq 0$ for all j. Since $\mathcal{I}$ is an ideal of $\mathcal{K}_0$, any commutator with x must be in $\mathcal{I}$ again. In particular,

$$[M_{n_N}^0, x] = \sum_{j=1}^{N} a_j(n_N - n_j)M_{n_j+n_N}^1 = \sum_{j=1}^{N-1} b_j M_{n_j+n_N}^1,$$

where each of $b_j = a_j(n_N - n_j)$, for $j = 1, 2, \cdots N-1$, is nonzero, must be an element of $\mathcal{I}$.

Similarly $[M_{n_N+n_{N-1}}^1, [M_{n_N}^0, x]] = \sum_{j=1}^{N-2} c_j M_{n_j+n_N+n_{N-1}}^1$ is also an element of $\mathcal{I}$. Repeating this procedure, we see that $\mathcal{I}$ contains some M_m^l. Then, using the commutation relation in $\mathcal{K}_l$, we see that $\mathcal{I}$ contains $\{M_n^{2l+1}\}_{n\neq 2m}$ and $\{M_n^{4l+3}\}_{n\in\mathbb{Z}}$. This implies that $\mathcal{I}$ includes $\mathcal{K}_{4l+3}$. □

To prove the next lemma, we need to recall that $\mathcal{K}_0$ is a subalgebra of smooth vector fields on S^1 and all the functionals $\{\phi_k\}_{k\in\mathbb{N}}$ have analytic interpretations as in subsection 2.3.1. There, we have identified the real line with the punctured circle, the point at infinity with the point $\theta = 0$. The algebra $\mathcal{K}_0$ is realized as a subalgebra of smooth functions on the circle vanishing at $\theta = 0$. Seen as the algebra of functions, their commutation relations are $[x, y] = xy' - x'y$.

Lemma 2.3.15. *Let $\mathcal{I}$ be a nontrivial ideal of $\mathcal{K}_0$ and let k be the smallest number such that $\mathcal{K}_k$ is included in $\mathcal{I}$ (this exists by lemma 2.3.14). If $k \geq 4$, then $\mathcal{I} = \mathcal{K}_k$.*

Proof. We will prove this lemma by contradiction. Let us assume that $\mathcal{I} \neq \mathcal{K}_k$ and that $x \in \mathcal{I} \setminus \mathcal{K}_k$. Possible cases are (1) $x \in \mathcal{K}_2$ (2) $x \in \mathcal{K}_0 \setminus \mathcal{K}_1$ (3) $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$. We treat these cases in this order.

If $x \in \mathcal{K}_2$, then there is l such that $2 \leq l < k$ and $x \in \mathcal{K}_l \setminus \mathcal{K}_{l+1}$. Let us take an element y from $\mathcal{K}_1 \setminus \mathcal{K}_2$. Then, since $\mathcal{K}_l$ is an ideal of $\mathcal{K}_0$ by the remark after proposition 2.3.3, we see $[x, y] \in \mathcal{K}_l$ and we calculate the derivatives at $\theta = 0$. By the assumption on x and y, the derivatives vanish up to certain orders and we have the following:

$$\begin{aligned}[x,y]^{(l+1)}(0) &= \sum_{k=0}^{l+1} {}_{l+1}C_k \left(y^{(k)}(0)x^{(l+1-k+1)}(0) - y^{(k+1)}(0)x^{(l+1-k)}(0)\right) \\ &= 0,\end{aligned}$$

$$\begin{aligned}[x,y]^{(l+2)}(0) &= \sum_{k=0}^{l+2} {}_{l+2}C_k \left(y^{(k)}(0)x^{(l+2-k+1)}(0) - y^{(k+1)}(0)x^{(l+2-k)}(0)\right) \\ &= ({}_{l+2}C_2 - {}_{l+2}C_1)y^{(2)}(0)x^{(l+1)}(0) \\ &= \frac{(l+2)(l-1)}{2} y^{(2)}(0)x^{(l+1)}(0).\end{aligned}$$

The latter cannot be zero by assumption and the fact $2 \leq l$. This means $[x,y]$ is in $\mathcal{K}_{l+1} \setminus \mathcal{K}_{l+2}$. Repeating this procedure, we obtain an element of $\mathcal{I}$ in $\mathcal{K}_{k-1} \setminus \mathcal{K}_k$. Therefore $\mathcal{I}$ contains $\mathcal{K}_{k-1}$ because by definition $\mathcal{I}$ contains $\mathcal{K}_k$ and $\mathcal{K}_{k-1}$ has codimension 1 in $\mathcal{K}_{k-1}$. But this contradicts the definition of k and we see that $x \in \mathcal{K}_2$ is impossible.

Next we $x \in \mathcal{K}_0 \setminus \mathcal{K}_1$. Then we can expand $x = a_0 M_0^0 + a_1 M_0^1 + y$ (here we use same symbols as before to save the number of characters) where $y \in \mathcal{K}_2$, hence a_0 is nonzero. If $a_1 \neq 0$ we have $[M_0^0, x] = a_1 M_1^1 + [M_0^0, y]$. If $a_1 = 0$ we have $[M_1^0, x] = a_0 M_1^1 + [M_1^0, y]$. Therefore at least one of these is in $\mathcal{K}_1 \setminus \mathcal{K}_2$ and we may assume that $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$.

Let us assume that $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$. Here we consider the following two cases, namely (3-1) $\phi_3(x) \neq 0$ (3-2) $\phi_3(x) = 0$. If $\phi_3(x) \neq 0$ and $y \in \mathcal{K}_0 \setminus \mathcal{K}_1$, then we see that $[x,y] + y'(0)x \in \mathcal{K}_2 \setminus \mathcal{K}_3$ (and this element is clearly in $\mathcal{I}$). In fact, by a direct calculation or by the Leibniz rule, we see

$$\begin{aligned}\left([x,y] + y'(0)x\right)^{(2)}(0) &= -y'(0)x^{(2)}(0) + y'(0)x^{(2)}(0) \\ &= 0, \\ \left([x,y] + y'(0)x\right)^{(3)}(0) &= -y'(0)x^{(2)}(0) + y'(0)x^{(2)}(0) \\ &= -2y'(0)x^{(3)}(0) + y'(0)x^{(3)}(0) \\ &= -y'(0)x^{(3)}(0).\end{aligned}$$

This implies that there is an element of $\mathcal{I}$ in $\mathcal{K}_2 \setminus \mathcal{K}_3$. By repeating the argument in the paragraph for the case $x \in \mathcal{K}_2$, we see again a contradiction. Hence we must have $\phi_3(x) = 0$.

By the calculation above, this time $[x,y] + y'(0)x \in \mathcal{K}_3$, but using $\phi_3(x) = 0$ we see

$$\begin{aligned}\left([x,y] + y'(0)x\right)^{(4)}(0) &= 2y^{(3)}(0)x^{(2)}(0) - 3y'(0)x^{(4)}(0) + y'(0)x^{(4)}(0) \\ &= 2y^{(3)}(0)x^{(2)}(0) - 2y'(0)x^{(4)}(0).\end{aligned}$$

Hence with an appropriate element y this does not vanish. That means $[x,y] + y'(0)x$ is an element of $\mathcal{K}_3 \setminus \mathcal{K}_4$. By the same argument as in the case of $x \in \mathcal{K}_2$, we see that this contradicts the definition of k and this completes the proof. □

We state now the final result of this subsection.

Theorem 2.3.16. *If $\mathcal{I}$ is an ideal of $\mathcal{K}_0$, then the possibilities are*

- $\mathcal{I} = \{0\}$
- $\mathcal{I} = \mathcal{K}_k$ *for some* $k \geq 0$

- $\mathcal{J} = \ker \phi_1 \cap \ker \phi_3$.

Proof. As before, we can define a number $k \geq 0$ as the smallest number such that $\mathcal{K}_k$ is included in $\mathcal{J}$.

If $k = 0$ or $k = 1$, then there is nothing to do because the former case means $\mathcal{J} = \mathcal{K}_0$ and in the latter case $\mathcal{K}_1$ has already codimension 1 and $\mathcal{J}$ must coincide with it.

Next we consider the case $k = 2$. Since $\mathcal{K}_2$ has codimension 2 in $\mathcal{K}_0$, it holds $\mathcal{J} = \mathcal{K}_2$ or $\mathcal{J}$ has an extra element. But the latter case cannot happen because if $x \in \mathcal{J} \setminus \mathcal{K}_2$ we can expand $x = a_0 M_0^0 + a_1 M_0^1 + y$ (the same symbols again, but the coefficients of a different element) where $y \in \mathcal{K}_2$ and $a_0 \neq 0$ (since otherwise $x \in \mathcal{K}_1$ and contradicts the assumption that $k = 2$). If $a_1 \neq 0$ then $[M_0^0, x] = a_1 M_1^1 + [M_0^0, y] \in \mathcal{K}_1 \setminus \mathcal{K}_2$. If $a_1 = 0$ then $[M_1^0, x] = a_0 M_1^1 + [M_1^0, y] \in \mathcal{K}_1 \setminus \mathcal{K}_2$. In both cases they contradict the assumption $k = 2$.

Let us assume $k = 3$ and $\mathcal{J} \neq \mathcal{K}_3$. We can take an element $x \in \mathcal{J} \setminus \mathcal{K}_3$ and expand it as

$$x = a_0 M_0^0 + a_1 M_0^1 + a_2 M_0^2 + y$$

(same symbols again to different coefficients) where $y \in \mathcal{K}_3$. By straightforward calculations we see that:

$$\begin{aligned}
[M_0^0, x] &= a_1 M_1^1 + 2a_2 M_1^2 + [M_0^0, y] \\
[M_1^0, x] &= (a_0 + a_1) M_1^1 + a_2 (M_1^2 + M_2^2)[M_1^0, y] \\
[M_1^0, [M_0^0, x]] &= a_1 M_3^1 + 4a_2 M_3^2 + [M_1^0, [M_0^0, y]] \\
[M_1^0, [M_1^0, x]] &= (a_0 + a_1) M_3^1 + a_2 (M_3^2 - 3M_4^2).
\end{aligned}$$

We note that all these elements are in $\mathcal{J}$ since it is an ideal. By comparing the first and third equations, we see

$$\begin{aligned}
[M_0^0, x] - [M_1^0, [M_0^0, x]] &= a_1 (M_1^2 + M_2^2) + 2a_2 (M_1^2 - 2M_3^2) + z \\
&= 2(a_1 - a_2) M_3^2 + a_1 (M_1^3 + 2M_2^3) + 2a_2 (M_1^3 + M_2^3) + z,
\end{aligned}$$

where z is the sum of commutators of y and hence again in $\mathcal{K}_3$. Now it is easy to see that this element is in $\mathcal{K}_3$ if and only if $a_1 = a_2$. And this must be in $\mathcal{K}_3$, since otherwise it is in $\mathcal{K}_2 \setminus \mathcal{K}_3$ and contradicts the assumption that $k = 3$. Therefore we have $a_1 = a_2$.

Next we consider the difference of the second and fourth equations with $a_1 = a_2$ above and we get

$$\begin{aligned}
[M_1^0, x] - [M_1^0, [M_1^0, x]] &= (a_0 + a_1)(M_1^2 + M_2^2) + \\
&\quad a_1 (M_1^2 + M_2^2 - M_3^2 - 3M_4^2) + z' \\
&= a_0 (M_1^2 + M_2^2) \\
&\quad + a_1 (2M_1^2 + 2M_2^2 - M_3^2 - 3M_4^2) + z' \\
&= a_0 (M_1^2 + M_2^2) + a_1 (2M_1^3 + 4M_2^3 + 3M_3^3) + z',
\end{aligned}$$

where z' is again an element of $\mathcal{K}_3$. As before it is in $\mathcal{J}$. By the assumption $k = 3$ it is contained in $\mathcal{K}_3$, therefore $a_0 = 0$. This indicates that an extra element of $\mathcal{J}$ must have the form

$$x = a_1 (M_0^1 + M_0^2)$$

and it is immediate to see this is in $\ker\phi_1 \cap \ker\phi_3$. Since $\mathcal{K}_3$ has codimension 1 in this intersection, $\mathcal{I}$ must be equal to $\ker\phi_1 \cap \ker\phi_3$.

By calculating derivatives, we can see that $\ker\phi_1 \cap \ker\phi_3$ is surely an ideal of $\mathrm{Vect}(S^1)_0$ and it is also the case even when restricted to $\mathcal{K}_0$.

The case $k \geq 4$ is already done in lemma 2.3.15. $\square$

2.3.5 The derived subgroup of B_0

As mentioned in the introduction, $\mathrm{Diff}(S^1)$ is the group of smooth, orientation preserving diffeomorphisms of S^1. The group B_0 is the subgroup of $\mathrm{Diff}(S^1)$ whose elements fix the point $\theta = 0$. Identifying S^1 and $\mathbb{R}/2\pi\mathbb{Z}$, we can think of an element of B_0 as a smooth function g on $\mathbb{R}$, satisfying $g(\theta+2\pi) = g(\theta)+2\pi$, $g(0) = 0$ and $g'(\theta) > 0$. The last condition comes from the fact that g has a smooth inverse. On the other hand, a function on $\mathbb{R}$ with the conditions above can be considered as an element of B_0. And it is easy to see that the composition operation of the group coincides with the composition of functions. In what follows we identify the group B_0 with the set of smooth functions with these conditions.

Under this identification, Lie algebra $\mathrm{Vect}(S^1)$ of B_0 is seen as the space of smooth functions f such that $f(0) = 0$ and $f(\theta + 2\pi) = f(\theta)$.

Proposition 2.3.17. *$B_1 := \{g \in B_0 : g'(0) = 1\}$ is a subgroup of B_0.*

Proof. By a simple calculation. $\square$

Proposition 2.3.18. *The derived group $[B_0, B_0]$ is included in B_1.*

Proof. Take elements g, h from B_0. It holds that

$$\begin{aligned}
\frac{d}{d\theta}[g,h](0) &= \frac{d}{d\theta}\left(g \circ h \circ g^{-1} \circ h^{-1}\right)(0) \\
&= g'(h(g^{-1}(h^{-1}(0)))) \times h'(g^{-1}(h^{-1}(0))) \\
&\quad \times (g^{-1})'(h^{-1}(0)) \times (h^{-1})'(0) \\
&= g'(0) \times h'(0) \times (g^{-1})'(0) \times (h^{-1})'(0) \\
&= 1,
\end{aligned}$$

where the last equality holds since the derivative of the inverse function on the corresponding point is the inverse number. $\square$

We need the following well-known result [88] [69] [35].

Theorem 2.3.19. $\mathrm{Diff}(\mathbb{R})_c$ *is a simple group, where* $\mathrm{Diff}(\mathbb{R})_c$ *is the group of smooth orientation-preserving diffeomorphisms of* $\mathbb{R}$ *whose supports are compact.*

Here, a support of a diffeomorphism means the closure of the set on which the given diffeomorphisms is not equal to the identity map.

Corollary 2.3.20. *Let B_c be the subgroup of B_0 whose elements have supports not containing $\theta = 0$. Then B_c is simple.*

Proof. There is a smooth diffeomorphism between $\mathbb{R}$ and $S^1 \setminus \{0\}$, for example, the stereographic projection. This diffeomorphism induces an isomorphism between $\mathrm{Diff}(\mathbb{R})_c$ and B_c. $\square$

The following is a result similar to the fact $[\mathcal{K}_0, \mathcal{K}_0] = K_1$ which we have proved in theorem 2.3.13.

Theorem 2.3.21. *$[B_0, B_0]$ is dense in B_1.*

Proof. By corollary 2.3.20, B_0 has a simple subgroup B_c. The simplicity of B_c implies $[B_0, B_0]$ includes $[B_c, B_c] = B_c$, since any commutator subgroup is normal. Hence we can freely use compactly supported diffeomorphisms.

Let g be an element of B_1. By the observation above, there is an element h of $[B_0, B_0]$ such that $g \circ h$ has compact support around 0. In other words, we may assume that g has a compact support around 0 and we only have to approximate g with elements in $[B_0, B_0]$.

By the stereographic projection in corollary 2.3.20, we can consider g as a diffeomorphism of $\mathbb{R}$. It is well-known that dilations of $\mathbb{R}$ are mapped by this isomorphism to elements of B_0. Let δ_t be the dilation by t. For $x \in \mathbb{R}$, it holds

$$\delta_t^{-1} \circ g^{-1} \circ \delta_t(x) = \frac{1}{t} g^{-1}(tx).$$

By assumption $g'(0) = 1$. It easy to see that for $t \to 0$ the functions $\frac{1}{t} g^{-1}(tx)$, its first derivative and higher-order derivatives converge to x, 1 and 0 respectively, uniformly on each compact set of $\mathbb{R}$. This means $\frac{1}{t} g^{-1}(tx)$ approximates the identity map around $x = 0$.

Let ϵ be a positive number. Let γ be a smooth function on $\mathbb{R}$ such that it is 1 on $[-\epsilon, \epsilon]$ and 0 on $x \leq -2\epsilon$ or $x \geq 2\epsilon$. And let us consider the following functions parametrized by t.

$$h_t(x) = x + \left(\frac{1}{t} g^{-1}(tx) - x \right) \gamma(x).$$

It is easy to see that h_t's are smooth, $h_t(0) = 0$, h_t's are equal to x outside a compact set and if t is sufficiently small then each of h_t has the first derivative which is strictly larger than 0. Hence we can consider h_t as a diffeomorphism of $\mathbb{R}$ with a compact support. From the observation above it is clear that h_t and its derivatives converge to x, 1, and 0 uniformly on $\mathbb{R}$, namely h_t converge to the identity element in the smooth topology.

An important fact is that h_t is equal to $\frac{1}{t} g^{-1}(t\cdot)$ on $[-\epsilon, \epsilon]$. The map $\delta_t^{-1} \circ g \circ \delta_t \circ h_t$ has a compact support which does not contain 0, hence it corresponds to an element of B_c. We denote it by f_t.

Now it is evident that $(g \circ \delta_t^{-1} \circ g^{-1} \circ \delta_t) \circ f_t = g \circ (\delta_t^{-1} \circ g^{-1} \circ \delta_t \circ f_t)$ is in $[B_0, B_0]$ because it is a product of a commutator and a diffeomorphism with compact support. It is equal to $g \circ h_t$ which converges to g with all its derivatives. This shows $[B_0, B_0]$ is dense in B_1. $\square$

Remark 2.3.22. The Lie group $\mathrm{Diff}(S^1)$ is simple [88] [69] [35], but the Lie algebra $\mathrm{Vect}(S^1)$ is not simple. This is easy to see: for example, we only have to consider the subalgebra of vector fields with compact supports in some fixed proper subinterval of S^1. By the commutation relation (2.4) this subalgebra is an ideal. This is closed in the smooth topology, hence $\mathrm{Vect}(S^1)$ is not even topologically simple.

On the other hand, the Witt algebra is simple. This can be seen by observing that the linear map $[L_0, \cdot]$ is diagonalized on the standard basis of Witt with no degeneration and that we can raise or lower the elements by commutating with L_n or L_{-n}. From this it is easy to see that any ideal containing nontrivial element must contain Witt.

2.4 The automorphism group of $\mathcal{K}$

In this section we will completely determine the *-automorphism group of $\mathcal{K}$, the unique central extension of $\mathcal{K}_0$ defined in section 2.2. However, this group is not necessarily a natural object. As we have seen in the introduction, the algebra $\mathcal{K}_0$ is a subalgebra of $\mathrm{Vect}(S^1)_0$, the Lie algebra of vector fields on S^1 which vanish at $\theta = 0$. On this algebra of vector fields the stabilizer subgroup B_0 of $\theta = 0$ of $\mathrm{Diff}(S^1)$ acts as automorphisms, but when we restrict these actions to $\mathcal{K}_0$, it does not necessarily globally stabilize $\mathcal{K}_0$. In fact, the group of *-automorphisms turns out to be very small. The situation is similar for the Virasoro algebra [99].

We will study this problem only for the interest of representation theory. Many things are known about the representation theory of Virasoro algebra. In particular, all the irreducible unitary highest weight representations are completely classified [49]. But for the algebra $\mathcal{K}$ the situation is different. Of course we can restrict any unitary representation of the Virasoro algebra to $\mathcal{K}$ to obtain a unitary representation of $\mathcal{K}$. But it is not known if there are other unitary representations which are not localized at the point at infinity.

On the other hand, if we make a composition of a (known) unitary representation with an endomorphism of $\mathcal{K}$ then we obtain a (possibly new) unitary representation. The result will show that this method is not productive and, in fact, all the representations obtained by this method are already known.

The algebra $\mathcal{K}$ has a natural decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus \mathbb{C}C$ where $\mathcal{K}_+ = \mathrm{span}\{K_n : n \geq 1\}$, $\mathcal{K}_- = \mathrm{span}\{K_n : n \leq -1\}$. Each of these direct summands is a subalgebra and it holds $\mathcal{K}_+^* = \mathcal{K}_-$.

Lemma 2.4.1. *Let K and K' be elements of $\mathcal{K}$. We expand them in the standard basis:*

$$\begin{aligned} K &= a_0C + a_{n_1}K_{n_1} + a_{n_2}K_{n_2} + \cdots + a_{n_k}K_{n_k}, \\ K' &= b_0C + b_{m_1}K_{m_1} + b_{m_2}K_{m_2} + \cdots + b_{m_l}K_{m_l}. \end{aligned}$$

We assume here that all a_{n_i} and b_{m_j} but a_0 and b_0 are not zero and that $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_l$. Suppose the expansion of $[K, K']$ in the standard basis does not contain terms K_i where $i > \max\{n_k, m_l\}$. If we decompose $K = K_+ + K_- + a_0C$ and $K'_+ + K'_- + b_0C$ according to the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \oplus C\mathbb{C}$, then K_+ and K'_+ are proportional.

Proof. We take a look of the commutation relations (2.1) of $\mathcal{K}$. It is easy to see that in $[K_m, K_n]$ the term with index higher than m and n appears if and only if m and n are positive. And in such a case, the term K_{m+n} appears if $m \neq n$.

We may assume n_k and m_l are positive, since otherwise the statement would be trivial.

From the observation above, we see that n_k must be equal to m_l. Otherwise, the term $K_{n_k+m_l}$ (which is larger than $\max\{n_k, m_l\}$) appears in $[K, K']$ and cannot be cancelled, but this contradicts the assumption that there is no term with index higher than n_k and m_l in the commutator.

Now K and K' have the following form:

$$\begin{aligned} K &= a_{n_1}K_{n_1} + a_{n_2}K_{n_2} + \cdots + a_{n_{k-1}}K_{n_{k-1}} + a_{n_k}K_{n_k}, \\ K' &= b_{m_1}K_{m_1} + b_{m_2}K_{m_2} + \cdots + b_{m_{l-1}}K_{m_{l-1}} + b_{m_l}K_{n_k}. \end{aligned}$$

In the commutator $[K, K']$, the terms with the highest indices are now $K_{n_k+n_{k-1}}$ and $K_{n_k+m_{l-1}}$ which appear from the commutators of K_{n_k} and $K_{n_{k-1}}$ or $K_{m_{l-1}}$. If one of n_{k-1} and m_{l-1} is still positive, then again by the assumption, the highest term in the commutator must be cancelled. This implies that again $n_{k-1} = m_{l-1}$ and $a_{n_k}b_{m_{l-1}} = b_{m_l}a_{n_{k-1}}$. This means that the last two terms of K and K' are proportional.

Next steps go similarly: we know the last two terms are proportional and their commutator vanishes. Again by considering the terms with highest indices which appear from the commutator $[K, K']$, we see also that the last three terms are proportional. Continuing this procedure, we can see that all the positive part of K and K' must be proportional. □

Note that with a completely analogous proof we can show a similar lemma for the negative parts.

Lemma 2.4.2. *If ρ is a *-endomorphism of $\mathcal{K}$, then there is an element K of $\mathcal{K}_+$ and $\lambda, \mu, \nu \in \mathbb{C}$ such that $\rho(K_1)$ takes the form*

$$\rho(K_1) = \lambda K + \mu K^* + \nu C.$$

Proof. Since ρ is a *-endomorphism, it holds that $\rho(K_{-1}) = \rho(K_1)^*$ and from (2.1)

$$[\rho(K_1), \rho(K_1)^*] = -\rho(K_1) - \rho(K_1)^*.$$

We can apply lemma 2.4.1 to see that the positive part of $\rho(K_1)$ is proportional to the positive part of $\rho(K_1)^*$. This is the statement of the lemma. □

With an analogous argument we have the following:

Lemma 2.4.3. *If ρ is a *-endomorphism of $\mathcal{K}$, then there is an element K^* of $\mathcal{K}_+$ and $\lambda', \mu', \nu' \in \mathbb{C}$ such that $\rho(K_2)$ takes the form:*

$$\rho(K_2) = \lambda' K' + \mu' K'^* + \nu' C.$$

By a direct calculation we see that the map τ defined by

$$\tau(K_n) = -K_{-n}, \tau(C) = -C$$

is a *-automorphism of $\mathcal{K}$ (it extends also to the Virasoro algebra).

It is also immediate that C is the unique central element up to a scalar. This means that any automorphism must map $\mathbb{C}C$ to $\mathbb{C}C$.

Lemma 2.4.4. *If ρ is a *-automorphism of $\mathcal{K}$, then there are two possibilities.*

1. *There are elements K, K' of $\mathcal{K}_+$ and $\nu, \nu' \in \mathbb{C}$ such that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$.*

2. *There are elements K, K' of $\mathcal{K}_-$ and $\nu, \nu' \in \mathbb{C}$ such that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$.*

Proof. By lemma 2.4.2, $\rho(K_1)$ takes the form $\rho(K_1) = \lambda K + \mu K + \nu C$ where $K \in \mathcal{K}_+$ and $\lambda, \mu, \nu \in \mathbb{C}$. By lemma 2.4.3 we have that $\rho(K_2) = \lambda' K' + \mu' K'^* + \nu'$. Let us recall that the following commutation relation holds.

$$[\rho(K_2), \rho(K_1)^*] = 3\rho(K_1) - 2\rho(K_2) - \rho(K_1)^*.$$

Note that $\rho(K_1)^* = \overline{\mu} K + \overline{\lambda} K^* + \overline{\nu} C$.

By considering the composition with τ, we may assume that $\lambda \neq 0$ ($\lambda = \mu = 0$ is impossible because it would mean that K_1 is mapped to a central element and ρ would not be an automorphism). We show that $\mu = 0$. If not, applying lemma 2.4.1 we see that K' must be proportional to K. But this is impossible because we would have

$$\begin{aligned} \rho(K_1) &= \lambda K + \mu K^* + \nu C, \\ \rho(K_1)^* &= \overline{\mu} K + \overline{\lambda} K^* + \overline{\nu} C, \\ \rho(K_2) &= \lambda' K + \mu' K^* + \nu' C, \\ \rho(K_2)^* &= \overline{\mu'} K + \overline{\lambda'} K^* + \nu' C, \end{aligned}$$

which are linearly dependent. The map ρ is an automorphism and this is a contradiction. Similarly we have $\mu' = 0$ applying lemma 2.4.1 to the negative parts of $\rho(K_1)^*$ and $\rho(K_2)$. This concludes the lemma. □

Now we can determine all the elements of the *-automorphism group of $\mathcal{K}$. Recall there is a family Λ of *-automorphisms parametrized by $\lambda \in \mathbb{R}$ defined in proposition 2.2.5.

Theorem 2.4.5. *If ρ is a *-automorphism of $\mathcal{K}$, then $\rho = \Lambda$ for some $\lambda \in \mathbb{R}$ or $\rho = \Lambda \circ \tau$.*

Proof. By lemma 2.4.4 and possibly a composition with τ, we may assume that $\rho(K_1) = K + \nu C$ and $\rho(K_2) = K' + \nu' C$ where K and K' are in $\mathcal{K}_+$.

Let us expand K and K' in the standard basis of $\mathcal{K}$,

$$K = \sum_{i=1}^{N} a_i K_i, K' = \sum_{j=1}^{M} b_j K_j,$$

and assume $a_N \neq 0 \neq b_M$.

Since ρ is a *-automorphism, it must hold that

$$[\rho(K_1), \rho(K_1)^*] = -\rho(K_1) - \rho(K_1)^*, \tag{2.5}$$

$$[\rho(K_2), \rho(K_2)^*] = -2\rho(K_2) - 2\rho(K_2)^* + \frac{C}{2}, \tag{2.6}$$

$$[\rho(K_2), \rho(K_1)^*] = 3\rho(K_1) - 2\rho(K_2) - \rho(K_1)^*. \tag{2.7}$$

Considering the terms K_N in the first equation, we see that $\sum_{i=1}^{N} a_i = \frac{1}{N}$. Similarly, considering the terms K_M in the second equation, we obtain $\sum_{j=1}^{M} b_j = \frac{2}{M}$. On the other hand, by comparing the terms K_{-N} in the third equation, it turns out that $-N\overline{a_N}\sum_{j=1}^{M} b_j = -\overline{a_N}$. Since we have the assumption that a_N is not zero, this implies that $2N = M$.

The subalgebra $\mathcal{K}_+$ is generated by K_1 and K_2 with the recursive formula

$$K_{n+1} = \frac{1}{n-1}\left([K_n, K_1] + nK_n - K_1\right).$$

From this formula we see by induction that the term with the highest index of $\rho(K_k)$ is K_{kN}. If N was larger than 1, these terms would not span all of $\mathcal{K}_+$ and ρ could not be surjective. Thus N must be 1.

Again, by equation (2.5) and by a direct calculation, we obtain $a_1 = 1$, namely:

$$\rho(K_1) = K_1 + \nu_1 C,$$

where ν is a pure imaginary number. Similarly we have two solution for equation (2.6):

$$\rho(K_2) = \begin{cases} K_2 + \nu_2 C, \\ -\frac{1}{3}K_1 + \frac{4}{3}K_2 + \nu_2 C. \end{cases}$$

The second solution does not satisfy equation (2.7). Then again by (2.7) we see $2\nu_1 = \nu_2$.

We have seen in Proposition 2.2.5 that this ρ can be extended to a *-automorphism of $\mathcal{K}$. Since K_1 and K_2 are the generators of $\mathcal{K}$ as a *-Lie algebra, this determines ρ uniquely. □

Corollary 2.4.6. *$Aut(\mathcal{K}) \cong \mathbb{R} \rtimes \mathbb{Z}_2$.*

Remark 2.4.7. It is also possible to determine the automorphism group of the Virasoro algebra [99]: it is generated by the extension of τ and one-parameter subgroup of rotation:

$$\begin{aligned} \rho_t(L_n) &= e^{itn} L_n, \\ \rho_t(C) &= C. \end{aligned}$$

It is again isomorphic to $\mathbb{R} \rtimes \mathbb{Z}_2$, but the action of the $\mathbb{R}$ part is different.

2.5 Generalized Verma modules

As we have seen in section 2.2, $\mathcal{K}_0$ has the unique (up to isomorphism) central extension which is the restriction of Vir. We denote it by $\mathcal{K}$. This section is an attempt to construct a family of unitary representations of $\mathcal{K}$.

We are going to construct modules $V_{h+ih',c,\lambda}$ parametrized by three complex numbers $h+ih', c, \lambda$, where $h, h' \in \mathbb{R}$ and $c, \lambda \in \mathbb{C}$. Every module has a "lowest weight vector" which satisfies $K_n v = (h+ih'+n\lambda)v$ for $n \geq 1$ and $Cv = cv$. If we restrict to the case $\lambda = 0$, this module reduces to the restriction of the Virasoro module to $\mathcal{K}$.

Recall that $\mathcal{K}$ is a *-Lie algebra. A sesquilinear form $\langle \cdot, \cdot \rangle$ on a module V is said to be contravariant if for any $v, w \in V$ and $x \in \mathcal{K}$ it holds $\langle xv, w \rangle = \langle v, x^* w \rangle$. In addition if this sesquilinear form is positive definite, then the representation of $\mathcal{K}$ on V is said to be unitary.

It turns out that for any set of values of h, h', c, λ we can construct a corresponding module. In addition, if c is real, there exists a contravariant sesquilinear form on the module. Then we arrive at natural problems, for example, when the contravariant form is unitary, when the representation of $\mathcal{K}$ integrates to the (projective unitary) representation of B_0 and when these representations are inequivalent, etc. The author hopes to return to these problems in future research.

Here we make some remarks. It is easy to see that these modules are inequivalent as representations of the Lie algebra $\mathcal{K}$, however, as we saw in the remark 2.2.7 (after proposition 2.2.6), the imaginary part of λ does not make difference for the corresponding projective representation of the group B_0. In addition, in [94] it has been proved that there are modules which integrate to equivalent projective representations of the group for some different values of h. Furthermore, as we will see in section 2.7, there exist true (non projective) representation of B_0 whose naturally corresponding representations of $\mathcal{K}$ are not lowest weight modules. In the case of $\mathrm{Diff}(S^1)$ there is a one-to-one correspondence between irreducible unitary positive energy projective representations of the group and irreducible lowest weight unitary representations of the Virasoro algebra. But for B_0 and $\mathcal{K}$ we cannot expect such a correspondence.

2.5.1 General construction of modules

We start with general notions. Let $\mathcal{L}_0$ be a Lie algebra, $U(\mathcal{L}_0)$ the universal enveloping algebra of $\mathcal{L}_0$, ψ_0 a nontrivial linear functional on $\mathcal{L}_0$ which vanishes on the commutator subalgebra $[\mathcal{L}_0, \mathcal{L}_0]$. In particular, we assume that $\mathcal{L}_0$ is not semisimple (otherwise ψ_0 would be trivial). Later $\mathcal{L}_0$ will be a upper-triangular subalgebra of a Lie algebra.

Lemma 2.5.1. *The linear functional ψ_0 extends to a homomorphism of the universal algebra $U(\mathcal{L}_0)$.*

Proof. Clearly ψ_0 extends to a homomorphism of the tensor algebra of $\mathcal{L}_0$. Now we only have to recall that $U(\mathcal{L}_0)$ is the quotient algebra by the two-sided ideal generated by elements of the form $a \otimes b - b \otimes a - [a, b]$ where $a, b \in \mathcal{L}_0$. By assumption, ψ_0 vanishes on these elements, hence on the ideal generated by them. This implies ψ_0 is well-defined on $U(\mathcal{L}_0)$. ☐

Lemma 2.5.2. *Let $\mathfrak{J}_0$ be the left ideal of $U(\mathcal{L}_0)$ (the subspace invariant under the multiplication from the left) generated by elements of the form $\psi_0(a) - a$ for $a \in \mathcal{L}_0$.*

Then $U(\mathcal{L}_0)/\mathfrak{J}_0$ is nontrivial if and only if ψ_0 vanishes on $[\mathcal{L}_0, \mathcal{L}_0]$. In this case $\mathfrak{J}_0 = \ker \psi_0$ and the quotient space is one-dimensional.

Proof. If ψ_0 vanishes on $[\mathcal{L}_0, \mathcal{L}_0]$, then by lemma 2.5.1 ψ_0 extends to $U(\mathcal{L}_0)$ and $\mathfrak{J}_0$ is included in $\ker \psi_0$. Since ψ_0 is nontrivial, $\ker \psi_0$ is nontrivial.

On the other hand, if ψ_0 doesn't vanish at $[\mathcal{L}_0, \mathcal{L}_0]$, then take $x, y \in \mathcal{L}_0$ such that $\psi_0([x, y]) \neq 0$. Then it holds that

$$[(\psi_0(x) - x)\,, (\psi_0(y) - y)] = [x, y] \in \mathfrak{J}_0,$$
$$\psi_0([x, y]) - [x, y] \in \mathfrak{J}_0.$$

Hence $\mathfrak{J}_0$ contains a nontrivial scalar and generates all.

To complete the proof, we only have to show that $\mathfrak{J}_0 \supset \ker \psi_0$ since the other inclusion has been done. Therefore it is enough to show that $\mathfrak{J}_0$ has codimension 1 in $U(\mathcal{L}_0)$. This is a rephrasing of the claim that any element of $U(\mathcal{L}_0)$ is equivalent to a scalar modulo $\mathfrak{J}_0$. This is easy to see since any element of $U(\mathcal{L}_0)$ is a linear combination of tensor products $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. By definition there is an element $a_1 \otimes a_2 \otimes \cdots \otimes (a_n - \psi_0(a_n))$ in $\mathfrak{J}_0$, therefore $a_1 \otimes a_2 \otimes \cdots \otimes a_n \equiv_{\mathfrak{J}_0} a_1 \otimes a_2 \otimes \cdots \otimes \psi_0(a_n)$. By repeating this procedure, we see that every element of $U(\mathcal{L}_0)$ is equivalent to a scalar. ☐

In the following we assume that $\mathcal{L}$ is a *-Lie algebra with a decomposition into Lie subalgebras $\mathcal{L} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$, where $(\mathcal{N}_+)^* = \mathcal{N}_-$, $(\mathcal{H})^* = \mathcal{H}$, and $\mathcal{H}$ is commutative.

Let ψ be a linear functional on $\mathcal{H} \oplus \mathcal{N}_+$ which vanishes on its commutator subalgebra. In other words, ψ is an element of $H^1(\mathcal{N}_+ \oplus \mathcal{H}, \mathbb{C})$. We will show that for any such ψ we have a left module on $\mathcal{L}$. Again let $U(\mathcal{L})$ be the universal enveloping algebra of $\mathcal{L}$. It is naturally a left module on $\mathcal{L}$.

Proposition 2.5.3. *Let $\mathfrak{J}$ be the left ideal of $U(\mathcal{L})$ generated by elements of the form $\psi(l_+) - l_+$, where $l_+ \in \mathcal{H} \oplus \mathcal{N}_+$. The subspace $\mathfrak{J}$ is a nontrivial submodule on $U(\mathcal{L})$.*

Proof. By the theorem of Poincarè-Birkhoff-Witt, it holds that $U(\mathcal{L}) = U(\mathcal{N}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{N}_+)$. By lemma 2.5.2, $\ker \psi$ has codimension one in $U(\mathcal{H}) \otimes U(\mathcal{N}_+)$. It is easy to see that $\mathfrak{J}$ takes the form $U(\mathcal{N}_-) \otimes \ker \psi$, hence it is nontrivial. ☐

For a fixed ψ we define the quotient module $V = U(\mathcal{L})/\mathfrak{J}$. Since $U(\mathcal{H}) \oplus U(\mathcal{N}_+)/\ker \psi$ is one dimensional, the module V is linearly isomorphic to $U(\mathcal{N}_-)$ and we identify them. There is a specified vector v which corresponds to $1 \in \mathbb{C} \subset U(\mathcal{N}_-)$ and, on v, an element x of $\mathcal{H} \otimes \mathcal{N}_+$ acts as $xv = \psi(x)v$.

Example 2.5.4. The Virasoro algebra has the following decomposition:

$$\mathrm{Vir} = \mathcal{V}_- \oplus \mathcal{H} \oplus \mathcal{V}_+,$$

where $\mathcal{V}_+ = \mathrm{span}\{L_n : n > 0\}$ and $\mathcal{H} = \mathrm{span}\{L_0, C\}$. It is easy to see that the commutator subalgebra $[\mathcal{H} \oplus \mathcal{V}_+, \mathcal{H} \oplus \mathcal{V}_+]$ is equal to $\mathcal{V}_+$. According to proposition 2.5.3, we obtain a module of Vir for any linear functional ψ on $\mathcal{H} \oplus \mathcal{V}_+$ vanishing on $\mathcal{V}_+$. The linear functional ψ is determined by the two values $c := \psi(C)$ and $h := \psi(L_0)$. It is well known that for some values of c and h we can define inner products on these modules and these representations integrate to representations of the group $\mathrm{Diff}(S^1)$ [43].

Example 2.5.5. The *-Lie algebra $\mathcal{K}$ has the decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-,$$

where $\mathcal{K}_+ = \mathrm{span}\{K_n : n > 0\}$ and $\mathcal{H} = \mathrm{span}\{C\}$. It can be shown that $H^1(\mathcal{K}_+ \oplus \mathcal{H}, \mathbb{C})$ is three dimensional and an element ψ in $H^1(\mathcal{K}_+ \oplus \mathcal{H}, \mathbb{C})$ takes the form

$$\psi(C) = c, \psi(K_n) = h + ih' + \lambda n \text{ where } c, \lambda \in \mathbb{C}, h, h', \in \mathbb{R}.$$

We denote this module on $\mathcal{K}$ by $V_{h+ih',c,\lambda}$. If $c \in \mathbb{C}$, $\psi(K_n) = h + ih' \in \mathbb{C}$ and $\lambda = 0$ then the modules $V_{h+ih',c,0}$ reduce to Verma modules on the Virasoro algebra (see proposition 2.5.9).

Let us return to general cases. From now on we assume that ψ is self-adjoint on $\mathcal{H}$ (namely, $\psi(h^*) = \overline{\psi(h)}$ for $h \in \mathcal{H}$). Recall that V is the quotient module $U(\mathcal{L})/\mathcal{J}$ as in the remark after proposition 2.5.3. Our next task is to define a contravariant sesquilinear form on V. Note that the *-operation extends naturally to $U(\mathcal{L})$.

We define a sesquilinear map on $V \times V$ ($= U(\mathcal{N}_+) \times U(\mathcal{N}_+)$) into $U(\mathcal{L})$ by

$$\alpha(L_1^-, L_2^-) = (L_2^-)^* \otimes L_1^-, \text{ for } L_1^-, L_2^- \in U(\mathcal{N}_-) = V.$$

On the other hand, we can define a linear form β on $U(\mathcal{L})$ using the decomposition $U(\mathcal{N}_-) \otimes U(\mathcal{H}) \otimes U(\mathcal{N}_+)$, by

$$\beta(L_- \otimes H \otimes L_+) = \overline{\psi\left((L_-)^*\right)}\psi(H)\psi(L_+).$$

It is easy to see that β is self-adjoint since ψ is self-adjoint on $\mathcal{H}$.

Theorem 2.5.6. *$\beta \circ \alpha := \gamma$ is contravariant.*

Proof. We have to show that for any $L \in \mathcal{L}$ it holds

$$\gamma(L \otimes L_1^-, L_2^-) = \gamma(L_1^-, L^* \otimes L_2^-).$$

As elements of $U(\mathcal{L})$, the we have the following decompositions by the Poincarè-Birkhoff-Witt theorem:

$$
\begin{aligned}
L \otimes L_1^- &= \sum_k L_k^- \otimes H_k \otimes L_k^+ \\
(L_2^-)^* \otimes L_k^- &= \sum_l L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \\
H_k \otimes L_k^+ \otimes H_{k,l} \otimes L_{k,l}^+ &= \sum_m H_{k,l,m} \otimes L_{k,l,m}^+,
\end{aligned}
\tag{2.8}
$$

where elements in the decompositions are $L_k^-, L_{k,l}^- \in U(\mathcal{N}_-)$, $H_k, H_{k,l}, H_{k,l,m} \in U(\mathcal{H})$ and $L_k^+, L_{k,l}^+, L_{k,l,m}^+ \in U(\mathcal{N}_+)$.

Now we calculate

$$
\begin{aligned}
\gamma(L \otimes L_1^-, L_2^-) &= \gamma\left(\sum_k L_k^- \psi(H_k \otimes L_k^+), L_2^-\right) \\
&= \sum_k \psi(H_k \otimes L_k^+) \beta\left((L_2^-)^* \otimes L_k^-\right)
\end{aligned}
$$

By substituting the expression in (2.8) to $(L_2^-)^* \otimes L_k^-$, we have

$$
\begin{aligned}
\gamma(L \otimes L_1^-, L_2^-) &= \sum_{k,l} \psi(H_k \otimes L_k^+) \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l} \otimes L_{k,l}^+) \\
&= \sum_{k,l} \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l} \otimes L_{k,l}^+ \otimes H_k \otimes L_k^+)
\end{aligned}
$$

By substituting the expression in (2.8) to $H_k \otimes L_k^+ \otimes H_{k,l} \otimes L_{k,l}^+$,

$$
\begin{aligned}
\gamma(L \otimes L_1^-, L_2^-) &= \sum_{k,l,m} \overline{\psi\left((L_{k,l}^-)^*\right)} \psi(H_{k,l,m} \otimes L_{k,l,m}^+) \\
&= \beta\left(\sum_{k,l,m} L_{k,l}^- \otimes H_{k,l,m} \otimes L_{k,l,m}^+\right) \\
&= \beta\left(\sum_{k,l} L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \otimes H_k \otimes L_k^+\right) \\
&= \beta\left(\sum_k (L_2^-)^* \otimes L_k^- \otimes H_k \otimes L_k^+\right) \\
&= \beta\left((L_2^-)^* \otimes L \otimes L_1^-\right).
\end{aligned}
$$

Similarly, in order to see $\beta\left((L_2^-)^* \otimes L \otimes L_1^-\right) = \gamma(L_1^-, (L)^* \otimes L_2^-)$ we need the following decompositions (we use same notations to save number of letters.).

$$
\begin{aligned}
L^* \otimes L_2^- &= \sum_k L_k^- \otimes H_k \otimes L_k^+ \\
(L_k^-)^* \otimes L_1^- &= \sum_l L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+ \\
(L_{k,l}^-)^* \otimes H_k \otimes L_k^+ &= \sum_m H_{k,l,m} \otimes L_{k,l,m}^+,
\end{aligned}
$$

where elements in the decompositions are $L_k^-, L_{k,l}^- \in U(\mathcal{N}_-)$, $H_k, H_{k,l}, H_{k,l,m} \in U(\mathcal{H})$ and $L_k^+, L_{k,l}^+, L_{k,l,m}^+ \in U(\mathcal{N}_+)$. Now the final computation goes as follows.

$$
\begin{aligned}
\gamma(L_1^-, L^* \otimes L_2^-) &= \gamma\left(L_1^-, \sum_k L_k^- \otimes H_k \otimes L_k^+\right) \\
&= \sum_k \overline{\psi(H_k \otimes L_k^+)}\beta\left((L_k^-)^* \otimes L_1^-\right) \\
&= \sum_{k,l} \overline{\psi(H_k)\psi(L_k^+)}\,\overline{\psi\left((L_{k,l}^-)^*\right)}\psi(H_{k,l})\psi(L_{k,l}^+) \\
&= \sum_{k,l} \overline{\psi\left((L_{k,l}^-)^* \otimes H_k \otimes L_k^+\right)}\psi(H_{k,l})\psi(L_{k,l}^+) \\
&= \sum_{k,l,m} \overline{\psi(H_{k,l,m})\psi(L_{k,l,m}^+)}\psi(H_{k,l})\psi(L_{k,l}^+).
\end{aligned}
$$

In the next step (and only here) we need the self-adjointness of ψ on $\mathcal{H}$. Continuing,

$$
\begin{aligned}
\gamma(L_1^-, L^* \otimes L_2^-) &= \sum_{k,l,m} \psi\left((H_{k,l,m})^*\right)\overline{\psi(L_{k,l,m}^+)}\psi(H_{k,l})\psi(L_{k,l}^+) \\
&= \beta\left(\sum_{k,l,m}(L_{k,l,m}^+)^* \otimes (H_{k,l,m})^* \otimes H_{k,l} \otimes L_{k,l}^+\right) \\
&= \beta\left(\sum_{k,l}(L_k^+)^* \otimes (H_k)^* \otimes L_{k,l}^- \otimes H_{k,l} \otimes L_{k,l}^+\right) \\
&= \beta\left(\sum_k (L_k^+)^* \otimes (H_k)^* \otimes (L_k^-)^* \otimes L_1^-\right) \\
&= \beta\left((L_2^-)^* \otimes L \otimes L_1^-\right).
\end{aligned}
$$

This completes the proof. □

In the case of Vir, $c = \psi(C)$ and $h = \psi(L_0)$ must be real for the sesquilinear form to be defined. For such ψ it has been completely determined when the sesquilinear forms are positive definite thanks to the Kac determinant formula [49].

In the case of $\mathcal{K}$, the only condition for the existence of sesquilinear form is that $\psi(C) \in \mathbb{R}$. Hence there are additional parameters $h' \in \mathbb{R}, \lambda \in \mathbb{C}$ for generalized Verma modules $V_{h+ih',c,\lambda}$ on $\mathcal{K}$.

2.5.2 Irreducibility of generalized Verma modules on $\mathcal{K}$

In this section, we completely determine for which values of $h + ih', c, \lambda$ the corresponding generalized Verma modules on $\mathcal{K}$ are irreducible. The proof heavily relies on the result of Feigin and Fuks [36] which has determined when the Verma modules on the Virasoro algebra are irreducible. To utilize their result, we extend the generalized Verma modules on $\mathcal{K}$ to (non-unitary) representations of the Virasoro algebra.

Let $V_{h+ih',c,\lambda}$ be a generalized Verma module on $\mathcal{K}$ and v be the corresponding lowest weight vector such that

$$K_n v = (h + ih' + n\lambda)v \text{ for } n \geq 1 \text{ and } Cv = cv. \tag{2.9}$$

First we observe that

$$K_n \mapsto K_n - n\lambda I, C \mapsto C,$$

where I is the identity operator on $V_{h+ih',c,\lambda}$, extends by linearity to a well-defined (non *-) representation (on the same space $V_{h+ih'+n\lambda}$) of $\mathcal{K}$ (the proof is the same as that of proposition 2.2.5). On the other hand, it is straightforward to see that this new representation is equivalent to $V_{h+ih',c,0}$. Irreducibility of a representation of an algebra is not changed even if we add the identity operator to the set of operators. Therefore the irreducibility of $V_{h+ih',c,\lambda}$ is equivalent to that of $V_{h+ih',c,0}$ and we may restrict the consideration to the latter case. We denote it $V_{h+ih',c}$.

Lemma 2.5.7. *For any $w \in V_{h+ih',c}$ there is $N \in \mathbb{N}$ such that $K_m w = K_n w$ for $m, n \geq N$.*

Proof. The module $V_{h+ih',c}$ is spanned by vectors $K_{n_1} \cdots K_{n_k} v$. We will show the lemma by induction with respect to k. If $w = v$, the lowest weight vector, then the lemma obviously holds with $N = 1$, hence the case $k = 0$ is done.

Assume that the lemma holds for w and put $\lim_m K_m w = w'$ (here lim has nothing to do with any topology, but simply means that "the equality holds for sufficiently large m"). We will show that it also holds for $K_n w$. Let us calculate

$$\begin{aligned} K_m K_n w &= ([K_m, K_n] + K_n K_m)w \\ &= ((m-n)K_{m+n} - mK_m + nK_n + K_n K_m)\, w, \end{aligned}$$

and for sufficiently large m this is equal to

$$(m-n)w' - mw' + nK_n w + K_n w' = -nw' + nK_n w + K_n w'.$$

and this does not depends on m. $\square$

Let us define $Dw = \lim_m K_m w$. Then, it is clear that D is a linear operator on $V_{h+ih',c}$ and it holds $Dv = (h+ih')v$.

Lemma 2.5.8. *The following commutation relation holds:*

$$[D, K_n] = n(K_n - D). \tag{2.10}$$

Proof. We only need to calculate

$$\begin{aligned} (DK_n - K_n D)w &= \lim_m (K_m K_n - K_n K_m)w \\ &= \lim_m ((m-n)K_{m+n} - mK_m + nK_n)\, w \\ &= n(K_n - D)w. \end{aligned}$$

□

The relation (2.10) can be rewritten as $[K_n - D, -D] = n(K_n - D)$.

Proposition 2.5.9. *The representation of $\mathcal{K}$ on $V_{h+ih',c,0}$ extends to a representation of* Vir*. This extension is the Verma module with $-h-ih', c$.*

Proof. We take a correspondence $L_0 \mapsto -D, L_n \mapsto K_n - D, C \mapsto C$. Now we know all the commutation relations between D and K_n, the confirmation that this correspondence is a representation is straightforward.

It is clear that the lowest weight vector is v and $-Dv = (-h-ih')v$, $(K_n - D)v = 0$ for $n \geq 0$, $Cv = cv$. We only have to show that all the vectors of the form $(K_{n_1} - D)\cdots(K_{n_k} - D)v$, where $n_1 \leq \cdots \leq n_k$, are linearly independent. But this is clear from the fact that these vectors are eigenvectors of D and the fact that $\{K_{n_1}\cdots K_{n_k} v\}$ are independent by definition. The former fact is shown by a straightforward induction. □

Here we remark that this extension of the representation does not change the irreducibility. If the module on $\mathcal{K}$ is irreducible, then clearly it is irreducible as a module on Vir. On the other hand the operator D above is defined as the limit of K_n's, hence if the module on $\mathcal{K}$ is reducible then it is still reducible as a module on Vir.

The following theorem is due to Feigin and Fuks [36].

Theorem 2.5.10. *For $h, c \in \mathbb{C}$, the Verma module $V_{h,c}$ on the Virasoro algebra is reducible if and only if there are natural numbers α, β such that*

$$\begin{aligned} \Phi_{\alpha,\beta}(h,c) &:= \left(h + \frac{1}{24}(\alpha^2 - 1)(c-13)\frac{1}{2}(\alpha\beta - 1)\right) \\ &\quad \times \left(h + \frac{1}{24}(\beta^2 - 1)(c-13)\frac{1}{2}(\alpha\beta - 1)\right) \\ &\quad + \frac{(\alpha^2 - \beta^2)^2}{16} = 0. \end{aligned}$$

The application of this to our case is now straightforward.

Corollary 2.5.11. *For $h, h' \in \mathbb{R}$, $c, \lambda \in \mathbb{C}$, the generalized Verma module $V_{h+ih',c,\lambda}$ on $\mathcal{K}$ is reducible if and only if there are natural numbers α, β such that*

$$\Phi_{\alpha,\beta}(-h-ih', c) = 0.$$

2.6 Endomorphisms of $\mathcal{K}$

This section is devoted to the study of *-endomorphisms of the algebra $\mathcal{K}$. As in the case of automorphisms, endomorphisms of $\mathcal{K}$ are not natural objects, but they are interesting from the viewpoint of representations. We remarked before that any composition of a *-endomorphism and unitary representation provides a unitary representation. In this way, we obtain a strange kind of representations of $\mathcal{K}$. We will also have a rough classification of endomorphisms.

It is well known (for example, see [65][94]) that the following maps are endomorphisms of the Virasoro algebra and they restrict to $\mathcal{K}$:

$$\begin{aligned}\delta_r(L_N) &= \frac{1}{r}L_{rn} + \frac{C}{24}\left(r - \frac{1}{r}\right),\\ \delta_r(C) &= rC,\end{aligned}$$

for any integer $r \in \mathbb{Z}$.

We have another type of *-endomorphisms of $\mathcal{K}$ parametrized by a complex number α. In the next section we will see that these endomorphisms are related to some unitary representation of $\mathrm{Diff(S^1)}_0$.

Proposition 2.6.1. *Let $\alpha \in \mathbb{C}$ and K be an element of $\mathcal{K}$ which satisfies $[K, K^*] = -K - K^*$. Define*

$$\begin{aligned}\sigma_\alpha(K_n) &= \left(\frac{n^2+n}{2}\alpha + \frac{n^2-n}{2}\overline{\alpha} - \frac{n^2-n}{2}\right)K\\ &\quad + \left(\frac{n^2+n}{2}\alpha + \frac{n^2-n}{2}\overline{\alpha} - \frac{n^2+n}{2}\right)K^*,\\ \sigma_\alpha(C) &= 0.\end{aligned}$$

*Then σ_α extends to a *-endomorphism of $\mathcal{K}$ by linearity.*

Remark 2.6.2. Examples of K in this proposition are $K = K_1, -K_{-1}, -\frac{1}{6}K_2 + \frac{2}{3}K_1$. Since the image of C is 0, σ_α extends also to a *-homomorphism of $\mathcal{K}_0$ into $\mathcal{K}$. Therefore, the kernel of σ_α is the direct sum of $\ker\sigma_\alpha$ as a homomorphism of $\mathcal{K}_0$ and $\mathbb{C}\cdot C$.

Proof. It is clear that σ_α preserves the *-operation. We only have to confirm that it preserves commutation relations and this is done by straightforward calculations. However, we will exhibit a clearer procedure.

Let us put $\beta = 3\alpha + \overline{\alpha} - 1$. The definition of σ_α can be rewritten as

$$\begin{aligned}\sigma_\alpha(K_n) = &\left(\frac{n^2-n}{2}\beta - (n^2-2n)\alpha\right)K\\ &+ \left(\frac{n^2-n}{2}\beta - (n^2-2n)\alpha - n\right)K^*.\end{aligned}$$

If we put $\gamma_n = \frac{n^2-n}{2}\beta - (n^2-2n)\alpha$, this takes the form $\sigma_\alpha(K_n) = \gamma_n K + (\gamma_n - n)K^*$. Now it is easy to see that

$$\begin{aligned}
[\sigma_\alpha(K_n), \sigma_\alpha(K_{-n})] &= [\gamma_n K + (\gamma_n - n)K^*, (\overline{\gamma_n} - n)K + \overline{\gamma_n}K^*] \\
&= (-|\gamma_n|^2 + |\gamma_n - n|^2)(K + K^*) \\
&= -n(2\mathrm{Re}\gamma_n - n)(K + K^*) \\
&= -n\left(\sigma_\alpha(K_n) + \sigma_\alpha(K_{-n})\right).
\end{aligned}$$

Next we calculate a general commutator, for $m \neq -n$,

$$\begin{aligned}
& [\sigma_\alpha(K_m), \sigma_\alpha(K_n)] \\
= & \left(m\left(\frac{n^2-n}{2}\beta - (n^2-2n)\alpha - n\right) - n\left(\frac{m^2-m}{2}\beta - (m^2-2m)\alpha\right)\right) \\
& \times(K + K^*) \\
= & \left(\frac{\beta}{2} - \alpha\right)(m^2n - mn^2)(K + K^*)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (m-n)\sigma_\alpha(K_{m+n}) - m\sigma_\alpha(K_m) + n\sigma_\alpha(K_n) \\
= & \left((m-n)\gamma_{m+n} - m\gamma_m + n\gamma_n - (m-n)(m+n) + m^2 - n^2\right) \\
& \times(K + K^*) \\
= & \left(\frac{\beta}{2} - \alpha\right)(m^2n - mn^2)(K + K^*)
\end{aligned}$$

and this completes the proof. □

Proposition 2.6.3. *Let us assume that* $K + K^* \neq 0$. *If* $\alpha \in \frac{1}{2} + i\mathbb{R}$, *then* $\ker(\sigma_\alpha)$ *is* $\mathcal{K}_1 \oplus \mathbb{C} \cdot C$ *(see section 2.3.1). Otherwise,* $\ker(\sigma_\alpha)$ *is* $\mathcal{K}_2 \oplus \mathbb{C} \cdot C$.

Proof. As we have noted in the remark 2.6.2, first we think σ_α as a homomorphism of $\mathcal{K}_0$.

By direct calculations, we have (see section 2.3.2),

$$\begin{aligned}
\rho(M_n^0) &= \left(-(n+1)\alpha - n\overline{\alpha} + n\right)K + \left(-(n+1)\alpha - n\overline{\alpha} + n + 1\right)K^*, \\
\rho(M_n^1) &= (\alpha + \overline{\alpha} - 1)(K + K^*), \\
\rho(M_n^2) &= 0.
\end{aligned}$$

The kernel of σ_α must be one of ideals in theorem 2.3.16. From this it is clear that $\ker(\sigma_\alpha)$ contains $\mathcal{K}_2$ and contains $\mathcal{K}_1$ if and only if $\mathrm{Re}\alpha = \frac{1}{2}$.

By the remark 2.6.2, the kernel of σ_α as a *-endomorphism is $\mathcal{K}_1 \oplus \mathbb{C} \cdot C$ or $\mathcal{K}_2 \oplus \mathbb{C} \cdot C$, respectively. □

We have a partial classification of endomorphisms of $\mathcal{K}$.

Proposition 2.6.4. *If ρ is a nontrivial *-endomorphism of $\mathcal{K}$, then the possibilities are:*

1. $\rho = \sigma_\alpha$ *with appropriate* K *and* $\alpha \in \frac{1}{2} + i\mathbb{R}$. *In this case,* $\ker(\rho) = \mathcal{K}_1 \oplus \mathbb{C}\cdot C$ *and* $\rho(K_1) = \alpha K + (\alpha - 1)K^*$.
2. $\rho = \sigma_\alpha$ *with appropriate* K *and* $\alpha \notin \frac{1}{2} + i\mathbb{R}$. *In this case,* $\ker(\rho) = \mathcal{K}_2 \oplus \mathbb{C}\cdot C$ *and* $\rho(K_1) = \alpha K + (\alpha - 1)K^*$.
3. $\rho(K_1) = \sum_{i=1}^N a_i K_i + a_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, $\rho(K_2) = \sum_{i=1}^{2N} b_i K_i + b_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, *where* $\sum_{i=1}^N a_i = \sum_{i=1}^{2N} b_i = \frac{1}{N}$. *In this case,* $\ker(\rho) = \{0\}$.
4. $\rho(K_1) = \sum_{i=-N}^{-1} a_i K_i + a_0 C \in \mathcal{K}_- \oplus \mathbb{C}C$, $\rho(K_2) = \sum_{i=2N}^{-1} b_i K_i + b_0 C \in \mathcal{K}_+ \oplus \mathbb{C}C$, *where* $\sum_{i=N}^{-1} a_i = \sum_{i=2N}^{-1} b_i = -\frac{1}{N}$. *In this case,* $\ker(\rho) = \{0\}$.
5. $\rho(K_n) = in\lambda C$ *for some* $\lambda \in \mathbb{R}$.

Proof. By lemma 2.4.2 and 2.4.3, it takes the form $\rho(K_1) = \lambda K + \mu K^* + \nu C$, $\rho(K_2) = \lambda' K' + \mu' K'^* + \nu' C$, where K and K' are elements of $\mathcal{K}_+$. Also by lemma 2.4.1 with the commutation relation of K_2 and K_{-1}, K and K'^* must be proportional.

If both of λ and μ are nonzero, then also K and K' must be proportional. By the commutation relation of K_1 and K_{-1} we see that some scalar multiple of K plus a central element (we call it temporarily $\tilde{K}$) satisfies $[\tilde{K}, \tilde{K}^*] = -\tilde{K} - \tilde{K}^*$. Hence from the beginning we may assume $[K, K^*] = -K - K^* + \kappa C$ for some $\kappa \in \mathbb{C}$. Then again by the commutation relation, $\mu = \lambda - 1$. Similarly, it holds $\mu' = \lambda' - 2$. By the commutation relation of K_2 and K_{-1} we see $\lambda' = 3\lambda + \overline{\lambda} - 1$. Then this is exactly the case (1) or (2). It depends on the value of λ whether it is (1) or (2).

Let one of λ and μ be zero. By composing an automorphism τ, we may assume $\mu = 0$ and we will show that we have the case (3). By the same argument of the beginning of theorem 2.4.5, $\rho(K_1)$ takes the form $\rho(K_1) = \sum_{i=1}^N a_i K_i + a_0 C, \rho(K_2) = \sum_{j=1}^{2N} b_j K_j + b_0 C$ and $\sum_{i=1}^N a_i = \frac{1}{N} = \sum_{i=1}^{2N} b_i$. Any finite set of $\rho(K_i)$'s is linearly independent (by considering the highest or lowest terms of $\rho(K_i)$ in the standard basis of $\mathcal{K}$) and we see $\ker(\rho) = \{0\}$.

If $\lambda = \mu = 0$, by the commutation relations 2.1, $\rho(K_2)$ must be mapped to a central element. By the same argument as that of 2.2.1, ρ is of the form $\rho(K_n) = in\lambda C$. □

Let $\mathfrak{p}$ be the Lie algebra of the group generated by translations and dilations in $\mathrm{Diff}(S^1)$. This algebra has a basis $\{T, D\}$ with the relation $[D, T] = T$ [63][57]. Its complexification (which we denote again $\mathfrak{p}$) is a *-Lie algebra with the *-operation $D^* = -D, T^* = -T$. By setting $K = -D + iT$, we have $[K, K^*] = -K - K^*$.

Lemma 2.6.5. *Any unitary representation φ' of $\mathfrak{p}$ produces a representation φ'_1 of $\mathcal{K}_0$ (or a representation of $\mathcal{K}$ with the central charge $c = 0$).*

Proof. It suffices to set

$$\varphi'_1(K_n) = \frac{n^2 + n}{2}\varphi'(K) + \frac{n^2 - n}{2}\varphi'(K^*).$$

We see that ϕ_1' preserves the commutation relations by the same computations in the proof of proposition 2.6.1 with $\alpha = 1$. $\square$

Remark 2.6.6. Any composition of a *-endomorphism and a unitary representation of $\mathcal{K}$ is again a unitary representation. As we shall see in the next section, a composition of an endomorphism of type (1) or (2) in proposition 2.6.4 and a lowest weight representation gives rise to a strange representation (in the sense that they are "localized at the point at infinity"). On the other hand, a composition of the type (3) endomorphism and a lowest weight representation contains at least one lowest weight vector in the sense of subsection 2.5.2, equation (2.9) which is the lowest weight vector of the original representation, and the value of $h + ih'$ is changed to $\frac{1}{N}(h + ih')$. If we start with the restriction to $\mathcal{K}$ of a unitary representation of Vir, representations with "complex energy" (namely, $h' \neq 0$) do not arise in this way.

2.7 Some unitary representations of B_0

In this section we will construct true (not projective) unitary representations of B_0. Symmetries in physics are in general described by unitary projective representations of a group [81]. From this point of view, one dimensional true representations are trivial, since they are equivalent to the trivial representations as projective representations. Nevertheless, we here exhibit a construction of a one dimensional representation. The author believes that this reveals the big difference between $\mathrm{Diff}(S^1)$ and B_0. In fact, $\mathrm{Diff}(S^1)$ does not admit any positive energy true representation (see [81]). This difference comes mainly from the fact that Diff(S^1) is simple but B_0 is not simple.

We identify B_0 with a space of functions on $\mathbb{R}$ as in section 2.3.5.

Proposition 2.7.1. *For any $\lambda \in \mathbb{R}$ the map*

$$\begin{array}{rcl} \varphi : B_0 & \to & S^1 \\ f & \mapsto & \exp(i\lambda \log f'(0)) \end{array}$$

is a (one-dimensional) unitary representation of B_0.

Proof. Recall that B_0 is the group of orientation preserving, 0-stabilizing diffeomorphisms of S^1. By the identification with the function space, the derivative of any element is everywhere (in particular at $\theta = 0$) positive, hence the map is properly defined.

By the formula

$$(f \circ g)'(0) = f'(0) \cdot g'(0),$$

we see the map φ above is multiplicative. $\square$

Remark 2.7.2. This φ is obviously irreducible and does not extend to $\mathrm{Diff}(S^1)$. In fact, φ is the integration of the one-dimensional representation of corollary 2.2.2. If $g \in B_0$ is localized on some closed interval which does not include 0, then $\varphi(g) = 1$. In this sense, φ is "localized at the point at infinity".

Next we need a general lemma.

Lemma 2.7.3. *Let G be a group, H a normal subgroup of G and π the quotient map $G \to G/H$. Let F be a subgroup of G such that $F \cap H = \{e\}$ and $\pi(F) = G/H$. Then G/H and F are isomorphic by a canonical isomorphism γ such that $\gamma \circ \pi|_F = \mathrm{id}$. If φ is a representation of F, it extends to a representation $\tilde{\varphi} := \varphi \circ \gamma \circ \pi$ of G.*

Let $B_2 = \{g \in B_0 : f'(0) = 1, f''(0) = 0\}$. It is easy to see that B_2 is a normal subgroup of B_0.

Let $G = B_0, H = B_2$ and $F = P$ be the subgroup generated by dilations and translations. It is obvious that any element of F can be written as a product of a dilation and a translation. The derivative of a dilation at point 0 is always 1, whereas a nontrivial translation has a derivative different from 1 at 0. From this, the intersection of F and H must be pure dilations. But then, any element of this intersection must have a vanishing second derivative at 0. This implies that the intersection is trivial.

By a similar consideration, it is not difficult to see that $\pi(P) = \pi(B_0)$. By the previous lemma, the unitary irreducible representation of $F = P$ extends to a unitary irreducible representation of B_0 having B_2 in the kernel.

Also this representation is "localized at the point at infinity", since if a diffeomorphism is localized in a closed interval which does not contain 0, then it is an element of B_2 and hence mapped to the identity operator.

Summing up, we have the following.

Theorem 2.7.4. *Any unitary representation φ of P canonically extends to a representation $\tilde{\varphi}$ of B_0 which is localized at the point at infinity.*

We describe the relation between this representation and the endomorphism of $\mathcal{K}$ constructed in section 2.6. The group P admits a unique irreducible positive energy (which means that the generator of translation is positive) true (not projective) representation [63]. This representation can be considered as the integration of several lowest weight representations of the Lie algebra $\mathfrak{p}$ of $\mathcal{P}$. In the following, we fix such a representation of $\mathfrak{p}$ and extend it to $\mathcal{K}$. The representation space of $\mathfrak{p}$ is a dense subspace of the representation space of P and it is the core of any generator of one-parameter subgroup of P (see [63]). Through $\tilde{\varphi}$, any one-parameter subgroup g_t of B_0 is first mapped to P by $\gamma \circ \pi$ and then represented as a one-parameter group of unitary operators. Hence any unbounded operator appearing here is in the representation of $\mathfrak{p}$ explained above and there arise no problems of domains or self-adjointness.

Proposition 2.7.5. *Let φ be a unitary representation of the Lie group P, φ' be the corresponding representation of the Lie algebra $\mathfrak{p}$ and φ'_1 be the extension to $\mathcal{K}$ in proposition 2.6.5, then φ'_1 integrates to $\tilde{\varphi}$ in the theorem 2.7.4.*

Proof. The quotient group B_0/B_2 is isomorphic to $\mathbb{R}_+ \rtimes \mathbb{R}$ with the group operation:

$$(X_1, X_2) \cdot (Y_1, Y_2) = (X_1Y_1, X_1Y_2 + Y_1^2X_2), \text{ for } X_1, Y_1 \in \mathbb{R}_+, X_2, Y_2 \in \mathbb{R}.$$

The isomorphism ρ is given by $f \mapsto (f'(0), f''(0))$.

It's Lie algebra has the structure $\mathbb{R} \oplus \mathbb{R}$ with

$$[(x_1, x_2), (y_1, y_2)] = (0, x_2 y_1 - x_1 y_2)] \text{ for } x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If g^s is a one-parameter subgroup in B_0 with generator v, then the corresponding element in the algebra is $\rho'(v) = (v'(0), v''(0))$, where ρ' is the derivative of ρ.

The generator of the one-parameter subgroup of dilations $D_s(\theta)$ is $\frac{1}{2}(K_1 - K_1^*)(\theta) =: d_1(\theta) = \sin\theta$ and the generator of translations $T_s(\theta)$ is $-\frac{i}{2}(K_1 + K_1^*) =: t_1(\theta) = 1 - \cos\theta$. Thus $\rho'(d_1) = (1, 0)$ and $\rho'(t_1) = (0, 1)$. Similarly, the generator $\frac{1}{2}(K_n - K_n^*)(\theta) =: d_n(\theta) = \sin n\theta$ is mapped to $(n, 0)$ and $-\frac{i}{2}(K_n + K_n^*) =: t_n(\theta) = 1 - \cos n\theta$ is mapped to $(0, n^2)$. In short, it holds that $\rho'(d_n) = n\rho'(d_1), \rho'(t_n) = n^2\rho'(t_1)$. Hence these relations hold also for the derivative of $\tilde{\varphi}$, namely $\tilde{\varphi}'(d_n) = n\tilde{\varphi}'(d_1), \tilde{\varphi}'(t_n) = n^2\tilde{\varphi}'(t_1)$.

On the other hand, for φ_1' we have

$$\begin{aligned}
\varphi_1'\left(\frac{1}{2}(K_1 - K_1^*)\right) &= \frac{1}{2}(K - K^*), \\
\varphi_1'\left(-\frac{i}{2}(K_1 + K_1^*)\right) &= -\frac{i}{2}(K + K^*), \\
\varphi_1'\left(\frac{1}{2}(K_n - K_n^*)\right) &= \frac{n}{2}(K - K^*) = n\varphi_1'\left(\frac{1}{2}(K_1 - K_1^*)\right), \\
\varphi_1'\left(-\frac{i}{2}(K_n + K_n^*)\right) &= -\frac{in^2}{2}(K + K^*) = n^2\varphi_1'\left(-\frac{i}{2}(K_1 + K_1^*)\right).
\end{aligned}$$

From this it is clear that φ_1' and $\tilde{\varphi}'$ are equivalent, since by definition $\varphi_1(d_1) = \tilde{\varphi}'(d_1)$ and $\varphi_1(t_1) = \tilde{\varphi}'(t_1)$ □

As remarked before, there is a unique irreducible positive energy representation of P. By the proposition above, it extends to an irreducible positive energy true representation of B_0.

2.8 Open problems

Construction and classification

A natural problem in the representation theory of a group is of course to construct and classify positive-energy representations. In particular, the generalized Verma modules we constructed here have a large family of candidates of such representations.

The group B_0 implements the diffeomorphism symmetry of conformal nets and the Virasoro net can be embedded in such nets, hence the representations of such larger nets could be used to construct new representations of B_0.

Further subalgebras

In this Chapter we considered the group B_0, the group of diffeomorphisms of S^1 which preserves the point at infinity. It is also natural to consider the group of the diffeomorphisms of $\mathbb{R}$ with compact support. On this group, $\mathbb{R}$ acts as translation and it is possible to define the notion of positivity of energy. The representation theory of this group would be interesting. As seen from an analogy with Chapter 3, such a representation is related with thermal states of the Virasoro nets.

Chapter 3

Ground state representations of loop algebras

Chapter Introduction

For a compact connected Lie group G, the group of smooth maps from the circle S^1 to G is called the loop group LG of G. Loop groups have been a subject of extensive research both from purely mathematical and physical viewpoints ([75], [91], [97], [41], [89], [29]). On the one hand, the representation theory of LG has a particularly simple structure. If we consider positive energy projective representations (defined below), and if G is simply connected, then such representations behave very much like ones of compact groups. They are completely reducible, irreducible representations are classified by their "lowest weights", and irreducible representations are realized as the spaces of complex line bundles on the group by analogy with Borel-Weyl theory [75]. On the other hand, any such representation can be considered as a charged sector of a conformal field theory.

It is a natural variant to think about the group of maps from the real line $\mathbb{R}$ into G. The natural group of covariance is now the translation group. Since S^1 is a one-point compactification of $\mathbb{R}$, we consider this group as a subgroup of LG. Then one would expect that there should arise several representations which do not extend to LG. This problem has been open for a long time [75].

The main objective of this Chapter is to show the contrary at the level of Lie algebra with the assumption of existence of an invariant vector: Namely, if a (projective) unitary representation of $\mathscr{S}\mathfrak{g}_\mathbb{C}$ (the Lie subalgebra of $L\mathfrak{g}_\mathbb{C}$ of Schwartz class elements, defined below) is covariant with respect to translation and admits a cyclic vector invariant under translation, then it extends to a representation of $L\mathfrak{g}_\mathbb{C}$. Then even a complete classification of such representations with a "ground state vector" follows due to the classification for $L\mathfrak{g}_\mathbb{C}$ by Garland [42] or at group level by Pressley and Segal [75].

Besides the interest from a purely representation-theoretic context, the study of positive energy representations with an invariant vector for translation is motivated by physics, in particular by chiral conformal field theory.

In the setting of algebraic quantum field theory, a chiral component of a conformal field theory is a net of von Neumann algebras on the circle (see Section 1.1.1). To construct examples of such nets, we can utilize positive energy representations of loop groups, and in fact these examples have played a key role in the classification of certain conformal field theories [52], [96].

For a certain class of representations of nets, a sophisticated theory has been established by Doplicher-Haag-Roberts (for its adaptation to chiral CFT, see Section 1.1.6 and [41]). The DHR theory is concerned with representations which are localized in some interval, i.e., unitarily equivalent to the original (vacuum) representations outside the interval of localization. These representations are considered to describe the states with finite charge.

On the other hand, in a physical context we are sometimes interested in a larger class of representations. A typical case occurs in the study of thermal equilibrium states. A thermal equilibrium state is invariant with time, thus in the context of one-dimensional chiral theory it is invariant under translation. By physical intuition, we would say that a state with a finite amount of charge cannot be invariant under translation. Then we should consider a more general class of representations. As explained later, an invariant state for translation whose GNS representation has positive-energy can be considered as an equilibrium state with temperature zero. Physicists call it a ground state.

Nets of von Neumann algebras generated by representations of loop groups are known to be completely rational (Section 1.1.5 and [55, 97]). This complete rationality implies that the net has only finitely many inequivalent irreducible DHR representations. Physically it means only finite amount of charge is possible in such a model. Then one would guess that any completely rational net has only equilibrium states without charge. We will prove a result on representations of the Lie algebras of loop groups which strongly supports this point of view, namely, we will show that any ground state representation of the loop algebra (in a certain sense clarified below) is the vacuum representation.

Similar lines of research will be conducted also for equilibrium states with finite temperature in Chapter 4, in which the we show that if a conformal net is completely rational then it admits the unique KMS state.

John E. Roberts has proved that for a general dilation-covariant net of observables there is a unique dilation-invariant state, the vacuum [80]. This in particular tells us that a ground state different from the vacuum cannot be dilation-invariant (although this never excludes the existence of other ground states). In fact, the composition of a ground state on the Virasoro nets with dilation is used to produce different ground states [86]. A similar technique is used in Chapter 4 to obtain continuously many different KMS states.

At the end of the introduction, I would like to note that the above result on KMS states has been proved with the techniques of operator algebras, in particular subfactors, and utilizes relationships between several nets. On the other hand, the present result on the uniqueness of ground states for loop algebras relies only on elementary facts on Lie algebras and gives a direct proof.

Unfortunately, the present result does not imply directly the uniqueness of ground state of nets of von Neumann algebras. There are still difficulties in the differentiability of given representations and extension to “Schwartz class” algebra. These problems will be discussed in Section 3.4.

In Section 3.1 we introduce the main object of this Chapter, the algebra $\mathscr{S}\mathfrak{g}$. In Section 3.2 we prove that translation-invariant 2-cocycle on $\mathscr{S}\mathfrak{g}$ is essentially unique up to scalar. In Section 3.3 we prove that ground states on $\mathscr{S}\mathfrak{g}$ can be classified only by the cocycle. In Section 3.4 we discuss the physical meaning of ground states and possible implications to the representation theory of conformal nets of von Neumann algebras. Section 3.5 summarizes open problems.

3.1 Preliminaries on the Schwartz class algebra $\mathscr{S}\mathfrak{g}$

As noted in the introduction, we will consider an analogous problem on infinite dimensional Lie algebras defined through the real line $\mathbb{R}$, instead of S^1. We identify the circle S^1 as the one-point compactification of the real line $\mathbb{R}$ by the Cayley transform:

$$t = i\frac{1+z}{1-z} \Longleftrightarrow z = \frac{t-i}{t+i}, \quad t \in \mathbb{R}, \;\; z \in S^1 \subset \mathbb{C}.$$

Here we denote by G a compact simple simply connected Lie group and by $\mathfrak{g}$ its Lie algebra. The Lie algebra $\mathfrak{g}$ is finite dimensional, hence for a map from $\mathbb{R}$ into $\mathfrak{g}$ we can define the rapidly decreasing property. As one of the simplest formulations, we take the following: Let n be the dimension of $\mathfrak{g}$. By fixing a basis in $\mathfrak{g}$, we can consider any map $\xi : \mathbb{R} \to \mathfrak{g}$ as the n-tuple of real-valued functions. Then we say ξ is rapidly decreasing if each component of ξ is rapidly decreasing. Of course this definition does not depend on the chosen basis. It is also straightforward to define a tempered distribution on $\mathscr{S}\mathfrak{g}$. A linear functional φ is said to be tempered if each restriction of φ to the subspaces of elements having nonzero value only on i-th component is a tempered distribution. Again this definition is independent of the choice of basis.

The main object of this Chapter is the following.

$$\begin{aligned}\mathscr{S}\mathfrak{g} := \{\xi : \mathbb{R} \to \mathfrak{g}, &\text{smooth, rapidly decreasing}\},\\ &[\xi,\eta](t) := [\xi(t),\eta(t)], t \in \mathbb{R}\end{aligned}$$

namely, the algebra of Schwartz class elements. Under the identification of the punctured circle and the real line, it is easy to see that this algebra is a closed subalgebra of $L\mathfrak{g}$. It is easy to see that as linear spaces $\mathscr{S}\mathfrak{g} = \mathfrak{g} \otimes \mathscr{S}$ and the Lie algebra operation is $[x \otimes f, y \otimes g] = [x,y] \otimes fg$.

The compact group G acts on $\mathfrak{g}$ by the adjoint action, hence also on $L\mathfrak{g}$ by the pointwise application. This action is smooth [75, Section 3.2]. Since $\mathscr{S}\mathfrak{g}$ is a closed subalgebra of $L\mathfrak{g}$, the restricted action of G on $\mathscr{S}\mathfrak{g}$ is also smooth. It is obvious that $\mathscr{S}\mathfrak{g}$ is invariant under G.

We are interested in positive-energy, unitary, projective representations. Recall that for $L\mathfrak{g}$ we considered the subalgebra of polynomial loops and all these notions are defined in purely algebraic terms. For $\mathscr{S}\mathfrak{g}$ we cannot take such an appropriate subalgebra. Instead, we need to formulate all these properties of representations with analytic terms from the beginning. Let $\mathcal{H}$ be a Hilbert space. Note this time that $\mathbb{R}$ acts on $\mathscr{S}\mathfrak{g}$ by translation:

$$\xi_a(t) := \xi(t-a).$$

Again it is straightforward to define the complexification of $\mathscr{S}\mathfrak{g}$ and it is identified with $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$. The $*$-operation is naturally defined.

Definition 3.1.1. *A projective unitary representation π with a 2-cocycle ω of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ assigns to any element ξ of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ a (possibly unbounded) linear operator $\pi(\xi)$ on $\mathcal{H}$ such that there is a common dense domain $V \subset \mathcal{H}$ for all $\{\pi(\xi) : \xi \in \mathscr{S}\mathfrak{g}_{\mathbb{C}}\}$ and on V it holds that*

$$\begin{aligned}\pi([\xi,\eta])v = (\pi(\xi)\pi(\eta) - \pi(\eta)\pi(\xi) + \omega(\xi,\eta))\, v,\\ \langle \pi(\xi)v_1, v_2\rangle = \langle v_1, \pi(\xi^*)v_2\rangle.\end{aligned}$$

A projective unitary representation of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ is said to have positive energy if there is a unitary representation U of $\mathbb{R}$ with positive spectrum such that $U(a)\pi(\xi)U(a)^* = \pi(\xi_a)$.

A projective unitary representation of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ is said to be smooth if for each v_1, v_2 in the common domain V the linear form $\langle \cdot v_1, v_2\rangle$ is tempered.

Remark 3.1.2. Let us make some remarks. By the same reason as in Remark 1.5.5, we can define an action of translation on the space of 2-cocycles on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ and for a positive energy representation the cocycle is invariant under translation.

If we have a representation of a group, it is natural to ask if this representation produces a representation of the Lie algebra by an appropriate derivation. And for LG the answer is yes. We can prove the existence of a common domain by utilizing finite dimensional subgroups in LG with common elements ([91, Section 1.8] or [24, Appendix]). We can define a corresponding group for $\mathscr{S}\mathfrak{g}$, but it is not clear if such a common domain exists for a representation of $\mathscr{S}\mathfrak{g}$. We will discuss on this problem in the final section.

There is also a problem on the smoothness of the representations. As explained in the final section, in the algebraic approach to CFT it is natural to consider the subalgebra of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ with compact support. On the other hand, for the moment we know the proof of uniqueness of ground state representations only for Schwartz class algebra. For the present proof it is essential since we exploit the Fourier transforms. Unfortunately we don't know to what extent it is natural to assume the continuity to the Schwartz class.

3.2 Uniqueness of translation invariant 2-cocycle

As we have seen in Remark 3.1.2, for a positive-energy representation the cocycle is always translation-invariant. Then we will restrict the consideration to translation-invariant cocycles. In this section, we will show that the Lie algebra $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ has the unique translation-covariant central extension. First of all, we can define an action of G on the space of cocycles by

$$(g\omega)(\xi,\eta) := \omega(g^{-1}\xi, g^{-1}\eta).$$

We show that we can restrict the consideration to G-invariant cocycles.

Lemma 3.2.1. *Any 2-cocycle ω on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ is local, namely, if ξ and η have disjoint supports, then $\omega(\xi,\eta) = 0$.*

Proof. Let us take $x, y, z \in \mathfrak{g}_{\mathbb{C}}$ and $f, g, h \in \mathscr{S}(\mathbb{R})$. Then by the Jacobi identity (1.2),

$$\begin{aligned} 0 &= \omega([x\otimes f, y\otimes g], z\otimes h) + \omega([y\otimes g, z\otimes h], x\otimes f) + \omega([z\otimes h, x\otimes f], y\otimes g) \\ &= \omega([x,y]\otimes fg, z\otimes h) + \omega([y,z]\otimes gh, x\otimes f) + \omega([z,x]\otimes hf, y\otimes g). \end{aligned}$$

Here, let the supports of f and g be disjoint and compact, and h be a function such that $h(t) = 1$ on $\mathrm{supp}(f)$ and $h(t) = 0$ on $\mathrm{supp}(g)$. Then the equality above transforms into

$$\omega([z,x]\otimes f, y\otimes g) = 0.$$

Since $\mathfrak{g}_{\mathbb{C}}$ is simple, $[z,x]$ spans the whole Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and the lemma is proved by noting that these elements span elements with compact support. □

Lemma 3.2.2. *Any translation-invariant continuous 2-cocycle ω on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ is equivalent up to coboundary to a G-invariant cocycle.*

Proof. We see that $g\omega - \omega$ is coboundary for any $g \in G$. Since G is connected, we can take a smooth path α such that $\alpha(0) = e$ and $\alpha(1) = g$. Then by the fundamental theorem of analysis it holds that

$$\begin{aligned} g\omega(\xi,\eta) - \omega(\xi,\eta) &= \alpha(1)\omega(\xi,\eta) - \alpha(0)\omega(\xi,\eta) \\ &= \int_0^1 \frac{d}{dt}\omega(\alpha^{-1}(t)\xi, \alpha^{-1}(t)\eta)dt. \end{aligned}$$

For the moment, let us assume that ξ and η have compact supports. Then there are elements $\delta(t)$ with support compact such that

$$\frac{d}{dt}\alpha^{-1}(t)\xi = [\delta(t), \alpha^{-1}(t)\xi], \quad \frac{d}{dt}\alpha^{-1}(t)\eta = [\delta(t), \alpha^{-1}(t)\eta].$$

In fact, it is enough to take an element of the form $x \otimes f$, where $x = \alpha^{-1}(t)'$ and $f(t) = 1$ on $\mathrm{supp}(\xi) \cup \mathrm{supp}(\eta)$.

Let $\mathscr{D}\mathfrak{g}_{\mathbb{C}}$ be the subalgebra of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ of elements with compact support. We define $\gamma_t : \mathscr{D}\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ by $\gamma_t(\xi) = \omega(\alpha^{-1}(t)\xi, \delta(t))$, where $\delta(t)$ depends on ξ as above. And this is well defined because ω is local by Lemma 3.2.1 and the variation of $\delta(t)$ outside the support of ξ does not change γ_t. Then γ_t is translation-invariant since $\delta(t)$ is defined in a translation-invariant way and ω is translation-invariant by assumption. And γ_t is continuous since ω is continuous by assumption and $\delta(t)$ is defined locally as an element in $\mathscr{D}\mathfrak{g}_{\mathbb{C}}$ and ω is local. Then $\mathscr{D}\mathfrak{g}_{\mathbb{C}}$ is the finite direct sum of test function spaces as a topological linear space, hence any translation-invariant continuous linear functional on this space is of the form

$$\int_{\mathbb{R}} \psi(\xi(s))ds,$$

where ψ is a linear functional on $\mathfrak{g}_{\mathbb{C}}$. Now it is obvious that γ_t extends to $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ by continuity.

As above, let ξ, η be elements with compact support. By the continuity of ω and the Jacobi identity (1.2) we see that

$$\begin{aligned}
\frac{d}{dt}\omega(\alpha^{-1}(t)\xi, \alpha^{-1}(t)\eta) &= \omega([\delta(t), \alpha^{-1}(t)\xi], \alpha^{-1}(t)\eta) + \omega(\alpha^{-1}(t)\xi, [\delta(t), \alpha^{-1}(t)\eta]) \\
&= -\omega([\alpha^{-1}(t)\xi, \alpha^{-1}(t)\eta], \delta(t)) \\
&= -\omega(\alpha^{-1}(t)[\xi, \eta], \delta(t)) \\
&= -\gamma_t([\xi, \eta]).
\end{aligned}$$

Now this equation extends to $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ since $\omega, \gamma, \alpha^{-1}$ are continuous. In short, we have

$$g\omega(\xi, \eta) - \omega(\xi, \eta) = -\int_0^1 \gamma_t([\xi, \eta])dt,$$

which shows that the difference between two cocycles is a linear functional of $[\xi, \eta]$, thus it is a coboundary.

Finally, obviously the averaged cocycle

$$\int_G g\omega dg$$

is a G-invariant cocycle. And the difference

$$\int_G (g\omega - \omega)dg$$

is a coboundary since the integrand is a coboundary. □

Then we can show that the translation-invariant 2-cocycle on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ is essentially unique.

Theorem 3.2.3. *If a translation-invariant continuous 2-cocycle ω is G-invariant, then $\omega(\xi, \eta)$ is proportional to the following one.*

$$\int \frac{1}{2\pi i}\langle \xi(t), \eta'(t)\rangle dt.$$

Proof. We fix Schwartz class functions $f, g \in \mathscr{S}(\mathbb{R})$. We define a bilinear form on $\mathfrak{g}_{\mathbb{C}}$

$$\omega_{f,g}(x, y) := \omega(x \otimes f, y \otimes g), \quad x, y \in \mathfrak{g}_{\mathbb{C}}.$$

Obviously, $\omega_{f,g}$ is G-invariant. Then, since G is simple, it is known that (see, for example, [75, Chapter 2]) any G-invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$ is proportional to the Killing form. The factor depends on f and g obviously in a linear way. Hence we find $\omega_{f,g}(x, y) = \langle x, y\rangle \gamma(f, g)$, where $\gamma(f, g)$ is a bilinear form on $\mathscr{S}(\mathbb{R})$.

Applying the Jacobi identity (1.2) to three elements $x \otimes f, y \otimes g, z \otimes h$, we see the following.

$$\begin{aligned} 0 &= \omega([x \otimes f, y \otimes g], z \otimes h) + \omega([y \otimes g, z \otimes h], x \otimes f) + \omega([z \otimes h, x \otimes f], y \otimes g) \\ &= \omega([x, y] \otimes fg, z \otimes h) + \omega([y, z] \otimes gh, x \otimes f) + \omega([z, x] \otimes hf, y \otimes g) \\ &= \langle [x, y], z \rangle \gamma(fg, h) + \langle [y, z], x \rangle \gamma(gh, f) + \langle [z, x], y \rangle \gamma(hf, g). \end{aligned}$$

By the invariance of the Killing form, we have $-\langle [x, y], z \rangle = \langle y, [x, z] \rangle$ and $\langle [y, z], x \rangle = -\langle z, [y, x] \rangle$. By the symmetry of the Killing form, it holds that

$$0 = \langle [x, y], z \rangle \left(\gamma(fg, h) + \gamma(gh, f) + \gamma(hf, g) \right).$$

Then by choosing appropriate x, y, z we see

$$\gamma(fg, h) + \gamma(gh, f) + \gamma(hf, g) = 0. \tag{3.1}$$

Let f and g be functions with disjoint supports $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, and let h be a function such that $h(t) = 1$ on $\mathrm{supp}(f)$ and $h(t) = 0$ on $\mathrm{supp}(g)$. By (3.1), we have $\gamma(0, h) + \gamma(0, f) + \gamma(f, g) = \gamma(f, g) = 0$. Namely, if the supports of f and g are disjoint, then $\gamma(f, g) = 0$. We call this property the locality of γ.

Now we fix a smooth function f with a compact support $\mathrm{supp}(f) \subset [-\frac{a}{2}, \frac{a}{2}]$. For $k \in \mathbb{Z}$, let e_k be a smooth function with a compact support such that on $[-\frac{a}{2}, \frac{a}{2}]$ it holds that $e_k(t) = e^{\frac{i2\pi tk}{a}}$.

Let g be some function in $\mathscr{S}(\mathbb{R})$. By the locality of γ, the value of $\gamma(f, g)$ does not depend on the form of g outside the support of f. Then again by the Jacobi identity for γ we see $\gamma(fe_k, e_1) + \gamma(e_{k+1}, f) + \gamma(fe_1, e_k) = 0$ or equivalently,

$$\gamma(f, e_{k+1}) = \gamma(fe_k, e_1) + \gamma(fe_1, e_k),$$

because values of functions outside the support of f do not affect the value of γ. Repeating this equality replacing f by fe_1 and k by $k - 1$, we have

$$\gamma(f, e_{k+1}) = \gamma(fe_k, e_1) + \gamma(fe_k, e_1) + \gamma(fe_2, e_{k-1}).$$

It is easy to see that $\gamma(f, e_0) = 0$. By induction it holds for $k \geq 1$ that

$$\gamma(f, e_k) = k\gamma(fe_{k-1}, e_1). \tag{3.2}$$

A similar argument holds also for $k \leq 0$.

We define $\varphi(f) := \gamma(f, e_1)$. By the translation-invariance of γ, we see that $\varphi(f) = c_a \int e^{\frac{i2\pi t}{a}} f(t)dt$ for some constant $c_a \in \mathbb{C}$. Then by the equality (3.2) we have

$$\gamma(f, e_k) = c_a \int k e^{\frac{i2\pi tk}{a}} f(t)dt.$$

Then for a smooth function g with support in $[-\frac{a}{2}, \frac{a}{2}]$, by considering its Fourier expansion $g(t) = \sum_k e^{\frac{i2\pi tk}{a}} g_k$, it holds that

$$\gamma(f,g) = \frac{c_a a}{2\pi i}\int g'(t)f(t)dt.$$

But the interval $[-\frac{a}{2}, \frac{a}{2}]$ is in reality arbitrary, then $c_a a$ does not depend on a and this equality holds for any compact support functions. Then by the continuity of γ it holds also for Schwartz class functions. □

We take the following as the standard normalization.

$$\omega_1(\xi,\eta) := \frac{1}{2\pi i}\int \langle \xi(t), \eta'(t)\rangle dt.$$

We say that a positive-energy representation has level c if its cocycle is $c\omega_1$.

3.3 Uniqueness of ground state representations

First of all, let us specify the class of representation in which we are interested.

Definition 3.3.1. *If a smooth positive-energy unitary projective representation π of $\mathscr{S}\mathfrak{g}_\mathbb{C}$ on the common domain $V \subset \mathcal{H}$ has a unique vector Ω (up to scalar) such that Ω is invariant under the unitary implementation U of the translation and V is algebraically generated by Ω, then π is said to be a ground state representation.*

Throughout this section, π is a ground state representation of $\mathscr{S}\mathfrak{g}_\mathbb{C}$ on $\mathcal{H}$, with a common domain V, Ω is the ground state vector, and U is the one-parameter group of unitary operators which implements the translation.

Note that any vacuum representation of $L\mathfrak{g}_\mathbb{C}$ is a ground state representation. Any vacuum representation of $L\mathfrak{g}_\mathbb{C}$ with a different value of the cocycle corresponds to a different conformal field theory. We will show the uniqueness of ground state for a CFT. In other words, any ground state representation of $\mathscr{S}\mathfrak{g}_\mathbb{C}$ with a fixed cocycle is the vacuum representation.

Note also that we assume from the beginning that the ground state vector Ω is cyclic and unique. Since we need to treat unbounded operators, it is not convenient to discuss decomposition of representations. We will return to this point in the final section.

Let us start with several observations similar to the classical argument in [40], which is originally given by Lüscher and Mack in their unpublished article. Let E be the spectral measure associated with U. If g is a smooth bounded function on $\mathbb{R}$, we denote by $g(U)$ the functional calculus associated with E, defined by

$$U(a) = \int e^{i2\pi pa} dE(p) \text{ for } a \in \mathbb{R},$$

$$g(U) = \int g(p)dE(p).$$

Lemma 3.3.2. *If the Fourier transform $\hat{f}$ of $f \in \mathscr{S}$ has support in $\mathbb{R}_+$, then it holds for any $x \in \mathfrak{g}_\mathbb{C}$ that $\pi(x \otimes f)\Omega = 0$.*

Proof. Recall that the Fourier transform is a homeomorphism of the space of Schwartz class functions $\mathscr{S}$. This holds also true for $\mathscr{S}\mathfrak{g}_\mathbb{C}$, since it is just the space of Schwartz class functions with several components. So we can define a Fourier transform $\hat{\xi}$ of an element $\xi \in \mathscr{S}\mathfrak{g}_\mathbb{C}$ as an element in $\mathscr{S}\mathfrak{g}_\mathbb{C}$ with the Fourier transformed functions in each component. To keep the notation simple, let us define $\hat{\pi}$ the Fourier transform of π, namely $\hat{\pi}(\hat{\xi}) := \pi(\xi)$.

The action of translation on $\mathscr{S}\mathfrak{g}_\mathbb{C}$ is as follows: $\xi_a(t) = \xi(t-a)$. In Fourier transform, it becomes

$$\widehat{\xi_a}(p) = \int e^{-i2\pi pt}\xi(t-a)dt = e^{-i2\pi pa}\hat{\xi}(p).$$

We introduce an obvious notation $g\xi(t) := g(t)\xi(t)$ where g is a smooth function on $\mathbb{R}$ and $\xi \in \mathscr{S}\mathfrak{g}_\mathbb{C}$. Then letting $e_a(p) := e^{i2\pi pa}$, we can write the relation above as $\widehat{g_a} = e_{-a}\hat{g}$. Let U be the unitary operators implementing translation. By the invariance of Ω, we can write this as follows.

$$U(a)\pi(\xi)\Omega = U(a)\pi(\xi)U(a)^*\Omega = \pi(\xi_a)\Omega = \hat{\pi}(e_{-a}\hat{\xi})\Omega.$$

Now let x and f be as in the statement and let g be a function in $\mathscr{S}$ such that its Fourier transform has $\hat{g}(p) = 1$ on $\operatorname{supp}(\hat{f})$ and has support in $[-\frac{S}{2}, \frac{S}{2}]$, where S is some positive number. The restriction of $\hat{g}$ to $[-\frac{S}{2}, \frac{S}{2}]$ can be expanded into a Fourier series

$$\hat{g}(p) = \sum_{k\in\mathbb{Z}} e^{\frac{i2\pi kp}{S}} g_{S,k}.$$

Recall that the convergence of the Fourier series is smooth (uniform on $[-\frac{S}{2}, \frac{S}{2}]$ for each derivative). If p is in the interval $[-\frac{S}{2}, \frac{S}{2}]$, then it holds that

$$\hat{f}(p) = \hat{f}(p)\hat{g}(p) = \hat{f}(p)\left(\sum_{k\in\mathbb{Z}} e_{\frac{k}{S}}(p)g_{S,k}\right) = \sum_{k\in\mathbb{Z}} \hat{f}(p)e_{\frac{k}{S}}(p)g_{S,k},$$

and the convergence in the last series is still smooth on $[-\frac{S}{2}, \frac{S}{2}]$, since $\hat{f}$ is a smooth function with a compact support in this interval, so the Leibniz rule shows the convergence. Then, looking at only the left and right hand sides we see that the equality above holds on the whole real line, simply because $\hat{f}(p) = 0$ outside the interval $[-\frac{S}{2}, \frac{S}{2}]$. The convergence is still smooth.

Since π is an operator valued distribution, so is $\hat{\pi}$, which is weakly continuous with respect to the smooth topology on $\mathscr{S}\mathfrak{g}_\mathbb{C}$. Then we find

$$\begin{aligned}
\pi(x \otimes f)\Omega &= \hat{\pi}(x \otimes \hat{f})\Omega \\
&= \sum_{k\in\mathbb{Z}} \hat{\pi}(x \otimes \hat{f}e_{\frac{k}{S}}g_{S,k})\Omega.
\end{aligned}$$

On the other hand, as a function on the whole real line $\mathbb{R}$, the series

$$\sum_{k\in\mathbb{Z}} e^{\frac{i2\pi kp}{S}} g_{S,k}$$

is uniformly convergent, since it is uniformly convergent on an interval $[-\frac{S}{2},\frac{S}{2}]$ because it is the Fourier expansion of $\hat{g}$, and uniformly convergent also on any translation of the interval $[-\frac{S}{2},\frac{S}{2}]$ since the series is obviously a function with a period S. It holds that

$$\sum_{k\in\mathbb{Z}} e^{\frac{-i2\pi kp}{S}} g_{S,k} = \sum_{k\in\mathbb{Z}} e^{\frac{i2\pi kp}{S}} g_{S,-k},$$

and it is also uniformly convergent. Let g_S be the function which has the series above as the Fourier transform. Then the series of operators $\sum_{k\in\mathbb{Z}} U\left(\frac{k}{S}\right) g_{S,-k} = g_S(U)$ is strongly convergent. Applying this equality to the vector $\pi(x\otimes f)\Omega$ we have

$$\begin{aligned} g_S(U)\pi(x\otimes f)\Omega &= \sum_{k\in\mathbb{Z}} U\left(\frac{k}{S}\right) g_{S,-k}\pi(x\otimes f)\Omega \\ &= \sum_{k\in\mathbb{Z}} g_{S,-k} U\left(\frac{k}{S}\right)\pi(x\otimes f)U\left(-\frac{k}{S}\right)\Omega \\ &= \sum_{k\in\mathbb{Z}} g_{S,-k}\pi\left(x\otimes f_{\frac{k}{S}}\right)\Omega, \end{aligned}$$

since Ω is invariant under translation and U implements it. Then by replacing k by $-k$ we can write it as follows.

$$\begin{aligned} g_S(U)\pi(x\otimes f)\Omega &= \sum_{k\in\mathbb{Z}} g_{S,k}\pi\left(x\otimes f_{-\frac{k}{S}}\right)\Omega \\ &= \sum_{k\in\mathbb{Z}} g_{S,k}\hat{\pi}\left(x\otimes e_{\frac{k}{S}}\hat{f}\right)\Omega \\ &= \hat{\pi}(x\otimes \hat{g}\hat{f})\Omega \\ &= \hat{\pi}(x\otimes \hat{f})\Omega \\ &= \pi(x\otimes f)\Omega. \end{aligned}$$

If we let S tend to ∞, $g_S(U)$ tends to an operator $\tilde{g}(U)$, where $\tilde{g}$ has the Fourier transform $\hat{g}(-p)$. Now recall that the condition on g is that its Fourier transform $\hat{g}$ has compact support and is equal to 1 on the support of $\hat{f}$. Then $\hat{\tilde{g}}$ is equal to 1 on $-\mathrm{supp}(\hat{f})$ and for such $\tilde{g}$ it holds $\tilde{g}(U)\pi(x\otimes f)\Omega = \pi(x\otimes f)$. Then the support of spectral measure of the vector $\pi(x\otimes f)\Omega$ with respect to U must be contained in $-\mathrm{supp}(\hat{f})$.

In particular, if $\mathrm{supp}(\hat{f})$ is compactly supported in $\mathbb{R}_+$, then the spectral measure of $\pi(x\otimes f)\Omega$ is compactly supported in $\mathbb{R}_-$, hence it is equal to 0 because of the positivity of the energy. Any function with support in $\mathbb{R}_+$ is smoothly approximated by a function compactly supported in $\mathbb{R}_+$, so the continuity of π as an operator valued distribution completes the lemma. $\square$

Let us define $\psi(\xi) := \langle \pi(\xi)\Omega, \Omega\rangle$. By definition Ω is unique for ground state representations, hence ψ is an invariant for this class of representations.

Lemma 3.3.3. *$\psi(\xi)$ depends only on $\hat{\xi}(0) \in \mathfrak{g}_{\mathbb{C}}$.*

Proof. We fix $x \in \mathfrak{g}_{\mathbb{C}}$ and consider the restriction

$$\begin{aligned} \psi_x : \mathscr{S} &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(x \otimes f). \end{aligned}$$

It is obvious that ψ_x is invariant under translation. Hence it has the form $\psi_x(f) = C_x \hat{f}(0)$, where C_x is a constant depending on x. The linear functional ψ can be reconstructed by such restrictions, hence ψ itself depends only on $\hat{\xi}(0)$. □

Lemma 3.3.4. *Let $\{\xi_n\}$ be a sequence of elements in $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ such that*

- *each $\hat{\xi}_n$ has a compact support.*
- *for $p \geq 0$, $\hat{\xi}_n(p) = \hat{\xi}_m(p)$ for any $n, m \in \mathbb{N}$.*
- *for $p < 0$, the norm of $\hat{\xi}_n(p) \in \mathfrak{g}_{\mathbb{C}}$ with respect to the Killing form is uniformly bounded and the Lebesgue measure of $\mathrm{supp}(\hat{\xi}_n) \cap \mathbb{R}_-$ tends to 0.*

Then $\pi(\xi_n)\Omega$ is convergent to $\langle \pi(\xi_n)\Omega, \Omega\rangle\Omega$.

Proof. By the proof of Lemma 3.3.2, $\pi(\xi_n)\Omega$ is contained in $\chi_{-\mathrm{supp}(\hat{\xi}_n)}(U)\mathcal{H}$. The intersection of these spaces is clearly the one-dimensional space $\mathbb{C}\Omega$. To see the convergence, we have to estimate the following.

$$\begin{aligned} \|\pi(\xi_m - \xi_n)\Omega\|^2 &= \langle \pi(\xi_m - \xi_n)^* \pi(\xi_m - \xi_n)\Omega, \Omega\rangle \\ &= \langle \pi(\xi_m - \xi_n)\pi(\xi_m - \xi_n)^*\Omega, \Omega\rangle + \langle [\pi(\xi_m - \xi_n)^*, \pi(\xi_m - \xi_n)]\Omega, \Omega\rangle. \end{aligned}$$

The first term vanishes by Lemma 3.3.2.

We can transform the second term using the commutation relation and obtain

$$\langle \pi([(\xi_m - \xi_n)^*, \xi_m - \xi_n])\Omega, \Omega\rangle - \omega((\xi_m - \xi_n)^*, \xi_m - \xi_n).$$

Let us estimate the first term of this difference. By Lemma 3.3.3, it is enough to estimate the value at 0 of the Fourier transform of $[(\xi_m - \xi_n)^*, \xi_m - \xi_n]$. By the assumption, the Fourier transform of $\xi_m - \xi_n$ is also bounded and the measure of its support tends to 0 as m, n tend to ∞. In general we have

$$\widehat{[\eta^*, \eta]}(0) = \int [\widehat{\eta^*}(p), \widehat{\eta}(-p)]dp.$$

If we apply this to $\eta = \xi_m - \xi_n$, the integral is bounded by (the square of the double of) the uniform bound of $\{\hat{\xi}_m\}$, the norm of the commutator of $\mathfrak{g}_{\mathbb{C}}$ and the measure of the support of $\hat{\xi}_m - \hat{\xi}_n$. Then this tends to 0. By continuity of π, this term tends to 0. For the second term, we know the concrete form of the cocycle ω and in the Fourier transform it takes

$$\omega(\eta^*, \eta) = \frac{c}{2\pi i} \int i2\pi p \langle \widehat{\eta^*}(-p), \widehat{\eta}(p) \rangle dt, = c \int p \langle \widehat{\eta^*}(-p), \widehat{\eta}(p) \rangle dt$$

then by a similar reasoning the corresponding term converges to 0.

Now that we know that the concerned sequence converges to a scalar multiple of Ω, it is enough to determine the coefficient $\langle \pi(\xi_n)\Omega, \Omega \rangle$. By Lemma 3.3.3 this is determined by $\hat{\xi}(0)$ and by the assumption this is constant. $\square$

Lemma 3.3.5. *Let $\{\xi_n\}$ be a sequence of elements in $\mathscr{S}\mathfrak{g}_\mathbb{C}$. Assume that components of $\hat{\xi}_n$ are uniformly bounded and convergent to a bounded function and the Lebesgue measure of the support of $\hat{\xi}_{n'} - \hat{\xi}_n$ is monotonically decreasing to 0. Then for any $\eta \in \mathscr{S}\mathfrak{g}_\mathbb{C}$, the commutator $[\xi_n, \eta]$ is smoothly convergent to an element in $\mathscr{S}\mathfrak{g}_\mathbb{C}$.*

Proof. It is enough to consider the case where $\xi_n = x \otimes f_n, \eta = y \otimes g$, since the general case is a finite linear combination of such elements and the convergence of the commutator follows immediately. In this case, the commutator is expressed with the Fourier transform as follows.

$$\widehat{[\xi_n, \eta]}(p) = [x, y] \otimes \int \hat{f}_n(s)\hat{g}(p - s)ds.$$

The convergence in the smooth topology is defined as the uniform convergence of the following functions.

$$p^l \widehat{[\xi_n, \eta]}^{(m)}(p) = [x, y] \otimes \int \hat{f}_n(s)p^l \hat{g}^{(m)}(p - s)ds.$$

Since the function in the integrand is uniformly bounded, and the measure of the support $\hat{\xi}_{n'} - \hat{\xi}_n$ is decreasing to 0, the integral is convergent uniformly. $\square$

Lemma 3.3.6. *The representation π is characterized by ψ and the level c. Namely, any two representations which correspond to the same functional ψ and the same level c are unitarily equivalent.*

Proof. We will show that the n-point function $\langle \pi(\xi_1)\pi(\xi_2) \cdots \pi(\xi_n)\Omega, \Omega \rangle$ is determined by ψ and c for any n. Since Ω is cyclic for π, this implies that any inner product of the form $\langle \pi(\xi_1) \cdots \pi(\xi_n)\Omega, \pi(\eta_1) \cdots \pi(\eta_m)\Omega \rangle$ is determined by ψ and c. If two representations π_1, π_2 have the same ψ and c, then the map

$$\pi_1(\xi_1) \cdots \pi_1(\xi_n)\Omega \longmapsto \pi_2(\xi_1) \cdots \pi_2(\xi_n)\Omega$$

is a unitary map intertwining the two representations since by the definition of ground state representation these vectors span the dense common domain V. Furthermore, by the continuity of π, we may assume that $\{\hat{\xi}_k\}$ have compact supports.

We show that for $n \geq 2$, the n-point function is reduced to $(n-1)$-point functions. Then an induction about n completes the proof. Let us decompose ξ_1 into two parts $\xi_1 = \xi_+ + \xi_-$

such that $\hat{\xi}_+$ has support in $\mathbb{R}_+$. By the Lemma 3.3.2 we know that $\pi(\xi_+)\Omega = 0$. In the n-point function, we can take $\pi(\xi_+)$ to the right using the commutation relation and annihilate it letting it act on Ω, so that the n-point function will be reduced to the sum of $(n-1)$-point functions and ξ_- part. Explicitly,

$$\begin{aligned}
\langle \pi(\xi_1)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle &= \langle \pi(\xi_+ + \xi_-)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&= \langle \pi(\xi_-)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&\quad + \langle [\pi(\xi_+),\pi(\xi_2)]\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&\quad + \langle \pi(\xi_2)\pi(\xi_+)\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&= \langle \pi(\xi_-)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&\quad + \sum_k \langle \pi(\xi_2)\cdots[\pi(\xi_+),\pi(\xi_k)]\cdots\pi(\xi_n)\Omega,\Omega\rangle.
\end{aligned}$$

This is equal to the following since π is a projective representation.

$$\begin{aligned}
\langle \pi(\xi_1)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle &= \langle \pi(\xi_-)\pi(\xi_2)\cdots\pi(\xi_n)\Omega,\Omega\rangle \\
&\quad + \sum_k \langle \pi(\xi_2)\cdots(\pi([\xi_+,\xi_k]) - \omega(\xi_+,\xi_k))\cdots\pi(\xi_n)\Omega,\Omega\rangle.
\end{aligned}$$

Now, let f_ϵ be a smooth function such that $f_\epsilon(p) = 1$ for $p \geq \epsilon$, $f_\epsilon(p) = 0$ for $p \leq 0$ and $0 < |f_\epsilon| < 1$ for $0 < p < \epsilon$. Let us make a decomposition of ξ_1 such that $\hat{\xi}_{\epsilon+}(p) = f_\epsilon(p)\hat{\xi}(p)$ and $\xi_{\epsilon-} = \xi_1 - \xi_{\epsilon+}$.

On the one hand, $\xi_{\epsilon+}$ satisfies the assumption of Lemma 3.3.5, hence by letting ϵ tend to 0, all the brackets above are convergent to images of some elements in $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$, hence there appear images by π and scalar multiples of c which depends only on the Lie algebra structure. On the other hand, $\xi_{\epsilon-}^*$ satisfies the assumption of Lemma 3.3.4 and $\pi(\xi_{\epsilon-}^*)\Omega = \pi(\xi_{\epsilon-})^*\Omega$ is convergent to $\psi(\xi_{\epsilon-}^*)\Omega$ (which does not depend on ϵ). This reduces every term in the n-point function to $(n-1)$-point functions, ψ and c. □

We have seen that ψ and c characterize the representation π. Finally we show that ψ is not necessary and π is determined only by c.

Theorem 3.3.7. *For any ground state representation, $\psi = 0$, thus $\langle \pi(\xi)\Omega,\Omega\rangle = 0$ for any $\xi \in \mathscr{S}\mathfrak{g}_{\mathbb{C}}$.*

Proof. We will show this by contradiction. To be precise, we assume that $\psi \neq 0$ and we show that representation is not unitary.

By definition it is easy to see that ψ is self-adjoint. Let $\xi \in \mathscr{S}\mathfrak{g}_{\mathbb{C}}$. As we have seen in Lemma 3.3.3, $\psi(\xi)$ is determined only by $\hat{\xi}(0)$. Let us define $\psi_0 : \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ such that $\psi(\xi) = \psi_0(\hat{\xi}(0))$.

By the assumption, there is an element x from $\mathfrak{g}_{\mathbb{C}}$ such that $\psi_0(x) \neq 0$. Since ψ is self-adjoint, so is ψ_0 and we may assume that x is self-adjoint and $\psi_0(x) \in \mathbb{R}$. Then there is a Cartan subalgebra which contains x. Let us consider the root decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to this Cartan subalgebra. Let α be an element in the root system Ψ, and let $\mathfrak{sl}(\alpha)$ be the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ associated to α. We define put $E_\alpha, F_\alpha, H_\alpha$ the elements in $\mathfrak{sl}(\alpha)$ corresponding to

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We may assume there is a root α such that $\psi_0(H_\alpha) \neq 0$, since the Cartan subalgebra is spanned by $\{H_\alpha\}_{\alpha\in\Psi}$.

As the first case, we assume $\psi_0(E_\alpha) > 0$. It holds that $[E_\alpha, F_\alpha] = H_\alpha$ and $E_\alpha^* = F_\alpha$. Let us take a smooth real function $f \in \mathscr{S}(\mathbb{R})$ with $\mathrm{supp}(f) \subset \mathbb{R}_-$. We will find a vector in $\mathcal{H}$ with negative norm. In fact, it holds that

$$\begin{aligned} \|\pi(E_\alpha \otimes f)\Omega\|^2 &= \langle \pi(E_\alpha \otimes f)\Omega, \pi(E_\alpha \otimes f)\Omega\rangle \\ &= \langle [\pi(F_\alpha \otimes \overline{f}), \pi(E_\alpha \otimes f)]\Omega, \Omega\rangle \\ &= \langle \pi([F_\alpha, E_\alpha] \otimes |f|^2)\Omega\rangle - \langle F_\alpha, E_\alpha\rangle \frac{c}{2\pi i}\int \overline{f(t)}f'(t)dt \\ &= \psi_0(-H_\alpha)\int |\hat{f}(p)|^2 dp - c\langle F_\alpha, E_\alpha\rangle \int p|\hat{f}(p)|^2 dp. \end{aligned}$$

Then if we take a function f such that $\hat{f}$ has support sufficiently near to 0 but nonzero, then the norm must be negative.

If $\psi_0(H_\alpha) < 0$, we only have to consider the norm of $\pi(F_\alpha \otimes f)\Omega$. □

Corollary 3.3.8. *All the ground states on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ are completely classified by c and such a representation is possible if and only if $c \in \mathbb{N}_+$.*

Proof. We have seen in Lemma 3.3.6 that ground states on $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ are completely classified by ψ and c, on the other hand Lemma 3.3.7 tells us that only the case $\psi = 0$ is possible.

For $L\mathfrak{g}_{\mathbb{C}}$ we know that lowest weight representations with invariant vector with respect to the Möbius group Möb are completely classified by c and the only possible values of c are positive integers. What remains to prove is that every ground state representation of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ extends to $L\mathfrak{g}_{\mathbb{C}}$. This is done by the repetition of the argument by [18, Section 4]. In fact, we know that a ground state representation π is determined by the value of c, and the cocycle ω is invariant under dilation. Also the positive and negative parts decomposition in the proof of Lemma 3.3.6 is not affected by dilation. Then it is straightforward to check that there is a unitary representation of dilation under which Ω is invariant and π is covariant. By analogy with the Lüscher-Mack theorem, all n-point functions extend to the circle S^1 and turn out to be invariant under Möb. Then by the reconstruction theorem, we obtain the representation of $L\mathfrak{g}_{\mathbb{C}}$.

It is known that in this case the level c must be a positive integer by Theorem 1.2.2 (due to [42]) or [75]. □

3.4 Ground states of conformal nets

3.4.1 Ground state representations of Lie algebras and conformal nets

The main result in the previous section is the uniqueness of ground state on the Schwartz class subalgebra $\mathscr{S}\mathfrak{g}_\mathbb{C}$ of $L\mathfrak{g}_\mathbb{C}$. Here we explain its (possible) physical implication. In our context, examples of conformal nets are given in terms of vacuum representations of loop groups (see Section 1.5.2, or for example [41]). Explicitly, let LG be the loop group of a certain simple simply connected Lie group G. We take a positive-energy vacuum representation π of LG at certain level k. Then we set

$$\mathcal{A}_{G,k}(I) := \{\pi(g) : \mathrm{supp}(g) \subset I\}''.$$

Isotony is obvious from the definition. Locality comes from the locality of cocycle. Each such vacuum representation is covariant under the diffeomorphism group $\mathrm{Diff}(S^1)$, in particular under Möb. Positivity of energy is readily seen. The lowest eigenvector of rotation behaves as the vacuum vector.

A conformal net is considered as a mathematical realization of a physical model. Several physical states are realized as states on the quasilocal C^*-algebra

$$\overline{\bigcup_{I\Subset\mathbb{R}} \mathcal{A}(I)}^{\|\cdot\|}.$$

We denote it simply by $\mathfrak{A}$. On this C^*-algebra, the group of translations acts as one-parameter automorphism τ.

Among all states on $\mathfrak{A}$, states which represent thermal equilibrium are of particular interest. The property of thermal equilibrium is characterized by the following KMS condition [9].

Definition 3.4.1. *A state φ on a C^*-algebra $\mathfrak{A}$ is called a β-KMS state (with respect to a one-parameter automorphism group τ) if for each pair $x, y \in \mathfrak{A}$ there is an analytic function $f(z)$ on $0 < \mathrm{Im}z < \beta$ and continuous on $0 \leq \mathrm{Im}z \leq \beta$ such that it holds for $t \in \mathbb{R}$*

$$f(t) = \varphi(x\tau_t(y)), \quad f(t + i\beta) = \varphi(\tau_t(y)x).$$

Here, $\frac{1}{\beta}$ is interpreted as the temperature of the state of equilibrium.

As easily seen, when the temperature goes to 0, β goes to the infinity and the domain of analyticity approaches to the half-plane. We simply take the following definition, and consider it as an equilibrium state with temperature zero.

Definition 3.4.2. *A state φ on a C^*-algebra $\mathfrak{A}$ is called a ground state with respect to τ if for each pair $x, y \in \mathfrak{A}$ there is an analytic function $f(z)$ on $0 < \mathrm{Im}z$ and continuous on $0 \leq \mathrm{Im}z$ such that it holds for $t \in \mathbb{R}$*

$$f(t) = \varphi(x\tau_t(y)).$$

In general, if φ is invariant under translation, the action of translation is implemented canonically by a one-parameter group of unitary operators in its GNS representation. It is known that this condition is characterized by its property in the GNS representation [9].

Theorem 3.4.3. *A translation-invariant state φ on $\mathfrak{A}$ is a ground state if and only if the generator of translation in the GNS representation has positive spectrum.*

No direct and obvious way to classify ground states on general conformal nets is at hand, but the case of loop group nets seems rather hopeful. Let π be a vacuum representation of loop group LG, φ be a ground state on $\mathcal{A}_{G,k}$ and π_φ be the GNS representation with respect to φ. Let us call temporarily $\mathscr{D}G$ the subgroup of LG with elements compactly supported in $\mathbb{R}$, with the identification of $\mathbb{R}$ as a part of S^1. Since $\mathcal{A}_{G,k}$ is generated by local operators, for any group element $\xi \in LG$ with support in $\mathbb{R}$ we have $\pi(\xi) \in \mathcal{A}_{G,k}$, hence the composition $\pi_\varphi \circ \pi$ is a representation of $\mathscr{D}G$, covariant under translation implemented by one-parameter unitary group with positive generator, containing a translation-invariant vector. Then to classify all ground state representations of $\mathcal{A}_{G,k}$, it is enough to classify ground state representations of $\mathscr{D}G$.

Hence the result of this Chapter can be considered as a first step towards the classification of ground states of loop groups. The remaining steps should be roughly the following.

- To show that every ground state representation of $\mathscr{D}G$ is differentiable and induces a representation of $\mathscr{D}\mathfrak{g}$.
- To show that every ground state representation of $\mathscr{D}\mathfrak{g}$ can be extended to $\mathscr{S}\mathfrak{g}$.

Combining it with the uniqueness result of this Chapter we would see the uniqueness of ground state on $\mathcal{A}_{G,k}$.

Unfortunately, I am not aware of any concrete strategy to these points. Recently a general theory about differentiability of representations of infinite dimensional groups was established by Neeb [71]. Detailed analysis for ground state representations could lead to general differentiability. For the second point, invariance of the ground state vector could imply the extension of operator valued distribution, in analogy of the case of distribution.

3.4.2 Irreducibility and factoriality of representations

In Section 3.3 we have classified representations of $\mathscr{S}\mathfrak{g}_\mathbb{C}$ with a cyclic ground state vector. We need to justify that the assumption of cyclicity is not essential. In fact we would like to show that any ground state representation should be decomposed into representations with cyclic ground state vector. This is a bit problematic at the level of Lie algebras, because operators are unbounded, hence not defined on the whole space. We would have to take care of commutation of unbounded operators, density of domain, existence of eigenvalues, etc. Instead, we content ourselves with considering the decomposition problem at the level of conformal net.

Here we just restate some well-known results, mainly taken from standard textbooks. The first two results come from [3, 1.2.3 Corollary and 1.2.7 Corollary, respectively].

Theorem 3.4.4. *A ground state representation with a unique ground state vector is irreducible.*

Theorem 3.4.5. *Let π be a ground state representation. If $\bigvee_{I \Subset \mathbb{R}} \pi(\mathcal{A}(I))$ is a factor then necessarily it is $B(\mathcal{H})$. In this case the ground state vector is unique.*

We remark that in the book [3], the statements are given for "vacuum representations" of two or higher dimensional Poincaré covariant nets of von Neumann algebras. In reality in the proofs of these results, covariance with respect to Lorentz transformations is not used, and adaptation to the one-dimensional case is straightforward.

For the following we refer the book [9, Section 5.3.3].

Theorem 3.4.6. *The following are equivalent.*

1. *The set of ground states is simplex.*
2. *If the von Neumann algebra generated by the GNS representation of a ground state is a factor, then it is $B(\mathcal{H})$.*

Then, a general ground state can be decomposed uniquely into extremal states (see [8, Theorem 4.1.15]). Any extremal states has a factorial representation (in fact, if the GNS representation is not factorial then a nontrivial central projection commutes with the representatives of translation [3, Theorem 1.1.1], thus the GNS vector decomposes into two ground state vectors), hence by the previous theorem it has a unique cyclic ground state vector.

Thanks to these general results, we can reduce a general ground state into a convex combination of pure ground states. A pure ground state has a unique ground state vector in its GNS representation. To classify ground states it is enough to find all pure ground states. Then it is natural to restrict also the study of Lie algebra representations to the case with a unique cyclic ground state vector.

3.5 Open problems

Ground states on conformal nets

It is desired to prove the uniqueness of (or to classify) the ground state representation of of the Lie group of $\mathscr{S}\mathfrak{g}_\mathbb{C}$. This is motivated from the algebraic approach to QFT, in particular the study of thermal states, as explained in the previous Section. Such a proof, however, would be plagued by the problem of domains of unbounded operators. Instead, a more direct approach based on operator-algebraic approach would be hopeful. Indeed, we will prove the uniqueness of thermal states with finite temperature of all completely rational models in Chapter 4.

More on positive-energy representations

In this Chapter we studied positive-energy representations with a ground state vector. A positive energy representation without a ground state representation is considered to represent a more general physical state. From the mathematical side, all the positive-energy representations of LG and $L\mathfrak{g}$ have been classified, hence one would expect a similar results also for $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$. It is even an open problem where there is a positive-energy representation of $\mathscr{S}\mathfrak{g}_{\mathbb{C}}$ which does not extend to $L\mathfrak{g}$.

Chapter 4

KMS states on conformal nets

Chapter Introduction

Although Quantum Field Theory is primarily designed to study finitely many particle states, the thermal aspects in QFT are of crucial importance for various reasons and one naturally aims at a general analysis of the thermal behavior starting from the basic properties shared by any QFT. As is known, at infinite volume the thermal equilibrium states are characterized by the Kubo-Martin-Schwinger condition (see [46]), in other words KMS states are Gibbs states for infinite volume systems. In this Chapter we carry out a general study of thermal states in CFT (conformal QFT), more precisely of the locally normal KMS states with respect to the translation one-parameter group.

Our first observation is that there always exists a canonical KMS state, that is constructed by a geometric procedure (Section 1.3.2). Indeed the restriction of the vacuum state to the von Neumann algebra associated with the positive real line is KMS with respect to the (rescaled) dilation group (Bisognano-Wichmann property [10, 41]); now the exponential map intertwines translations with dilations and one can use it to pull back the vacuum state and define the geometric KMS state w.r.t. translations.

One may ask whether this geometric KMS state is the only one or there are other locally normal KMS states (different phases, in physical terms). Indeed in general there are many KMS states.

In this Chapter we first concentrate on the case where the net $\mathcal{A}$ is a completely rational. We show that there exists exactly one locally normal KMS state φ with respect to the translation group $\mathrm{Ad}U(\tau)$, the geometric state. As we shall see, the proof of this result is obtained in several steps by a crucial use of the thermal completion net and an inductive extension procedure. This is in accordance with the previous result which showed the uniqueness of ground state (which is considered as a state with zero temperature) on loop algebras in Chapter 3.

Our results extends to the case of a local conformal net $\mathcal{A}$ of von Neumann algebras on the two-dimensional Minkowski spacetime. We shall show that, if $\mathcal{A}$ is completely rational, there exists a unique KMS state w.r.t. the time-translation one-parameter group. Also in this case the KMS state has a geometric origin.

Then we shall study the set of KMS states for local conformal nets that are not rational. In contrast to completely rational case, we shall see that there are non-rational nets with continuously many KMS states.

We shall focus our attention to two important models. The first one is the free field, i.e. the net generated by the $U(1)$-current. In this model we manage to classify KMS states. We shall show that the primary (locally normal) KMS states of the $U(1)$-current net are in one-to-one correspondence with real numbers $q \in \mathbb{R}$. As we shall see, each state φ^q is uniquely and explicitly determined by its value on the Weyl operators. The geometric KMS state is $\varphi_{\text{geo}} = \varphi^0$ and any other primary KMS state is obtained by composition of the geometric one with the automorphisms γ_q of the net (see Section 4.7.2): $\varphi^q = \varphi_{\text{geo}} \circ \gamma_q$.

The second model we study is the Virasoro net Vir_c, the net generated by the stress-energy tensor with a given central charge c. This net is fundamental and is contained in any local conformal net [52]. If c is in the discrete series, thus $c < 1$, the net Vir_c is completely rational, so there exists a unique KMS state by the previous Sections of this Chapter. In the case $c = 1$ we are able to classify all KMS states. The primary (locally normal) KMS states of the Vir_1 net w.r.t. translations are in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ is uniquely determined by its value on the stress-energy tensor

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f \, dx.$$

The geometric KMS state corresponds to $q = 0$, because it is the restriction of the geometric KMS state on the $U(1)$-current net, and the corresponding value of the 'energy density' $\frac{\pi}{12\beta^2} + \frac{q^2}{2}$ is the lowest in the set of the KMS states. However we construct these KMS states by composing the geometric state with automorphisms on the larger $U(1)$-current net.

We mention that, as a tool here, we adapt the Araki-Haag-Kastler-Takesaki theorem [1] to locally normal system with the help of split property. We show that, if we have an inclusion of split nets with a conditional expectation, then any extremal invariant state on the smaller net extends to the larger net.

Finally we consider the case $c > 1$. In this case we produce a continuous family which is probably exhaustive. While we leave the problem of the completeness of this family, we mention that the formulae on polynomials of fields should be useful. There is a set of primary (locally normal) KMS states of the Vir_c net with $c > 1$ w.r.t. translations in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ can be evaluated on the stress-energy tensor

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f \, dx$$

and the geometric KMS state corresponds to $q = \frac{1}{\beta}\sqrt{\frac{\pi(c-1)}{6}}$ and energy density $\frac{\pi c}{12\beta^2}$. It is even possible to evaluate $\varphi^{|q|}$ on polynomials of the stress-energy tensor and these values are already determined by the value above on $T(f)$, hence by the number $|q|$. This should give an important information for the complete classification.

4.1 Preliminaries for the uniqueness results

4.1.1 Pimsner-Popa inequality and normality

We discuss here some properties of finite-index expectation needed in this Chapter, cf. [48] for related facts.

Suppose $\mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras and $E : \mathcal{M} \to \mathcal{N}$ is an expectation. Let

$$E = E_n + E_s$$

be the (unique) decomposition of E into the sum of a normal and a singular $\mathcal{M} \to \mathcal{N}$ positive map (with E_n standing for the normal part and E_s for the singular part). As is known, one of the equivalent definitions of singularity is that for any P nonzero ortho-projection there is a nonzero subprojection $Q \leq P$ such that $E_s(Q) = 0$.

Lemma 4.1.1. *$E_n(AX) = AE_n(X)$ and $E_n(XA) = E_n(X)A$ for all $A \in \mathcal{N}$ and $X \in \mathcal{M}$.*

Proof. Let $T, S \in \mathcal{N}$ with $TS = ST = \mathbb{1}$ and $\Phi(\cdot) := T \cdot T^*$. Then $\Phi^{-1}(\cdot) = S \cdot S^*$ and both Φ and Φ^{-1} are faithful positive normal maps. It follows that $\Phi \circ E_n \circ \Phi^{-1}$ is a normal positive map and it is also clear that $\Phi \circ E_s \circ \Phi^{-1}$ is a positive map. We shall now show that this latter one is actually a singular map.

It is rather evident that if $E_s \circ \Phi^{-1}$ is singular then so is $\Phi \circ E_s \circ \Phi^{-1}$. So let $P \in \mathcal{M}$ be a nonzero ortho-projection. Then $\Phi^{-1}(P) = SPS^*$ is a nonzero positive operator so its spectral projection Q associated to the interval $[a/2, a]$ where $a = \|SPS^*\|$ is nonzero and we have that $SPS^* \geq (a/2)Q$. By singularity of E_s, there exists a nonzero subprojection $Q_0 \leq Q$, $Q_0 \neq 0$ such that $E_s(Q_0) = 0$. Then TQ_0T^* is a nonzero positive operator so again we shall consider its spectral projection R associated to the interval $[b/2, b]$ where $b = \|TQ_0T^*\|$. Again, it is nonzero and we have that $\Phi(Q_0) = TQ_0T^* \geq (b/2)R$. Putting together the inequalities, we have

$$R \leq \frac{2}{b}\Phi(Q_0) \leq \frac{2}{b}\Phi(Q) \leq \frac{2}{b}\frac{2}{a}\Phi(\Phi^{-1}(P)) = \frac{4}{ab}P$$

and it is easy to see that if for two ortho-projections P_1, P_2 the inequality $P_1 \leq tP_2$ holds for *some* $t > 0$, then actually $P_1 \leq P_2$. So we have that R is a nonzero subprojection of P, and since $E_s \circ \Phi^{-1}$ is a positive map, by the listed inequality we also have that

$$E_s \circ \Phi^{-1}(R) \leq \frac{2}{b}E_s \circ \Phi^{-1}(\Phi(Q_0)) = \frac{2}{b}E_s(Q_0) = 0.$$

Thus $E_s \circ \Phi^{-1}$ — and hence $\Phi \circ E_s \circ \Phi^{-1}$, too — are indeed singular. However,

$$\Phi \circ E_n \circ \Phi^{-1} + \Phi \circ E_s \circ \Phi^{-1} = \Phi \circ E \circ \Phi^{-1} = E$$

since $TE(SXS^*)T^* = TSE(X)S^*T^* = E(X)$ for all $X \in \mathcal{M}$ and $S \in \mathcal{N}$. Hence, by the uniqueness of the decomposition, we have that $\Phi \circ E_s \circ \Phi^{-1} = E_s$ and $\Phi \circ E_n \circ \Phi^{-1} = E_n$ or, equivalently, $\Phi \circ E_n = E_n \circ \Phi$. So we have that

$$TE_n(X)T^* = E_n(TXT^*) \tag{4.1}$$

for all $X \in \mathcal{M}$. Now let $A \in \mathcal{N}$ be a strictly positive element (i.e. $0 \notin \mathrm{Sp}(A) \subset \mathbb{R}+$). Then $T := A$ and $\tilde{T} := \mathbb{1} + iA$ are invertible elements in $\mathcal{N}$ with bounded inverse and so equation (4.1) can be applied for both. After a straightforward calculation we obtain that for all $X \in \mathcal{M}$

$$[A, E_n(X)] = E_n([A, X]),$$

where $[Y, Z] = YZ - ZY$ is the commutator. On the other hand, replacing $\tilde{T}$ by $\tilde{T} = \mathbb{1} + A$ and repeating the previous argument we also find that for all $X \in \mathcal{M}$

$$\{A, E_n(X)\} = E_n(\{A, X\}),$$

where $\{Y, Z\} = YZ + ZY$ is the anti-commutator. So actually we have shown that E_n commutes with both taking commutators and taking anti-commutators with an arbitrary strictly positive operator $A \in \mathcal{N}$. Then the claimed bimodule property follows, since the linear span of strictly positive elements is dense in $\mathcal{N}$ and E_n is normal. □

Let now $F : \mathcal{M} \to \mathcal{N}$ be a positive map satisfying a Pimsner-Popa type inequality [73]; i.e. we suppose that there exists a $\lambda > 0$ such that

$$F(X^*X) \geq \lambda X^*X$$

for all $X \in \mathcal{M}$. Now consider the decomposition $F = F_n + F_s$ into the sum of a normal and a singular positive maps. F_n must be faithful. Indeed, an easy argument relying on the normality of F_n shows that, if there is a positive nonzero element which is annihilated by F_n, then there is also a nonzero ortho-projection P which is annihilated by F_n. However, there is a subprojection $Q \leq P$, $Q \neq 0$ such that on this subprojection also F_s is zero. Thus $F(Q) = F_n(Q) + F_s(Q) = 0$ in contradiction with the assumed inequality. Actually we can say much more.

Lemma 4.1.2. *The normal part F_n of F satisfies the Pimsner-Popa inequality with the same constant λ.*

Proof. By assumption we know that $K := F - \lambda \cdot \mathrm{id}$ is a positive map. Our goal is to show that $\tilde{K} := F_n - \lambda \mathrm{id} = K - F_s$ is also a positive map. Since $\tilde{K}$ is evidently normal, it is enough to show that if $P \in \mathcal{M}$ is an ortho-projection then $\tilde{K}(P) \geq 0$. So let $P \in \mathcal{M}$ be an ortho-projection and

$$\mathcal{S} := \{Q \in \mathcal{M} |\, Q^2 = Q = Q^*, Q \leq P, \tilde{K}(Q) \geq 0\}.$$

Now $\mathcal{S}$ can be viewed as a partially ordered set (with the ordering given by the operator ordering) and, if $\{Q_\alpha\}$ is a chain in $\mathcal{S}$, then — by the normality of $\tilde{K}$ — $Q := \vee_\alpha Q_\alpha$ is still an element of $\mathcal{S}$. Hence, by an application of the Zorn lemma, there is a maximal element in $\mathcal{S}$; say $Q \in \mathcal{S}$ is such an element.

If $Q = P$, we have finished. So assume by contradiction that $P - Q$ is nonzero. Then there exists a nonzero subprojection $R \leq P - Q$ such that $F_s(R) = 0$. Hence $\tilde{K}(R) = K(R) - F_s(R) = K(R)$ and

$$\tilde{K}(Q + R) = \tilde{K}(Q) + \tilde{K}(R) = \tilde{K}(Q) + K(R) \geq \tilde{K}(Q) + \lambda R \geq 0,$$

implying that $Q + R \in \mathcal{S}$ in contradiction with the maximality of Q. □

Let us return now to discussing expectations $E : \mathcal{M} \to \mathcal{N}$ (non necessarily normal), with normal-singular decomposition $E = E_n + E_s$.

Theorem 4.1.3. *Suppose E satisfies the Pimsner-Popa inequality with constant $\lambda > 0$. Then $Z := E_n(\mathbb{1})$ is a strictly positive and hence invertible element in the center of $\mathcal{N}$ and $\tilde{E} := Z^{-1}E_n$ is a normal expectation from $\mathcal{M}$ to $\mathcal{N}$ satisfying the Pimsner-Popa inequality with the same constant $\lambda > 0$.*

Proof. By Lemma 4.1.1 we have that

$$AZ = AE_n(\mathbb{1}) = E_n(A) = E_n(\mathbb{1})A = ZA$$

for all $A \in \mathcal{N}$, showing that Z is indeed a central element.

We may estimate Z from above by considering that $\mathbb{1} = E(\mathbb{1}) = E_n(\mathbb{1}) + E_s(\mathbb{1}) = Z + E_s(\mathbb{1})$ and the fact that E_s is a positive map. From below, we may apply our previous lemma. Putting them together, we have

$$\lambda^{-1}\mathbb{1} \leq Z = \mathbb{1} - E_s(\mathbb{1}) \leq \mathbb{1}.$$

One of the inequalities shows that Z^{-1} is bounded, whereas the other shows that $Z^{-1} \geq \mathbb{1}$ and so $Z^{-1}E_n$ still satisfies the Pimsner-Popa inequality with the same λ. The rest of the statement – namely that $Z^{-1}E_n$ is a normal expectation – follows easily from the facts so far established in this Section. □

Now it turns out that the normal part is in fact the expectation itself. The argument here is due to Kenny De Commer.

Corollary 4.1.4. *If a conditional expectation $E : \mathcal{M} \to \mathcal{N}$ satisfies the Pimsner-Popa inequality with the constant $\lambda > 0$, then any conditional expectation $F : \mathcal{M} \to \mathcal{N}$ is normal.*

Proof. As we have seen in Theorem 4.1.3, there is a normal conditional expectation $\tilde{E} : \mathcal{M} \to \mathcal{N}$ which satisfies the Pimsner-Popa inequality with the same constant λ. Let us suppose that there is another conditional expectation F. To show that F is normal, it is enough to see that for a bounded increasing net $\{x_\alpha\}$ of positive elements in $\mathcal{M}$ it holds that $\lim_\alpha F(x_\alpha) = F(\lim_\alpha x_\alpha)$ in σ-weak topology. In fact, by replacing x_α with $x - x_\alpha$, it is equivalent to show that if x_α is decreasing to 0, then $F(\lim x_\alpha) = F(0) = 0$.

By the Pimsner-Popa inequality for $\tilde{E}$, we have $x_\alpha \leq \lambda^{-1}\tilde{E}(x_\alpha)$. We apply F to the both sides to obtain

$$F(x_\alpha) \leq F(\lambda^{-1}\tilde{E}(x_\alpha)) = \lambda^{-1}F(\tilde{E}(x_\alpha)) = \lambda^{-1}\tilde{E}(x_\alpha),$$

since the image of $\tilde{E}$ is contained in $\mathcal{N}$ and F is an expectation $\mathcal{M} \to \mathcal{N}$. The normality of $\tilde{E}$ implies that the right-hand side tends to 0, so does the left-hand side. This proves the normality of F. □

4.1.2 Irreducible inclusion of factors

Here we collect some observations on irreducible subfactors with a conditional expectation. Throughout this Section, $\mathcal{N} \subset \mathcal{M}$ is an irreducible inclusion of factors, E is the unique conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, φ is a faithful normal state on $\mathcal{N}$ and $\hat{\varphi} = \varphi \circ E$.

Lemma 4.1.5. *If α is an automorphism of $\mathcal{M}$ which preserves $\mathcal{N}$ and the restriction to $\mathcal{N}$ preserves φ, then α commutes with the modular automorphism group $\sigma_t^{\hat{\varphi}}$.*

Proof. Since α preserves $\mathcal{N}$, $\alpha \circ E \circ \alpha^{-1}$ is a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. By the irreducibility such a conditional expectation is unique, hence $\alpha \circ E \circ \alpha^{-1} = E$, or $\alpha \circ E = E \circ \alpha$. We claim that α preserves $\hat{\varphi}$. Indeed, we have

$$\hat{\varphi}(\alpha(x)) = \varphi(E(\alpha(x))) = \varphi(\alpha(E(x))) = \varphi(E(x)) = \hat{\varphi}(x).$$

From this it follows that α commutes with $\sigma_t^{\hat{\varphi}}$ (see, for example, [82, chapter VIII, Cor. 1.4]). □

We insert a purely group-theoretic observation.

Lemma 4.1.6. *Let G be a group and $\pi : \mathbb{R} \to G$ be a group-homomorphism. If there exists $n \in \mathbb{N}$ such that for any $t \in \mathbb{R}$ it holds that $\pi(t)^{m_t} = e$ for some $m_t \leq n$ where e is the unit element in G, then $\pi(t) = e$, in other words π is trivial.*

Proof. Let us assume the contrary, namely that there were a t such that $\pi(t) \neq e$. Then $\pi(\frac{t}{n!}) \neq e$, since otherwise $\pi(t) = \pi(\frac{t}{n!})^{n!} = e$. But by assumption there exists $m_t \leq n$ such that

$$\pi\left(\frac{t}{n(n-1)\cdots\hat{m_t}\cdots 2\cdot 1}\right) = \pi\left(\frac{t}{n!}\right)^{m_t} = e,$$

where $\hat{m_t}$ means the omission of m_t in the product. This is a contradiction because the $n(n-1)\cdots\hat{m_t}\cdots 2\cdot 1$-th power of the left hand side is $\pi(t) \neq e$. □

Lemma 4.1.7. *Let the inclusion $\mathcal{N} \subset \mathcal{M}$ have finite index. If $\{\alpha_t\}$ is one-parameter group of automorphisms of $\mathcal{M}$ which preserve $\mathcal{N}$ and if it holds that $\alpha_t|_{\mathcal{N}} = \sigma_t^{\varphi}$, then $\alpha_t = \sigma_t^{\hat{\varphi}}$.*

Proof. By Lemma 4.1.5, α_s commutes with $\sigma_t^{\hat\varphi}$. Hence $\beta_t := \alpha_{-t} \circ \sigma_t^{\hat\varphi}$ is again a one-parameter group of automorphisms of $\mathcal{M}$, preserving $\mathcal{N}$, and its restriction to $\mathcal{N}$ is trivial by assumption.

We claim that the one-parameter automorphism $\{\beta_t\}$ is inner. Once we know this, the lemma follows since the implementing unitary operators should be in the relative commutant, which is trivial for an irreducible inclusion.

Suppose the contrary, namely that there were a $t \in \mathbb{R}$ such that β_t is outer. Let π be the natural homomorphism $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Out}(\mathcal{M})$.

We show that the order of $\pi(\beta_t)$ is smaller than the index $[\mathcal{M}, \mathcal{N}]$. Indeed, if $\pi(\beta_t)$ has order $p > [\mathcal{M}, \mathcal{N}]$, then $\gamma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{M})$, $\gamma(n) := \beta_{nt}$ is an outer action of $\mathbb{Z}_p$ on $\mathcal{M}$. If $\pi(\beta_t)$ has infinite order, then $\gamma(n) := \beta_{nt}$ is an outer action of $\mathbb{Z}$. In any case, the subfactor $\mathcal{B}^\gamma \subset \mathcal{B}$ has the index larger than $[\mathcal{M}, \mathcal{N}]$. But this is a contradiction, since we have $\mathcal{N} \subset \mathcal{M}^\gamma \subset \mathcal{M}$ and the index of $\mathcal{M}^\gamma \subset \mathcal{M}$ has to be smaller than or equal to $[\mathcal{M}, \mathcal{N}]$.

Having seen that the order of any element $\pi(\beta_t)$ is smaller than or equal to $[\mathcal{M}, \mathcal{N}]$, we infer that $\pi(\beta_t)$ is the unit element in $\mathrm{Out}(\mathcal{M})$ by Lemma 4.1.6, which means β_t is inner for each t. □

Finally we put a simple remark on a group of automorphisms of irreducible inclusion $\mathcal{N} \subset \mathcal{M}$ with finite index.

Lemma 4.1.8. *Let G be the group of automorphisms of $\mathcal{M}$ which act identically on $\mathcal{N}$. Then $|G| \leq [\mathcal{M}, \mathcal{N}]$. In particular, if $\{\beta_t\}$ is a continuous family of such automorphisms, then it is constant.*

Proof. Note that any nontrivial element in G is outer. In fact, if it were inner, it would be implemented by an unitary $U \in \mathcal{M}$ which commutes with $\mathcal{N}$, hence by the assumed irreducibility of $\mathcal{N} \subset \mathcal{M}$ it must be scalar. By considering the inclusion $\mathcal{N} \subset \mathcal{M}^G \subset \mathcal{M}$ we see that the order of G cannot exceed the index of $\mathcal{N} \subset \mathcal{M}$. The second statement follows immediately. □

4.1.3 KMS condition on locally normal systems

In the present Chapter we consider KMS states on the quasilocal algebra of conformal nets with respect to translations or dilations. The typical systems, treated e.g. in [9, Section 5.3.1], are C^*- or a W^*-dynamical systems, but they are not directly applicable to our case. Indeed, the algebra concerned is the quasilocal C^*-algebra generated by local von Neumann algebras; on the other hand, the automorphisms concerned are translations or dilations, which are not norm-continuous. Although the modification is rather straightforward, for the readers' convenience we give a variation of the standard results in [9] in a form applicable to conformal nets.

Let $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n \subset \cdots$ be a growing sequence of von Neumann algebras and $\mathfrak{M}$ be the "quasilocal algebra" $\overline{\bigcup_n \mathcal{M}_n}^{\|\cdot\|}$. We consider a state φ on $\mathfrak{M}$ which is normal and faithful on each $\mathcal{M}_n$, i.e. "locally normal and locally faithful". (When we state some property with the adverb "locally", we mean that the property holds if restricted to each local algebra $\mathcal{M}_n$). Let σ^n be the modular automorphism of $\mathcal{M}_n$ with respect to φ. We assume that, for each k, $\sigma_t^n(\mathcal{M}_k) \subset \mathcal{M}_{k+1}$ for sufficiently small t irrespective of $n > k$. We assume also that σ^n converges to some one-parameter automorphism σ pointwise *-strongly, σ_t is a locally normal map for each t and $t \mapsto \sigma_t$ is strongly continuous. Let us call such a dynamical system a **locally normal system**. From these definitions, it is easy to see that σ preserves φ.

Definition 4.1.9. *Suppose that $\mathfrak{M}$ is a C^* (or a W^*) algebra, σ is a norm (resp. σ-weakly) continuous one-parameter group of automorphisms and ψ is a state (resp. a normal state) on $\mathfrak{M}$. If for any $x, y \in \mathcal{M}$ and any function g on $\mathbb{R}$ which is the Fourier transform of a compactly supported function it holds that*

$$\int g(t)\psi(x\sigma_t(y))dt = \int g(t+i\beta)\psi(\sigma_t(y)x)dt,$$

then we say that ψ satisfies the **smeared KMS condition** *with respect to σ.*

In each case, C^*-dynamical system or W^*-dynamical system, the usual KMS condition is equivalent to the smeared condition [9]. We use the same term for a locally normal system as well.

Lemma 4.1.10. *The state φ satisfies the smeared KMS condition with respect to σ.*

Proof. For each $x, y \in \mathcal{M}_k$, φ satisfies the smeared condition with respect to σ^n where $n \geq k$. Namely, it holds that

$$\int g(t)\varphi(x\sigma_t^n(y))dt = \int g(t+i\beta)\varphi(\sigma_t^n(y)x)dt.$$

We assumed that, for a fixed t, $\sigma_t^n(y)$ converges strongly to $\sigma_t(y)$. Then the condition for σ follows by the Lebegues' dominated convergence theorem.

A general element in $\mathfrak{M}$ can be approximated from $\{\mathcal{M}_n\}$ by norm. □

We fix an element $y \in \mathcal{M}_n$ and define the analytic elements

$$y_\varepsilon := \int \sigma_t(y)\sqrt{\frac{\pi}{\varepsilon}}\exp\left(-\frac{t^2}{\varepsilon}\right)dt. \tag{4.2}$$

s.t. $y_\varepsilon \to y$ *-strongly for $\varepsilon \to 0$. These are well-defined as elements of $\mathfrak{M}$. Indeed, if we truncate the integral to a compact interval, then the integrand lies in some local algebra and the integral defines a local element. Such truncated integrals converge in norm because of the Gaussian factor, hence define an element of the C^*-algebra.

Lemma 4.1.11. *For any locally normal state ψ, $\psi(\sigma_t(y_\varepsilon))$ continues to an entire function of t.*

Proof. By the assumed local normality of ψ, for a truncated integral, the integral and ψ commute. The full integral is approximated by norm, hence the full integral and ψ commute as well. Namely, for $z \in \mathbb{C}$, we have

$$\psi\left(\int \sigma_t(y)\sqrt{\frac{\pi}{\varepsilon}}\exp\left(-\frac{(t-z)^2}{\varepsilon}\right)\right)dt = \int \psi(\sigma_t(y))\sqrt{\frac{\pi}{\varepsilon}}\exp\left(-\frac{(t-z)^2}{\varepsilon}\right)dt.$$

The right hand side is analytic and the left hand side is equal to $\psi(\sigma_z(y_\varepsilon))$ when z is real. □

Lemma 4.1.12. *For $x, y \in \mathcal{M}_n$, there is an analytic function f such that*

$$f(t) = \varphi(x\sigma_t(y_\varepsilon)), f(t+i\beta) = \varphi(\sigma_t(y_\varepsilon)x).$$

Proof. We define f by the first equation. We saw that f is entire in Lemma 4.1.11. By Lemma 4.1.10, for any g, $\hat{g} \in \mathscr{D}$, it holds that

$$\int g(t+i\beta)\varphi(\sigma_t(y_\varepsilon)x)dt = \int g(t)\varphi(x\sigma_t(y_\varepsilon))dt =$$

$$= \int g(t)f(t)dt = \int g(t+i\beta)f(t+i\beta)dt.$$

Since g is arbitrary under the condition above, we obtain the second equation. □

Lemma 4.1.13. *For $x, y \in \mathcal{M}_n$, the sequence $\varphi(x\sigma_t(y_\varepsilon))$ (respectively $\varphi(\sigma_t(y_\varepsilon)x)$) converges to $\varphi(x\sigma_t(y))$ (respectively $\varphi(\sigma_t(y)x)$) uniformly on t.*

Proof. We just prove the first, since the second is analogous by the assumed *-strong convergence of the modular automorphisms. Note that, by the Schwarz inequality and by the invariance of φ with respect to σ, we have

$$\|\varphi(x\sigma_t(y_\varepsilon - y))\|^2 \leq \varphi(x^*x)\varphi\left((y_\varepsilon - y)^*(y_\varepsilon - y)\right),$$

hence the uniformity is not a problem once we show the convergence of the right hand side.

By hypothesis, there is a $\delta > 0$ s.t. $\sigma_t(M_n) \subset M_{n+1}$ for $|t| \leq \delta$. Let us define $\tilde{y}_\varepsilon$ by the truncation of the integral in (4.2) to the subset $[-\delta, \delta] \subset \mathbb{R}$. It follows that $\tilde{y}_\varepsilon \in M_{n+1}$, $\|\tilde{y}_\varepsilon\| \leq \|y\|$ and, as the norm difference $\|\tilde{y}_\varepsilon - y_\varepsilon\|$ tends to 0, it is enough to show the convergence of the right hand side with the local elements $\tilde{y}_\varepsilon$ in place of y_ε. The restriction of φ to M_{n+1} is normal and can be approximated in norm by linear combinations of weakly continuous functionals of the form $\langle \xi, \cdot\, \eta\rangle$ with a pair of vectors ξ, η. Since $\langle (y - \tilde{y}_\varepsilon)\xi, (y - \tilde{y}_\varepsilon)\eta\rangle$ is convergent to 0 and the sequence $\tilde{y}_\varepsilon$ is bounded the desired convergence follows. □

Proposition 4.1.14. *The state φ satisfies the KMS condition with respect to σ.*

Proof. As we saw in Lemma 4.1.12, the KMS condition is satisfied for any pair x, y_ε where $x, y \in \mathcal{M}_n$. As ε tends to 0, the analytic function $\varphi(x\sigma_t(y_\varepsilon))$ tends to $\varphi(x\sigma_t(y))$ uniformly on the strip by Lemma 4.1.13 and by the three-line theorem. The limit function connects $\varphi(x\sigma_t(y_\varepsilon))$ and $\varphi(\sigma_t(y_\varepsilon)x)$. Any pair of elements in $\mathfrak{M}$ can be approximated in norm by elements in $\mathcal{M}_n$, hence the same reasoning completes the proof. □

4.1.4 Remarks on local diffeomorphisms

We consider diffeomorphisms of $\mathbb{R}$. We say simply a sequence of diffeomorphisms $\{\eta_n\}$ converges smoothly to a diffeomorphism η when $\{\eta_n\}$ and all their derivatives converge to η uniformly on each compact set. Recall that any diffeomorphism is a smooth (C^∞) function $\mathbb{R} \to \mathbb{R}$ with strictly positive derivative.

Lemma 4.1.15. *For each interval I, there is a diffeomorphism $\tilde{\tau}_s$ with compact support which coincides with translation τ_s on I.*

Proof. We may assume $s = 1$. There is a smooth non-negative function with a compact support whose value is strictly less than 1. By dilating this function, we may assume that its integral over $\mathbb{R}$ is 1. By considering its indefinite integral, we obtain a smooth non-negative function which is 0 on $\mathbb{R}_-$ and 1 on some half-line $\mathbb{R}_+ + a$, $a > 0$, with derivative strictly less than 1. Similarly we obtain a smooth non-negative function which is 1 on $\mathbb{R}_-$ and 0 on $\mathbb{R}_+ + a$ with derivative strictly larger than -1. By translating and multiplying these functions, we obtain a non-negative function with compact support with derivative larger than -1 which is 1 on I. The desired diffeomorphism is the function represented by this function added by the identity function $\mathrm{id}(t) = t$. $\square$

Lemma 4.1.16. *If a sequence of diffeomorphisms η_n of $\mathbb{R}$ converges smoothly to translation τ_s, then for any interval I there is an interval $\tilde{I} \supset I$ and a smoothly convergent sequence of diffeomorphisms $\tilde{\eta}_n$ with support in $\tilde{I}$ which coincides with η_n on I (hence converges smoothly on I to τ_s).*

Proof. Note that $\eta_n \circ \tau_{-s}$ converges smoothly to the identity map id. Let g_n be functions which represent $\eta_n \circ \tau_{-s}$. And h be a function with a compact support such that $h(t) = 1$ on I. Let us define

$$\hat{g}_n(t) = (g_n(t) - t)h(t) + t.$$

Since $\{g_n\}$ converges to id smoothly, for sufficiently large n their derivatives are strictly positive and define diffeomorphisms $\hat{\eta}_n$. The function $\hat{g}_n$ coincides with g_n on I by the definition of h. Let $\tilde{\tau}_s$ be the local diffeomorphism constructed in Lemma 4.1.15. The composition $\tilde{\eta}_n := \hat{\eta}_n \circ \tilde{\tau}_s$ gives the required sequence. $\square$

By the exponential map (or by an analogous proof) we obtain the corresponding construction for dilation.

Lemma 4.1.17. *If a sequence of diffeomorphisms η_n of $\mathbb{R}_+$ converges smoothly to dilation δ_s, then for any interval $I \Subset \mathbb{R}_+$ there is an interval $\tilde{I} \supset I$ and a smoothly convergent sequence of diffeomorphisms $\tilde{\eta}_n$ with support in $\tilde{I}$ which coincides with η_n on I (hence converges smoothly on I to δ_s).*

We apply these to the case of dilations of intervals. The standard dilation (restricted to $\mathbb{R}_+$) is the map $\delta_s : \mathbb{R}_+ \ni t \mapsto e^s t \in \mathbb{R}$. A dilation δ_s^I of an interval I is defined by $(\eta^I)^{-1} \circ \delta_s^I \circ \eta^I$, where χ^I is a linear fractional transformation which maps I to $\mathbb{R}_+$. This is well-defined, since any other such linear fractional transformation is a composition of the χ^I and a standard dilation.

Lemma 4.1.18. *If $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset \mathbb{R}_+$ is an increasing sequence of intervals with $\bigcup_n I_n = \mathbb{R}_+$, then for any fixed s, $\{\delta_s^{I_n}\}$ smoothly converge to δ_s.*

Proof. Let us put $I_n = (a_n, b_n)$, hence $a_n \to 0$ and $b_n \to \infty$. We take the fractional linear transformations as follows:

$$\chi^{I_n}(t) = \frac{t - a_n}{b_n - t}, (\chi^{I_n})^{-1}(t) = \frac{b_n t + a_n}{t + 1}.$$

Then we can calculate the dilation of I_n concretely:

$$\delta_s^{I_n}(t) \;=\; (\chi^{I_n})^{-1} \circ \delta_s^I \circ \chi^{I_n} \;=\; \frac{e^s b_n(t - a_n) + a_n(b_n - t)}{e^s(t - a_n) + b_n - t} \;=\; \frac{e^s(t - a_n) + a_n(1 - \frac{t}{b_n})}{1 + \frac{e^s(t-a_n)-t}{b_n}}.$$

From this expression it is easy to see that $\delta_s^{I_n}(t)$ converge smoothly to $\delta_s(t) = e^s t$, since the numerator tends smoothly to $e^s t$ and the denominator tends to 1 smoothly. □

We summarize these remarks to obtain the following.

Proposition 4.1.19. *For each s and $I \Subset \mathbb{R}_+$, there is a $\tilde{I} \Subset \mathbb{R}_+$ and a smoothly convergent sequence of diffeomorphisms $\eta_s^{I_n}$ with support in $\tilde{I}$ which converge to δ_s and coincide with $\delta_s^{I_n}$ on I.*

4.2 The thermal completion and the role of relative commutants

Let φ be a locally normal state on the quasi-local algebra $\mathfrak{A}_\mathcal{A}$ associated to a conformal net $(\mathcal{A}, U)$ and π_φ be the GNS representation with respect to φ. For an $I \subset \mathbb{R}$ we shall set

$$\mathcal{A}_\varphi(I) \equiv \{ \bigcup_{I \supset \tilde{I} \Subset \mathbb{R}} \pi_\varphi(\mathcal{A}(\tilde{I}))\}''. \tag{4.3}$$

Note that, when I is a finite length (open) interval, $\mathcal{A}_\varphi(I)$ is simply the image of $\mathcal{A}(I)$ under the representation π_φ; however, $\mathcal{A}_\varphi$ is defined even for infinite length intervals.

Recall that representatives of local diffeomorphisms are contained in $\mathcal{A}$ (see Section 1.1.4). Similarly as above, to simplify notations, for a diffeomorphism $\eta : \mathbb{R} \to \mathbb{R}$ localized in some finite length interval $I \Subset \mathbb{R}$ we shall set $U_\varphi(\eta) := \pi_\varphi(U(\eta))$. The following basic properties can be easily checked.

- $\mathcal{A}_\varphi$ is *local* and *isotonous*: $[\mathcal{A}_\varphi(I_1), \mathcal{A}_\varphi(I_2)] = 0$ whenever $I_1 \cap I_2 = \emptyset$ and $\mathcal{A}_\varphi(I_1) \subset \mathcal{A}_\varphi(I_2)$ whenever $I_1 \subset I_2$.
- $U_\varphi(\eta)\mathcal{A}_\varphi(K)U_\varphi(\eta)^* = \mathcal{A}_\varphi(\eta(K))$ for every diffeomorphism η localized in some finite length interval and for every $K \subset \mathbb{R}$.

- If $\mathcal{A}$ is *strongly additive*, then so is $\mathcal{A}_\varphi$: we have that $\mathcal{A}_\varphi(r,t) \vee \mathcal{A}_\varphi(t,s) = \mathcal{A}_\varphi(r,s)$ for all $r < t < s$, $r,t,s \in \mathbb{R} \cup \{\pm\infty\}$.
- Assuming that $\mathcal{A}$ is strongly additive, if $\mathcal{A}_\varphi(\mathbb{R}) = \pi_\varphi(\mathfrak{A}_\mathcal{A})''$ is a *factor*, then so are the algebras $\mathcal{A}_\varphi(t+\mathbb{R}_+), \mathcal{A}_\varphi(t+\mathbb{R}_-)$ $(t \in \mathbb{R})$, too (notice that $\mathcal{A}_\varphi(\mathbb{R}_+) \cap \mathcal{A}_\varphi(\mathbb{R}_+)' \subset \mathcal{A}_\varphi(\mathbb{R}_-)' \cap \mathcal{A}_\varphi(\mathbb{R}_+)' = (\mathcal{A}_\varphi(\mathbb{R}_-) \vee \mathcal{A}_\varphi(\mathbb{R}_+))' = \mathcal{A}_\varphi(\mathbb{R})'$).

Suppose φ is a primary KMS state on $\mathfrak{A}_\mathcal{A}$ w.r.t. the translations $t \mapsto \mathrm{Ad}U(\tau_t)$ and π_φ is the GNS representation associated to φ with GNS vector Φ. Then one can easily find that $(\Phi, \mathcal{A}_\varphi(\mathbb{R}_+) \subset \mathcal{A}_\varphi(\mathbb{R}))$ is a standard half-sided modular inclusion (see Section 1.1.7, [95, 2]) and, by the last listed property, it is actually an inclusion of factors. In this situation, there exists a unique (possibly not "fully" diffeomorphism covariant) Möbius covariant, strongly additive net $(\hat{\mathcal{A}}_\varphi, \hat{U}_\varphi)$ such that

- $\hat{U}_\varphi(g)\Phi = \Phi$ for every Möbius transformation g,
- $\hat{\mathcal{A}}_\varphi(\mathbb{R}_+) = \mathcal{A}_\varphi(\mathbb{R})$ and $\hat{\mathcal{A}}_\varphi(1+\mathbb{R}_+) = \mathcal{A}_\varphi(\mathbb{R}_+)$.

The net $(\hat{\mathcal{A}}_\varphi, \hat{U}_\varphi)$ is called the **thermal completion** of $\mathcal{A}$ w.r.t. to the primary KMS state φ and it was previously studied in [61][1]. One has that

$$\hat{\mathcal{A}}_\varphi(e^{2\pi t}, e^{2\pi s}) = \mathcal{A}^d_\varphi(t,s) \tag{4.4}$$

where

$$\mathcal{A}^d_\varphi(t,s) = \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,\infty)' \qquad (t < s,\ t,s \in \mathbb{R} \cup \{\pm\infty\}).$$

Note that $\mathcal{A}_\varphi(t,s) \subset \mathcal{A}^d_\varphi(t,s)$ and, by Remark 1.3.6, if $\mathcal{A}$ is strongly additive and φ is the geometric KMS state, this inclusion is actually an equality:

$$\mathcal{A}^d_{\mathrm{geo}}(t,s) = \mathcal{A}_{\mathrm{geo}}(t,s). \tag{4.5}$$

Theorem 4.2.1. *Let $\mathcal{A}$ be a conformal net satisfying the split property, φ a primary KMS state on $\mathfrak{A}_\mathcal{A}$ with GNS representation π_φ, and assume that*

$$\mathcal{A}_\varphi(t,s) = \mathcal{A}^d_\varphi(t,s) \tag{4.6}$$

for some $t < s$, $t,s \in \mathbb{R}$. Then $\mathcal{A}$ is strongly additive and φ is of the form $\varphi = \varphi_{\mathrm{geo}} \circ \alpha$ where φ_{geo} is the geometric KMS state and $\alpha \in \mathrm{Aut}(\mathfrak{A}_\mathcal{A})$ such that

- *$\alpha(\mathcal{A}(I)) = \mathcal{A}(I)$ for all $I \Subset \mathbb{R}$*
- *$\alpha \circ \mathrm{Ad}U(\tau_t) = \mathrm{Ad}U(\tau_t) \circ \alpha$ for all $t \in \mathbb{R}$.*

In particular, in this case the thermal completion and the original net in the vacuum representation, as Möbius covariant nets, are unitarily equivalent.

[1]The notion of thermal completion was proposed in [24] based on heuristic considerations.

Proof. By (local) diffeomorphism covariance, if the assumption regarding the relative commutant holds for a particular $t < s$, $t, s \in \mathbb{R}$, then it holds for all such pairs. So fix $t_1 < t_2 < t_3$, $t_1, t_2, t_3 \in \mathbb{R}$; then by the strong additivity of the thermal completion we have that

$$\mathcal{A}_\varphi(t_1, t_2) \vee \mathcal{A}_\varphi(t_2, t_3) = \hat{\mathcal{A}}_\varphi(e^{2\pi t_1}, e^{2\pi t_2}) \vee \hat{\mathcal{A}}_\varphi(e^{2\pi t_2}, e^{2\pi t_3}) = \hat{\mathcal{A}}_\varphi(e^{2\pi t_1}, e^{2\pi t_3}) = \mathcal{A}_\varphi(t_1, t_3).$$

Since $\pi_{\varphi,I} \equiv \pi_\varphi|_{\mathcal{A}(I)}$ is a unitarily implementable isomorphism for any finite length interval $I \Subset \mathbb{R}$, the above equation shows that $\mathcal{A}$ is strongly additive. A similar argument shows that the split property of $\mathcal{A}$ implies the split property of the thermal completion.

Consider the GNS representations π_φ and π_{geo} and the thermal completions $\hat{\mathcal{A}}_\varphi$ and $\hat{\mathcal{A}}_{\varphi_{\mathrm{geo}}}$ associated to φ and φ_{geo}, respectively. By (4.4), (4.5) and point (1) of Prop. 1.3.4, the thermal completion given by the geometric KMS state is equivalent to the *(strongly additive) dual* of the original net in the vacuum representation, so, in our case, simply to the original net (which is already strongly additive):

$$\hat{\mathcal{A}}_{\mathrm{geo}}(e^{2\pi t}, e^{2\pi s}) = \mathcal{A}^d_{\mathrm{geo}}(t, s) = \mathcal{A}_{\mathrm{geo}}(t, s) = \mathcal{A}(e^{2\pi t}, e^{2\pi s}).$$

Fix a nonempty, finite length open interval $I \Subset \mathbb{R}$. Since both $\pi_{\varphi,I}$ and $\pi_{\mathrm{geo},I}$ are unitarily implementable, there exists a unitary V such that

$$\mathrm{Ad}(V)|_{\mathcal{A}_\varphi(I)} = \pi_{\mathrm{geo},I} \circ \pi^{-1}_{\varphi,I}$$

and one has that for all $t, s \in I$

$$\begin{aligned} V\hat{\mathcal{A}}_\varphi(e^{2\pi t}, e^{2\pi s})V^* = V\mathcal{A}^d_\varphi(t, s)V^* = V\mathcal{A}_\varphi(t, s)V^* = \\ = \pi_{\varphi_{\mathrm{geo}},I}(\mathcal{A}(t, s)) = \mathcal{A}_{\varphi_{\mathrm{geo}}}(t, s) = \mathcal{A}^d_{\mathrm{geo}}(t, s) = \hat{\mathcal{A}}_{\varphi_{\mathrm{geo}}}(e^{2\pi t}, e^{2\pi s}). \end{aligned}$$

Thus, by [92, Thm. 5.1], it follows that two thermal completions are equivalent: there exists a unitary operator W such that $W\hat{\mathcal{A}}_\varphi(a, b)W^* = \hat{\mathcal{A}}_{\varphi_{\mathrm{geo}}}(a, b)$ for *all* $a, b \in \mathbb{R}$ and $W\hat{U}_\varphi(g)W^* = \hat{U}_{\varphi_{\mathrm{geo}}}(g)$ for all Möbius transformations g. (Note that this latter fact implies that $\mathrm{Ad}(W)$ also connects the respective vacuum states of the two thermal completions.) Then, using that both $\mathcal{A}^d_{\varphi_{\mathrm{geo}}}(I) = \mathcal{A}_{\varphi_{\mathrm{geo}}}(I)$ and $\mathcal{A}^d_\varphi(I) = \mathcal{A}_\varphi(I)$, one sees that the automorphism of $\mathcal{A}(I)$

$$\alpha_I := \pi^{-1}_{\mathrm{geo},I} \circ \mathrm{Ad}(W^*) \circ \pi_{\varphi,I}$$

is well-defined (i.e. $W^*\mathcal{A}_\varphi(I)W = \mathcal{A}_{\varphi_{\mathrm{geo}}}(I)$) for every $I \Subset \mathbb{R}$. Moreover, it is also clear that $\alpha_I = \alpha_K|_{\mathcal{A}(I)}$ whenever $I \subset K$, hence that it defines an automorphism α of $\mathfrak{A}_\mathcal{A}$ which preserves every local algebra $\mathcal{A}(I)$, $I \Subset \mathbb{R}$.

The fact that $\mathrm{Ad}(W)$ connects the relevant representations of the Möbius group shows that α commutes with the one-parameter group of translations $t \mapsto \mathrm{Ad}U(\tau_t)$. Moreover, since $\mathrm{Ad}(W)$ also connects the vacuum states of the two thermal completions, one can also easily verify that $\varphi_{\mathrm{geo}} \circ \alpha = \varphi$. □

As will be shown by examples later in this Chapter, without the assumption of the previous theorem the inclusion $\mathcal{A}_\varphi(t,s) \subset \mathcal{A}^d_\varphi(t,s) \equiv \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,\infty)'$ is not necessarily an equality. We shall now investigate the completely rational case.

Lemma 4.2.2. *Let π_φ be the GNS representation of a primary KMS state φ on $\mathfrak{A}_\mathcal{A}$. If $\mathcal{A}$ is completely rational, then $\mathcal{A}_\varphi(t,s) \subset \mathcal{A}^d_\varphi(t,s)$ is a finite index irreducible inclusion.*

Proof. We noted at the beginning of this section some basic properties of $\mathcal{A}_\varphi$. In particular, the strong additivity of $\mathcal{A}$ implies the strong additivity of $\mathcal{A}_\varphi$, hence $\mathcal{A}_\varphi(t,\infty) = \mathcal{A}_\varphi(t,s) \vee \mathcal{A}_\varphi(s,\infty)$ and the relative commutant of the inclusion in question is simply the center of $\mathcal{A}_\varphi(t,\infty)$. On the other hand, when our KMS state is primary, the algebra $\mathcal{A}_\varphi(t,\infty)$ is a factor. So our inclusion is indeed irreducible:

$$\begin{aligned}\mathcal{A}^d_\varphi(t,s) \cap \mathcal{A}_\varphi(t,s)' &= (\mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,\infty)') \cap \mathcal{A}_\varphi(t,s)' = \\ &= \mathcal{A}_\varphi(t,\infty) \cap (\mathcal{A}_\varphi(s,\infty) \vee \mathcal{A}_\varphi(t,s))' = \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}(t,\infty)' = \mathbb{C}\mathbb{1}.\end{aligned}$$

Let now $n,m \in \mathbb{N}$ with $0 < n < m$. Since locally π_φ is a unitarily implementable isomorphism, the index of the inclusion

$$\mathcal{N}_{n,m} := \mathcal{A}_\varphi(t,s) \vee \mathcal{A}_\varphi(s+n,s+m) \subset \mathcal{A}_\varphi(t,s+m) \cap \mathcal{A}'_\varphi(s,s+n) =: \mathcal{M}_{n,m} \tag{4.7}$$

is simply the so called μ-index $\mu_\mathcal{A}$ of the completely rational net $\mathcal{A}$. Now it is clear that, as m increases, both sides of (4.7) increase, whereas, as n increases, both sides of (4.7) decrease. So let us set

$$\begin{aligned}\mathcal{N}_n &:= \{\cup_{m>n}\mathcal{N}_{n,m}\}'', \qquad \mathcal{M}_n := \{\cup_{m>n}\mathcal{M}_{n,m}\}'', \text{ and in turn} \\ \mathcal{N} &:= \cap_n \mathcal{N}_n, \qquad \mathcal{M} := \cap_n \mathcal{M}_n.\end{aligned}$$

Fixing the value of n and considering the sequence of inclusions $m \mapsto (\mathcal{N}_{n,m} \subset \mathcal{M}_{n,m})$, by [54, Prop. 3] we have that there is an expectation $E_n : \mathcal{M}_n \to \mathcal{N}_n$ satisfying the Pimsner-Popa inequality with constant $1/\mu$. Note that even without *a priori* assuming the normality of E_n, this implies that the index of $\mathcal{N}_n \subset \mathcal{M}_n$ is less or equal to μ; see Section 4.1.1. Then in turn, considering the sequence $n \mapsto (\mathcal{N}_n \subset \mathcal{M}_n)$, we find that the index of the inclusion $\mathcal{N} \subset \mathcal{M}$ is also smaller or equal to μ.

Now it is rather straightforward that $\mathcal{N}_n = \mathcal{A}_\varphi(t,s) \vee \mathcal{A}_\varphi(s+n,\infty)$. Moreover, we have $\cap_n \mathcal{A}_\varphi(s+n,\infty) = \mathbb{C}\mathbb{1}$ since the intersection in question is clearly in the center of the factor $\mathcal{A}_\varphi(\mathbb{R}) = \pi_\varphi(\mathfrak{A}_\mathcal{A})''$. It is not obvious whether the order of the operations "$\vee$" and "$\cap$" can be inverted:

$$\mathcal{N} = \cap_n \big(\mathcal{A}_\varphi(t,s) \vee \mathcal{A}_\varphi(s+n,\infty)\big) \stackrel{?}{=} \mathcal{A}_\varphi(t,s) \vee \big(\cap_n \mathcal{A}_\varphi(s+n,\infty)\big) = \mathcal{A}_\varphi(t,s).$$

We shall now show that using the split property the above equation can be justified. Indeed, by the split property, there exists a pair of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary operator W such that

$$W\mathcal{A}_\varphi(t,s)W^* \subset \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C}\mathbb{1}_{\mathcal{H}_2}, \quad \text{and} \quad W\mathcal{A}_\varphi(t-1,s+1)'W^* \subset \mathbb{C}\mathbb{1}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2).$$

In particular, $W\mathcal{A}_\varphi(t,s)W^* = \mathcal{K} \otimes \mathbb{C}\mathbb{1}_{\mathcal{H}_2}$ for some $\mathcal{K} \subset \mathcal{B}(\mathcal{H}_1)$. Now, if $n \geq 1$, then by locality the algebras $\mathcal{A}_\varphi(s+n,\infty)$ and $\mathcal{A}_\varphi(t-1,s+1)$ commute and hence

$$W\mathcal{A}_\varphi(s+n,\infty)W^* \subset \mathbb{C}\mathbb{1}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)$$

implying that $W\mathcal{A}_\varphi(s+n,\infty)W^* = 1_{\mathcal{H}_1} \otimes \mathrm{R}_n$ for some $\mathrm{R}_n \subset \mathcal{B}(\mathcal{H}_2)$. Since it holds that $\cap_n \mathcal{A}_\varphi(s+n,\infty) = \mathbb{C}\mathbb{1}$, we have that $\cap_n \mathrm{R}_n = \mathbb{C}\mathbb{1}_{\mathcal{H}_2}$ and

$$W(\cap_n \mathcal{N}_n)W^* = \cap_n (W\mathcal{N}_n W^*) = \cap_n (\mathcal{K} \otimes \mathrm{R}_n) = \mathcal{K} \otimes (\cap_n \mathrm{R}_n) = \mathcal{K} \otimes \mathbb{1}_{\mathcal{H}_2} = W\mathcal{A}_\varphi(t,s)W^*$$

which justifies that $\mathcal{N} = \cap_n \mathcal{N}_n = \mathcal{A}_\varphi(t,s)$. By a similar argument, again relying on the split property, we can also show that $\mathcal{M}_n = \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,s+n)'$ and hence that

$$\mathcal{M} = \cap_n \mathcal{M}_n = \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,\infty)' = \mathcal{A}^d_\varphi(t,s)$$

which concludes our proof. □

Theorem 4.2.3. *Let π_φ be the GNS representation of a primary KMS state φ on $\mathfrak{A}_\mathcal{A}$. If $\mathcal{A}$ is completely rational, then the thermal completion $(\hat{\mathcal{A}}_\varphi, \hat{U}_\varphi)$, as a Möbius covariant net, is conformal and unitarily equivalent to an irreducible local extension of the original net $(\mathcal{A}, U)$. Moreover, this extension is trivial (i.e. coincides with the original net in the vacuum representation) if and only if $\mathcal{A}^d_\varphi(t,s) = \mathcal{A}_\varphi(t,s)$ for some (and hence for all) $t < s$, $t,s \in \mathbb{R}$.*

Proof. First note that by strong additivity, for all $r \in (t,s)$, we have that

$$\begin{aligned}\mathcal{A}^d_\varphi(t,s) \cap \mathcal{A}_\varphi(r,s)' &= (\mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(r,s)') \cap \mathcal{A}_\varphi(s,\infty)' = \\ &= \mathcal{A}_\varphi(t,\infty) \cap (\mathcal{A}_\varphi(r,s) \vee \mathcal{A}_\varphi(s,\infty))' = \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(r,\infty)' = \mathcal{A}^d_\varphi(t,r). \quad (4.8)\end{aligned}$$

Similarly, we have that $\mathcal{A}^d_\varphi(t,s) \cap \mathcal{A}(t,r)' = \mathcal{A}^d_\varphi(r,s)$, too. Now consider the faithful normal state $\tilde{\varphi}_{\rm geo} \circ E$ on $\mathcal{A}^d_\varphi(t,s)$, where $E : \mathcal{A}^d_\varphi(t,s) \to \mathcal{A}_\varphi(t,s)$ is the (unique) faithful normal expectation whose existence is guaranteed by Lemma 4.2.2 and the state $\tilde{\varphi}_{\rm geo}$ on $\mathcal{A}_\varphi(t,s)$ is defined by the formula

$$\tilde{\varphi}_{\rm geo}(x) := \varphi_{\rm geo}(\pi_\varphi^{-1}(x)),$$

$\forall x \in \mathcal{A}_\varphi(t,s)$. Note that the above formula indeed well-defines a faithful normal state since π_φ is locally an isomorphism. Being a faithful normal state on $\mathcal{A}^d_\varphi(s_1,s_2)$, it gives rise to a one-parameter group of modular automorphisms $t \mapsto \tilde{\sigma}_t$. By construction, $t \mapsto \tilde{\sigma}_t$ preserves $\mathcal{A}_\varphi(s_1,s_2)$ and on this subalgebra it acts like its modular group associated to the state $\tilde{\varphi}_{\rm geo}$.

Locally, both π_φ and $\pi_{\rm geo}$ (the GNS representations associated to φ and $\varphi_{\rm geo}$, respectively) are isomorphisms and the algebras $\pi_{\rm geo}(\mathcal{A}(s,r))$ are local algebras of the thermal completion net $\hat{\mathcal{A}}_{\varphi_{\rm geo}}$. Hence, by the Bisognano-Wichmann property, it follows that

$$\tilde{\sigma}_t(\mathcal{A}_\varphi(s_1,r)) = \mathcal{A}_\varphi(s_1,f_t(r)) \quad \text{and} \quad \tilde{\sigma}_t(\mathcal{A}_\varphi(r,s_2)) = \mathcal{A}_\varphi(f_t(r),s_2).$$

Note that the actual formula of the function $f_t : (s_1, s_2) \to (s_1, s_2)$ could be easily worked out (we would then also need to take account of the fact that, when passing to the thermal completion net, one needs to perform a re-parametrization). However, in what follows, we shall not need a concrete formula for f_t, so for simplicity of the discussion we leave the expression in this abstract form. Note further that, by eq. (4.8), our previous formula holds for the dual algebras, too:

$$\tilde{\sigma}_t(\mathcal{A}^d_\varphi(s_1, r)) = \mathcal{A}^d_\varphi(s_1, f_t(r)) \quad \text{and} \quad \tilde{\sigma}_t(\mathcal{A}^d_\varphi(r, s_2)) = \mathcal{A}^d_\varphi(f_t(r), s_2). \tag{4.9}$$

Let now Φ be the GNS vector given by the state φ in its GNS representation π_φ. Since $\mathcal{A}^d_\varphi(s_1, s_2)$ is a local algebra of the thermal completion net $\hat{\mathcal{A}}_\varphi$ and Φ is the vacuum-vector of this net, the modular group of unitaries $t \mapsto \Delta^{it}_\Phi$ associated to $(\Phi, \mathcal{A}^d_\varphi(s_1, s_2))$ also acts in a "geometrical manner" on $\mathcal{A}^d_\varphi(s_1, r)$ and we have that

$$\Delta^{it}_\Phi \mathcal{A}^d_\varphi(s_1, r) \Delta^{-it}_\Phi = \mathcal{A}^d_\varphi(s_1, f_t(r)). \tag{4.10}$$

Consider the inclusion of factors $\mathcal{A}^d_\varphi(t, r_0) \subset \mathcal{A}^d_\varphi(t, s)$ for some fixed $t < r_0 < s$. It becomes a standard half-sided modular inclusion of factors both when it is considered with the state $\tilde{\varphi}$ given by the vector Φ and with the state $\tilde{\varphi}_{\text{geo}} \circ E$. Indeed, it has been already shown that is is a half-sided modular inclusion. Standardness with respect to Φ follows from the Reeh-Schlieder property for KMS states (see Section 1.3.1). As for $\tilde{\varphi}_{\text{geo}} \circ E$, let Φ' be the GNS vector in the GNS representation π'. The subspace generated by $\pi'(\mathcal{A}_\varphi(t, s))$ and Φ' is equivalent to the representation space with respect to φ_{geo}, hence it holds that $\overline{\pi'(\mathcal{A}_\varphi(t, r_0))\Phi'} = \overline{\pi'(\mathcal{A}_\varphi(r_0, s))\Phi'}$ again by the Reeh-Schlieder property. Note that $\overline{\pi'(\mathcal{A}^d_\varphi(t, r_0))\Phi'} = \overline{\pi'(\mathcal{A}^d_\varphi(t, r_0) \vee \mathcal{A}_\varphi(r_0, s))\Phi'}$, since $\mathcal{A}^d_\varphi(t, r_0)$ commutes with $\mathcal{A}_\varphi(r_0, s)$ and $\overline{\pi'(\mathcal{A}_\varphi(r_0, s))\Phi'}$ is already included in $\overline{\pi'(\mathcal{A}^d_\varphi(t, r_0))\Phi}$. By strong additivity of $\mathcal{A}$, $\mathcal{A}^d_\varphi(t, r_0) \vee \mathcal{A}_\varphi(r_0, s)$ includes $\mathcal{A}_\varphi(t, s)$, in particular the representatives of local diffeomorphisms supported in (t, s). Therefore it holds that $\mathcal{A}^d_\varphi(t, r_0) \vee \mathcal{A}_\varphi(r_0, s) = \mathcal{A}^d_\varphi(t, s)$ and this implies the cyclicity of Φ' for $\mathcal{A}^d_\varphi(t, r_0)$. The cyclicity for $\mathcal{A}^d_\varphi(r_0, t)$ can be proved analogously.

Thus we can construct two Möbius covariant nets. Of course, the one constructed with $\tilde{\varphi}$ simply gives back the thermal completion $\hat{\mathcal{A}}_\varphi$. The other one, constructed with the help of $\tilde{\varphi}_{\text{geo}} \circ E$, is easily seen to be a local extension of the net obtained by the inclusion $(\tilde{\varphi}_{\text{geo}}, \mathcal{A}_\varphi(t, r_0) \subset \mathcal{A}_\varphi(t, s))$ which in turn is equivalent to the thermal completion obtained with φ_{geo} and hence with the original net $\mathcal{A}$ (in the vacuum representation).

However, as we have seen their modular actions in equations (4.9) (4.10), both constructed nets will have $\mathcal{A}^d_\varphi(t, r)$ as the local algebra corresponding to the interval $(e^{2\pi t}, e^{2\pi r})$ for all $r \in [t, s]$. Furthermore, it turns out that the extension of $\mathcal{A}$ is split. Indeed, we have already observed in Lemma 4.2.2 that the inclusion is irreducible and of finite index, and the original net $\mathcal{A}$ is completely rational by assumption. Then by [62] a finite index extension is split as well. Hence the two strongly additive split nets coincide on all intervals $(e^{2\pi r_1}, e^{2\pi r_2})$, with $t < r_1, r_2 < s$, and thus by an application of [92, Thm. 5.1] they are equivalent. At this point we can infer that the extension $\mathcal{A}^d$ is conformal. Indeed, it includes $\mathcal{A}$ as a subnet, in particular its Virasoro subnet, hence there is a representation of $\mathrm{Diff}(S^1)$. Local representatives of $\mathrm{Diff}(S^1)$ supported in (t,s) act covariantly on $\mathcal{A}^d_\varphi(t,s)$. Any interval in S^1 can be obtained from (t,s) and an action of Möbius group, any local diffeomorphism acts covariantly. The group $\mathrm{Diff}(S^1)$ is generated by local diffeomorphisms, hence diffeomorphism covariance holds.

We have obtained that the thermal completion constructed with φ is a local extension of the original net (in the vacuum representation). If $\mathcal{A}^d_\varphi(t,s) = \mathcal{A}_\varphi(t,s)$, then of course the extension is trivial. On the other hand, a completely rational net cannot be equivalent to a nontrivial extension of itself since we have the formula [54, Prop. 24] relating the μ-indices of the net and of the extension to the index of the extension. □

A Möbius covariant net for which the only irreducible local extension is the trivial one (i.e. itself) is said to be a **maximal** net. Putting together the two presented theorems, the following conclusion can be drawn.

Corollary 4.2.4. *Let $\mathcal{A}$ be a conformal net and φ a primary KMS state on its quasi-local algebra $\mathfrak{A}_\mathcal{A}$ w.r.t. the translations $t \mapsto \mathrm{Ad}U(\tau_t)$. If $\mathcal{A}$ is completely rational and maximal, then there exists an automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A}_\mathcal{A})$ satisfying*

- $\alpha(\mathcal{A})(I) = \mathcal{A}(I)$ *for all* $I \Subset \mathbb{R}$
- $\alpha \circ \mathrm{Ad}U(\tau_t) = \mathrm{Ad}U(\tau_t) \circ \alpha$ *for all* $t \in \mathbb{R}$

such that $\varphi = \varphi_{\mathrm{geo}} \circ \alpha$ where φ_{geo} is the geometric KMS state.

4.3 Uniqueness results

4.3.1 Maximal completely rational nets

As seen in Section 4.2, any KMS state φ on a completely rational maximal net is a composition of the geometric KMS state φ_{geo} and an automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A}_\mathcal{A})$ such that $\alpha \circ \mathrm{Ad}U(\tau_t) = \mathrm{Ad}U(\tau_t) \circ \alpha$ for all $t \in \mathbb{R}$ and $\alpha(\mathcal{A}(I)) = \mathcal{A}(I)$ for all $I \Subset \mathbb{R}$. From now on, we simply call such α an **automorphism of the net** $\mathcal{A}|_\mathbb{R}$ commuting with translations. Here we study these automorphisms.

As noted in the introduction, among many examples, completely rational nets are of particular interest. A completely rational net admits only finitely many sectors [54]. In this subsection we will show the uniqueness of KMS state in cases where the net is completely rational and maximal with respect to extension. To obtain the uniqueness, we need to connect automorphisms on $\mathbb{R}$ and sectors (on S^1 by definition). Proposition 4.3.6 will demonstrate that there is a nice correspondence between them.

Let us begin with simple observations on automorphisms which commute with rotations or translations.

Proposition 4.3.1. *Let σ_1 and σ_2 be two automorphisms of the net $\mathcal{A}$ commuting with rotations. If they are in the same sector, namely if there is a unitary operator W which intertwines σ_1 and σ_2, then actually $\mathrm{Ad}(W)$ is an inner symmetry.*

Proof. By the definition of inner symmetry, we have just to prove that the vacuum vector Ω is invarian for W.

For any local element x of $\mathcal{A}$ it holds that $W\sigma_1(x)W^* = \sigma_2(x)$. Since σ_1 is an automorphism and surjective, this is equivalent to $\mathrm{Ad}(W)(x) = \sigma_2\sigma_1^{-1}(x)$. By assumption σ_1 and σ_2 commute with rotations, so does $\mathrm{Ad}(W)$.

Let L_0 be the generator of rotations. The observation above implies that $\mathrm{Ad}(W) \circ \mathrm{Ad}(e^{itL_0}) = \mathrm{Ad}(e^{itL_0}) \circ \mathrm{Ad}(W)$, for $t \in \mathbb{R}$, or, by setting $L_0' := W^*L_0W$, that $\mathrm{Ad}(e^{itL_0'}) = \mathrm{Ad}(e^{itL_0})$. Since the net is irreducible in the vacuum representation, this in turn shows that $e^{itL_0'}$ is a scalar multiple of e^{itL_0}. Let us denote the scalar by $\lambda(t)$.

It is immediate that $t \mapsto \lambda(t)$ is a continuous homomorphism from the group $\mathbb{R}$ to the group of complex numbers of modulus 1. Thus it follows that $L_0' = W^*L_0W = L_0 + \epsilon$ where $\lambda(t) = e^{it\epsilon}$. On the other hand, by the positivity of energy, the spectrum of L_0 is bounded below. But L_0 and L_0' must have the same spectrum since they are unitarily equivalent, hence ϵ must be 0. Namely, W commutes with L_0. This implies in particular that W preserves Ω, an eigenvector of L_0 with multiplicity one. □

Proposition 4.3.2. *If an automorphism α of $\mathcal{A}|_\mathbb{R}$ preserves the vacuum state ω, then α commutes with any diffeomorphism and it preserves also the geometric KMS state.*

Proof. The second part of the statement follows immediately from the first part, since the geometric state is a "composition of the vacuum with diffeomorphism", as seen from the construction in Section 1.3.2.

To show the first part, we observe that α_I is implemented by a unitary operator W, since it preserves the vector state ω and this implementation does not depend on the interval I, by the Reeh-Schlieder property. Since, for any $I \subset \mathbb{R}$, $\mathcal{A}(I)$ is preserved by $\mathrm{Ad}(W)$, so is $\mathcal{A}(I')$ $(= \mathcal{A}(I)'$ by the Haag duality), where I' is the complementary interval on S^1. Any interval on S^1 is either of the form I or I' with $I \subset \mathbb{R}$. By [26, Corollary 5.8], W commutes with all the diffeomorphisms. □

By an analogous proof as Proposition 4.3.1, we easily obtain the following proposition for the net on the real line $\mathbb{R}$.

Proposition 4.3.3. *Let α_1 and α_2 be two automorphisms of the net $\mathcal{A}|_\mathbb{R}$ commuting with translations. If they are unitarily equivalent, then the unitary operator W which intertwines α_1 and α_2 implements an inner symmetry.*

The following lemmas will serve to connect different KMS states and inequivalent automorphisms.

Lemma 4.3.4. *If a locally normal state ψ on $\mathcal{A}|_\mathbb{R}$ is invariant under dilation $\mathrm{Ad}U(\delta_s)$ with some $s \in \mathbb{R} \setminus \{0\}$, then ψ is equal to the vacuum state ω.*

Proof. It is obvious that ψ is invariant under $\delta_{ns}, n \in \mathbb{Z}$. Hence we may assume that $s > 0$.

Let us consider intervals $I_T = [-T, T]$. As noted in [92, Lemma 4.1], the norm-difference of restrictions $\psi|_{\mathcal{A}(I_T)}, \omega|_{\mathcal{A}(I_T)}$ tends to 0 when T decreases to 0. On the other hand, ψ and

ω are invariant under $\mathrm{Ad}U(\delta_{ns})$ by assumption and definition respectively. Therefore the norm-difference on $\mathcal{A}(e^{ns}I)$ is the same as on $\mathcal{A}(I)$ by the invariance. Namely,

$$\left\|\psi|_{\mathcal{A}(I_T)} - \omega|_{\mathcal{A}(I_T)}\right\| = \left\|\psi \circ \mathrm{Ad}U(\delta_{ns})|_{\mathcal{A}(I_{e^{-ns}T})} - \omega \circ \mathrm{Ad}U(\delta_{ns})|_{\mathcal{A}(I_{e^{-ns}T})}\right\| =$$
$$= \left\|\psi|_{\mathcal{A}(I_{e^{-ns}T})} - \omega|_{\mathcal{A}(I_{e^{-ns}T})}\right\| \to 0,$$

which shows that the two states are the same state when restricted to I_T. As T is arbitrary, they are the same. □

Lemma 4.3.5. *Let α be an automorphism of $\mathcal{A}|_{\mathbb{R}}$ commuting with translations. Let us denote the "dilated" automorphism $\mathrm{Ad}U(\delta_s) \circ \alpha \circ \mathrm{Ad}U(\delta_{-s})$ by α_s. If α does not preserve the vacuum state ω, then the automorphisms of the family $\{\alpha_s\}_{s\in\mathbb{R}_+}$ are mutually unitarily inequivalent.*

Proof. By assumption $\omega \circ \alpha$ is different from ω. Thus Lemma 4.3.4 implies that the states of the family $\{\omega \circ \alpha \circ \mathrm{Ad}U(\delta_s^{-1})\}_{s\in\mathbb{R}}$ are mutually different. We recall that ω is invariant under dilations, hence this family is the same as the family $\{\omega \circ \alpha_s\}_{s\in\mathbb{R}}$.

It is immediate that all the automorphisms $\{\alpha_s\}_{s\in\mathbb{R}}$ commute with translations. Then, by Proposition 4.3.3, any two of such automorphisms are unitarily equivalent if and only if they are conjugate by an inner symmetry. If there were such a pair of automorphisms, then their compositions with the vacuum state ω would be equal, but this contradicts the observation in the first paragraph. □

Next we construct a correspondence from automorphisms on $\mathcal{A}|_{\mathbb{R}}$ to automorphic sectors of $\mathcal{A}$.

Proposition 4.3.6. *For any automorphism α on $\mathcal{A}|_{\mathbb{R}}$ which commutes with translations, there corresponds an automorphism σ_α of $\mathcal{A}$ which commutes with rotations. The images σ_{α_1} and σ_{α_2} are unitarily equivalent if and only if α_1 and α_2 are unitarily equivalent.*

Proof. Recall that the real line $\mathbb{R}$ is identified with a subset of S^1 as explained in Section 1.1.1. First we fix an open interval I_0 whose closure does not contain the point at infinity and has the length 2π in the real line picture. Note that $S^1\setminus\{\infty\}$ is naturally diffeomorphic to an interval I_0 of length 2π. Indeed, there is a diffeomorphism from $S^1 \setminus \{\infty\}$ onto I_0 which preserves the lengths in the circle picture of $S^1 \setminus \{\infty\}$ with respect to the lengths in the real-line picture of I_0. Let us call this diffeomorphism η_0. Let p be a point in $S^1 \setminus \{\infty\}$. If $s_p > 0$ (or $s_p < 0$) is small enough so that for any $0 \leq s' \leq s_p$ (or $0 \geq s' \geq s_p$) it holds that $\rho_{s'}(p) \in S^1 \setminus \{\infty\}$, then it is easy to see that $\eta_0 \circ \rho_{s'}(p) = \tau_{s'} \circ \eta_0(p)$.

We have to define an automorphism σ_α through α. Let us take an interval $I \subset S^1$. We can choose a rotation ρ_s such that $\overline{\rho_s(I)}$ is inside $S^1\setminus\{\infty\}$. It is again easy to see that there is a diffeomorphism η of S^1 which coincides with η_0 on $\rho_s(I)$. The desired automorphism is defined by

$$\sigma_{\alpha,I} := (\mathrm{Ad}(U(\rho_s)))^{-1} \circ (\mathrm{Ad}(U(\eta)))^{-1} \circ \alpha \circ \mathrm{Ad}(U(\eta)) \circ \mathrm{Ad}(U(\rho_s)).$$

Since α preserves each algebra $\mathcal{A}(I)$ on any interval I, this is an automorphism. We must check that this definition does not depend on s and η and that $\sigma_{\alpha,I}$ satisfy the consistency condition w.r.t. inclusions of intervals.

Let us fix s which satisfies the condition that $\rho_s(I)$ does not touch the point at infinity. A different choice of η under the condition that η coincides with η_0 on $\rho_s(I)$ does not matter at all. Indeed, let η' be another diffeomorphism which complies with the condition. Then $\eta^{-1}\circ\eta'$ does not move points in $\rho_s(I)$; in other words, the support of $\eta^{-1}\circ\eta'$ is in the complement of $\rho_s(I)$. Since U is a projective unitary representation, it holds that $U(\eta') = c\cdot U(\eta)U(\eta^{-1}\circ\eta')$, where c is a scalar with modulus 1, hence the adjoint actions of $U(\eta')$ and $U(\eta)$ on $\mathcal{A}(\rho_s(I))$ are the same by the locality of the net.

We consider next different choices $s_1 < s_2$ of rotations. A rotation of 2π is just the identity, thus we may assume that $s_2 < 2\pi$ and that, for any $s_1 \leq s \leq s_2$, the interval $\rho_s(I)$ never contains ∞. Then, for any point p of I and for any $0 \leq t \leq s_2 - s_1$, it holds that $\eta_0\circ\rho_t\circ\rho_{s_1}(p) = \tau_t\circ\eta_0\circ\rho_{s_1}(p)$. The adjoint action of a diffeomorphism on $\mathcal{A}(I)$ is determined by the action of the diffeomorphism on I (by a similar argument to that in the previous paragraph), so it holds that

$$\mathrm{Ad}(U(\eta))\circ\mathrm{Ad}(U(\rho_t))|_{\mathcal{A}(\rho_{s_1}(I))} = \mathrm{Ad}(U(\tau_t))\circ\mathrm{Ad}(U(\eta))|_{\mathcal{A}(\rho_{s_1}(I))}.$$

By assumption α commutes with $\mathrm{Ad}(U(\tau_t))$ for any t, hence, putting $t = s_2 - s_1$, we have on $\mathcal{A}(I)$

$$\begin{aligned}
& \mathrm{Ad}(U(\rho_{s_2}))^{-1}\circ\mathrm{Ad}(U(\eta))^{-1}\circ\alpha\circ\mathrm{Ad}(U(\eta))\circ\mathrm{Ad}(U(\rho_{s_2})) \\
= \ & \mathrm{Ad}(U(\rho_{s_1}))^{-1}\circ\mathrm{Ad}(U(\rho_t))^{-1}\circ\mathrm{Ad}(U(\eta))^{-1}\circ\alpha\circ\mathrm{Ad}(U(\eta))\circ\mathrm{Ad}(U(\rho_t))\circ\mathrm{Ad}(U(\rho_{s_1})) \\
= \ & \mathrm{Ad}(U(\rho_{s_1}))^{-1}\circ\mathrm{Ad}(U(\eta))^{-1}\circ\mathrm{Ad}(U(\tau_t))^{-1}\circ\alpha\circ\mathrm{Ad}(U(\tau_t))\mathrm{Ad}(U(\eta))\circ\mathrm{Ad}(U(\rho_{s_1})) \\
= \ & \mathrm{Ad}(U(\rho_{s_1}))^{-1}\circ\mathrm{Ad}(U(\eta))^{-1}\circ\alpha\circ\mathrm{Ad}(U(\eta))\mathrm{Ad}(U(\rho_{s_1})).
\end{aligned}$$

This completes the proof of well-definedness of $\sigma_{\alpha,I}$.

Let us check the consistency w.r.t. inclusions of intervals. If $I \subset J$, then the η and ρ_s chosen for the larger interval J still work also for I and their action on I is just a restriction.

To confirm that σ_α commutes with rotations, let us fix an interval I. Let us choose η and s as above. If t is small enough so that $\rho_t(\rho_s(I))$ does not touch ∞, then a similar calculation as above shows that $\mathrm{Ad}(U(\rho_t))$ commutes with $\sigma_{\alpha,I}$. By repeating a small rotation we obtain arbitrary rotations. We just have to check that the set of allowed t above, for $\rho_{s'}(I)$ ($s' \in \mathbb{R}$), depends on the length of I and not on the position of $\rho_{s'}(I)$. Indeed, for any s', we can choose s so that $\rho_s\,(\rho_{s'}(I))$ is at the same fixed distance from ∞.

Automorphisms on S^1 commuting with rotations (respectively on $\mathbb{R}$ commuting with translations) are unitarily equivalent if and only if they are conjugated by an inner symmetry by Proposition 4.3.1 (respectively Proposition 4.3.3). An inner symmetry commutes with any diffeomorphism, on the other hand the correspondence $\alpha \mapsto \sigma_\alpha$ is constructed with composition with diffeomorphisms. From this it is immediate to see the last statement. $\square$

Let us conclude this subsection with a uniqueness result for maximal rational nets.

Theorem 4.3.7. *If a net $\mathcal{A}$ is completely rational and maximal, then it admits a unique KMS state, the geometric state φ_{geo}.*

Proof. We have seen in Cor. 4.2.4 that any primary KMS state φ on such a net is a composition of the geometric state with an automorphism α commuting with translations. Let us assume that φ were different from the geometric state. Then by Proposition 4.3.2, α must change the vacuum state ω. Then Lemma 4.3.5 would imply that all the automorphisms $\{\alpha_s\}$ are mutually unitarily inequivalent. From these automorphisms we could construct mutually inequivalent sectors by Proposition 4.3.6. This contradicts with the finiteness of the number of sectors in a completely rational net. Thus if a KMS state φ is primary, then it is the geometric state.

An arbitrary KMS state is a convex combination of primary KMS states [83], hence in this case the geometric state itself. □

4.3.2 General completely rational nets

Here we show the uniqueness of KMS state for general completely rational nets. In the previous section we have proved that any maximal completely rational net admits only the geometric state. One would naturally expect that, if one has an inclusion of nets with finite index, then every KMS state on the smaller net should extend to the larger net, thereafter the uniqueness would follow from the uniqueness for maximal nets. Unfortunately I am not aware of such a general statement. Instead, we will see that if we have a KMS state then its thermal completion admits some KMS state. We repeat this procedure and arrive at the maximal net, where any KMS state is geometric, and find that the initial state was in fact geometric as well.

Extension trick

Let $\mathcal{A}$ be a completely rational net and φ be a KMS state on $\mathcal{A}$. In this case, as we saw in Theorem 4.2.3, the thermal completion $\hat{\mathcal{A}}_\varphi$ of $\mathcal{A}$ with respect to φ is identified with an extension of the net $\mathcal{A}$. The objective here is to construct another KMS state on $\hat{\mathcal{A}}_\varphi$.

By Lemma 4.2.2 and eq. 4.4, $\mathcal{A}_\varphi(a,b) \subset \hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b})$ is an irreducible finite index inclusion for each interval (a,b); therefore, there is a unique conditional expectation

$$E_{(a,b)} : \hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b}) \longmapsto \mathcal{A}_\varphi(a,b).$$

It is easy to see that this is a consistent family w.r.t. inclusions of intervals. We denote simply by E the map defined on the closed union $\overline{\bigcup_{I\Subset\mathbb{R}_+} \hat{\mathcal{A}}_\varphi(I)}^{\|\cdot\|}$. Let us define the state

$$\hat{\omega} = \omega \circ \mathrm{Exp} \circ {\pi_\varphi}^{-1} \circ E \tag{4.11}$$

on $\overline{\bigcup_{I\Subset\mathbb{R}_+} \hat{\mathcal{A}}_\varphi(I)}^{\|\cdot\|}$. We will show that $\hat{\omega}$ is a KMS state with respect to dilations. We collected general remarks in Sections 4.1.2 and 4.1.3.

First of all, we recall that the original net $\mathcal{A}$ in the vacuum representation is diffeomorphism covariant. Even in the GNS representation π_φ with respect to φ, as explained at the beginning of Section 4.2, local diffeomorphisms act covariantly on each intervals, implemented by U_φ: $U_\varphi(\eta)\mathcal{A}_\varphi(I)U_\varphi(\eta)^* = \mathcal{A}_\varphi(\eta(I))$. Since the extended net $\mathcal{A}^d_\varphi$ is defined as the relative commutant $\mathcal{A}^d_\varphi(a,b) := \mathcal{A}_\varphi(a,\infty) \cap \mathcal{A}_\varphi(b,\infty)'$, local diffeomorphisms U_φ respect the structure of intervals:

$$U_\varphi(\eta)\mathcal{A}^d_\varphi(a,b)U_\varphi(\eta)^* = \mathcal{A}^d_\varphi(\eta(a),\eta(b)).$$

In particular, if a diffeomorphism η preserves an interval of finite length I, then it acts on $\mathcal{A}^d_\varphi(I)$ as an automorphism.

On the original net $\mathcal{A}$, we know that the modular automorphism of $\mathcal{A}(I)$ with respect to the vacuum ω acts as the dilation associated to $I = (a,b)$. On $\mathcal{A}(I)$ such dilation can be implemented by local diffeomorphisms η_t. In fact, the dilation preserves I, hence it is enough to modify this outside I so that the support is compact. If we restrict $\hat{\omega} = \omega\circ\mathrm{Exp}\circ\pi_\varphi^{-1}\circ E$ to $\mathcal{A}_\varphi(a,b)$, where Exp is defined in Prop. 1.3.4, the modular automorphism is

$$(\pi_\varphi \circ \mathrm{Exp}^{-1}) \circ \mathrm{Ad}U(\delta_t^{\exp I}) \circ (\mathrm{Exp} \circ \pi_\varphi^{-1}),$$

where $\delta_t^{\exp I}$ is the dilation associated to $(e^{2\pi a}, e^{2\pi b})$. Take diffeomorphisms η_t with the condition specified above and notice that, although exp and log are diffeomorphisms only locally, $\log\circ\eta_t\circ\exp$ are global diffeomorphisms. It holds on $\mathcal{A}_\varphi(a,b)$ that

$$\begin{aligned}(\pi_\varphi \circ \mathrm{Exp}^{-1}) \circ \mathrm{Ad}U(\delta_t^{\exp I}) \circ (\mathrm{Exp} \circ \pi_\varphi^{-1}) &= (\pi_\varphi \circ \mathrm{Exp}^{-1}) \circ \mathrm{Ad}U(\eta_t) \circ (\mathrm{Exp} \circ \pi_\varphi^{-1}) = \\ &= \pi_\varphi \circ \mathrm{Ad}U(\log\circ\eta_t\circ\exp)\circ\pi_\varphi^{-1} = \mathrm{Ad}U_\varphi(\log\circ\eta_t\circ\exp).\end{aligned}$$

By Lemma 4.1.7, we see that $\mathrm{Ad}(U_\varphi(\log\circ\eta_t\circ\exp))$ is the modular automorphism of $\hat{\mathcal{A}}_\varphi(I)$ with respect to $\hat{\omega}$. Let us assume that there is a sequence of local diffeomorphisms $\zeta_t^{I_n}$ supported in $\mathbb{R}_+$ whose actions on $I_n := [\frac{1}{n}, n]$ are dilation by e^t. The adjoint action $\mathrm{Ad}(U_\varphi(\log\circ\zeta_t^{I_n}\circ\exp))$ of diffeomorphisms on a local algebra $\hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b})$ is determined by the action of $\zeta_t^{I_n}$ on $(e^{2\pi a}, e^{2\pi b})$, hence we can consider the limit of these adjoint actions and we denote it by σ_t.

On the other hand, translation on $\mathcal{A}_\varphi(a,b)$ is implemented by unitaries $V_\varphi(t)$ in this GNS representation (note that a translation is not a local diffeomorphism, hence we cannot define the representative through π_φ). This in turn shows that $\mathrm{Ad}(V_\varphi(t))$ takes $\hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b})$ to $\hat{\mathcal{A}}_\varphi(e^{2\pi(a+t)}, e^{2\pi(b+t)})$, by recalling the definition of $\hat{\mathcal{A}}_\varphi$.

We show that the two actions $\mathrm{Ad}(V_\varphi(t))$ and σ_t are the same even on the thermal completion $\hat{\mathcal{A}}_\varphi$. In fact, these two actions take $\hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b})$ to $\hat{\mathcal{A}}_\varphi(e^{2\pi(a+t)}, e^{2\pi(b+t)})$, hence the composition $\mathrm{Ad}(V_\varphi(t))\circ\sigma_t^{-1}$ is an automorphism of $\hat{\mathcal{A}}_\varphi(e^{2\pi a}, e^{2\pi b})$ and σ-weakly continuous. It is obvious that this composition acts identically on $\mathcal{A}_\varphi(a,b)$, by considering the two actions in the original representation, and if $t = 0$ it is the identity. Then by Lemma 4.1.8, second statement, it is constant for all t.

Proposition 4.3.8. *The state $\hat{\omega}$ on $\overline{\bigcup_{I \Subset \mathbb{R}_+} \hat{\mathcal{A}}_\varphi(I)}^{\|\cdot\|}$ defined in (4.11) is a KMS state with respect to dilations.*

Proof. To apply the general statement of Proposition 4.1.14 to the inclusion of factors $\hat{\mathcal{A}}_\varphi(\frac{1}{2}, 2) \subset \hat{\mathcal{A}}_\varphi(\frac{1}{3}, 3) \subset \cdots \hat{\mathcal{A}}_\varphi(\frac{1}{n}, n) \subset \cdots$, and $\hat{\omega}$, we need to confirm that for each interval $I \Subset \mathbb{R}_+$ the action of the modular automorphisms of $\hat{\mathcal{A}}_\varphi(\frac{1}{n}, n)$ with respect to $\hat{\omega}$ (for sufficiently large n) on $\hat{\mathcal{A}}_\varphi(I)$ is *-strongly convergent and the limit is normal. As remarked above, the action of the modular automorphisms is implemented by local diffeomorphisms U_φ and by Section 4.1.4 we may assume that these diffeomorphisms $\eta_t^{I_n}$ are smoothly convergent. Then the representatives $U_\varphi(\log \circ \eta_t^{I_n} \circ \exp)$ are strongly convergent, hence their adjoint actions are *-strongly convergent as well, and the limit is normal.

Moreover, in this way we find diffeomorphisms $\zeta_t^I = \lim_n \eta^{I_n}$ which appeared in the previous remarks. Thus, when I tends to $(0, \infty)$, the limit of these adjoint actions $\mathrm{Ad}(U_\varphi(\log \circ \zeta_t^I \circ \exp))$ is σ_t, which in turn is equal to $\mathrm{Ad}V_\varphi(t)$. □

Proof of uniqueness

We continue to use the same notations as in Section 4.3.2.

Lemma 4.3.9. *The extended state $\hat{\omega}$ is the vacuum if and only if φ is the geometric KMS state.*

Proof. If φ is geometric, then, as we saw in Section 1.3.2, we have $\pi_{\varphi_{\mathrm{geo}}} = \mathrm{Exp}$ and the conditional expectation E is trivial. Hence $\hat{\omega} = \omega \circ \mathrm{Exp} \circ \mathrm{Exp}^{-1} = \omega$.

Conversely, suppose that $\hat{\omega}$ is the vacuum of the extended net. We note that

$$\hat{\omega}|_{\mathcal{A}_\varphi(a,b)} = \omega \circ \mathrm{Exp} \circ {\pi_\varphi}^{-1}|_{\mathcal{A}_\varphi(a,b)} = \varphi_{\mathrm{geo}} \circ {\pi_\varphi}^{-1}|_{\mathcal{A}_\varphi(a,b)},$$

but the vacuum of the extended net is the vector state $\langle \Phi, \cdot \Phi \rangle$; when restricted to $\mathcal{A}_\varphi(a, b)$ we have $\langle \Phi, \cdot \Phi \rangle = \varphi \circ {\pi_\varphi}^{-1}(\cdot)$. Hence this in turn means that the initial KMS state φ is in fact φ_{geo}. □

Theorem 4.3.10. *Any completely rational net $\mathcal{A}$ admits only the geometric KMS state φ_{geo}.*

Proof. Any completely rational net has only finitely many irreducible extensions with finite index. Let us consider a sequence of conformal extensions $\mathcal{A}_1 := \mathcal{A} \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_n$, where $\mathcal{A}_n$ is maximal. By the remarked finiteness of extensions, the number of such sequences is finite. Let $N_\mathcal{A}$ be the length of the longest sequence. If $\mathcal{A}$ is maximal, then $N_\mathcal{A}$ is 1. We will show the theorem by induction with respect to $N_\mathcal{A}$. For the case $N_\mathcal{A} = 1$ we have already proved the thesis in Theorem 4.3.7.

We assume that the proof is done for nets with $N_\mathcal{A} < k$. Let φ be a primary KMS state on $\mathcal{A}$, where $N_\mathcal{A} = k$. We perform the thermal completion $\hat{\mathcal{A}}_\varphi$ with respect to φ. If $\hat{\mathcal{A}}_\varphi$ is not a proper extension, the same reasoning as in Section 4.3.1 shows that $\varphi = \varphi_{\mathrm{geo}}$.

Hence we may assume that $\hat{\mathcal{A}}_\varphi$ is a proper extension of $\mathcal{A}$. Let $\hat{\omega}$ be the KMS state on $\overline{\bigcup_{I\Subset\mathbb{R}_+}\hat{\mathcal{A}}_\varphi(I)}^{\|\cdot\|}$ with respect to dilations of Prop. 4.3.8. Recall that there is a one-to-one correspondence between KMS states on the half-line with respect to dilation and KMS states on the real-line with respect to translation (see Proposition 1.3.7). By definition of N, $N_\mathcal{A} = k$ implies $N_{\hat{\mathcal{A}}_\varphi} < k$. It follows from the assumptions of induction that $\hat{\mathcal{A}}_\varphi$ admits only one KMS state on the half-line, hence $\hat{\omega}$ is the vacuum. In this case Lemma 4.3.9 tells us that the primary KMS state φ is the geometric state on $\mathcal{A}$. An arbitrary KMS state is a convex combination of primary states, hence it is necessarily geometric. This concludes the induction. $\square$

4.3.3 The uniqueness of KMS state for extensions

In this section we consider the following situation. Let $\mathcal{A} \subset \mathcal{B}$ be a finite-index inclusion of conformal nets. We assume that $\mathcal{A}$ admits a unique KMS state. Any conformal net has the geometric KMS state φ_{geo}, hence the unique state is this. We will show that the geometric state on $\mathcal{A}$ extends only to the geometric state on $\mathcal{B}$; in other words $\mathcal{B}$ admits a unique KMS state, too.

We note that the construction of the geometric KMS state works for any diffeomorphism covariant net (thus relatively local w.r.t. the Virasoro subnet). The result in this section is true if $\mathcal{B}$ is not necessarily local. We will use this fact for the analysis of two-dimensional conformal nets in next section.

Theorem 4.3.11. *If $\mathcal{A}$ admits a unique KMS state and $\mathcal{A} \subset \mathcal{B}$ is of finite index, then $\mathcal{B}$ admits a unique KMS state as well (which is again the geometric state).*

Proof. Let φ_0 be the unique KMS state of $\mathcal{A}$, namely the geometric state of $\mathcal{A}$. By construction, with E the unique conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$, the geometric KMS state φ of $\mathcal{B}$ satisfies

$$\varphi = \varphi_0 \circ E \ .$$

Let ψ be a KMS state on $\mathcal{B}$. By the uniqueness of the KMS state on $\mathcal{A}$ we have

$$\psi|_\mathcal{A} = \varphi_0 = \varphi|_\mathcal{A} \ .$$

Let $\lambda > 0$ be the Pimsner-Popa bound for E, we have

$$\varphi(x) = \varphi_0 \circ E(x) = \psi \circ E(x) \geq \lambda\psi(x)$$

for all positive elements $x \in \mathcal{B}$. Therefore ψ is dominated by φ. As φ is extremal, being the geometric KMS state, we then have $\psi = \varphi$. $\square$

4.4 KMS states for two-dimensional nets

Here we consider two-dimensional conformal nets (see Section 1.1.8).

Given a Möbius covariant net $\mathcal{A}$ on $\mathcal{M}$ and a bounded interval $I \subset \mathrm{L}_+$ we set

$$\mathcal{A}_+(I) \equiv \bigcap_{\mathcal{O}=I\times J} \mathcal{A}(\mathcal{O}) \tag{4.12}$$

(intersection over all intervals $J \subset \mathrm{L}_-$), and analogously define $\mathcal{A}_-$ (see also Section 5.2.1). By identifying $\mathrm{L}_\pm$ with $\mathbb{R}$ we then get two local nets $\mathcal{A}_\pm$ on $\mathbb{R}$, *the chiral components of* $\mathcal{A}$. They extend to local nets on S^1 which satisfy the axioms of Möbius covariant local nets, but for the cyclicity of Ω. We shall also denote $\mathcal{A}_\pm$ by $\mathcal{A}_R$ and $\mathcal{A}_L$. By the Reeh-Schlieder theorem the cyclic subspace $\mathcal{H}_\pm \equiv \overline{\mathcal{A}_\pm(I)\Omega}$ is independent of the interval $I \subset \mathrm{L}_\pm$ and $\mathcal{A}_\pm$ restricts to a (cyclic) Möbius covariant local net on the Hilbert space $\mathcal{H}_\pm$. Since Ω is separating for every $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, the map $X \in \mathcal{A}_\pm(I) \mapsto X \restriction \mathcal{H}_\pm$ is an isomorphism for any interval I, so we will often identify $\mathcal{A}_\pm$ with its restriction to $\mathcal{H}_\pm$.

Proposition 4.4.1. *Let $\mathcal{A}$ be a local conformal net on $\mathcal{M}$. Setting $\mathcal{A}_0(\mathcal{O}) \equiv \mathcal{A}_+(I_+) \vee \mathcal{A}_-(I_-)$, $\mathcal{O} = I_+ \times I_-$, then $\mathcal{A}_0$ is a conformal, irreducible subnet of $\mathcal{A}$. There exists a consistent family of vacuum preserving conditional expectations $\epsilon_\mathcal{O} : \mathcal{A}(O) \to \mathcal{A}_0(\mathcal{O})$ and the natural isomorphism from the product $\mathcal{A}_+(I_+) \cdot \mathcal{A}_-(I_-)$ to the algebraic tensor product $\mathcal{A}_+(I_+) \odot \mathcal{A}_-(I_-)$ extends to a normal isomorphism between $\mathcal{A}_+(I_+) \vee \mathcal{A}_-(I_-)$ and $\mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-)$.*

Thus we may identify $\mathcal{H}_+ \otimes \mathcal{H}_-$ with $\mathcal{H}_0 \equiv \overline{\mathcal{A}_0(\mathcal{O})\Omega}$ and $\mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-)$ with $\mathcal{A}_0(\mathcal{O})$.

Let $\mathcal{A}$ be a local conformal net on the two-dimensional Minkowski spacetime $\mathcal{M}$. We shall say that $\mathcal{A}$ is **completely rational** if the two associated chiral nets $\mathcal{A}_\pm$ in (4.12) are completely rational.

Proposition 4.4.2. *If $\mathcal{A}$ is completely rational the following three conditions hold:*

a) Haag duality on $\mathcal{M}$. *For any double cone $\mathcal{O}$ we have $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$. Here $\mathcal{O}'$ is the causal complement of $\mathcal{O}$ in $\mathcal{M}$*

b) Split property. *If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and the closure $\bar{\mathcal{O}}_1$ of $\mathcal{O}_1$ is contained in $\mathcal{O}_2$, the natural map $\mathcal{A}(\mathcal{O}_1) \cdot \mathcal{A}(\mathcal{O}_2)' \to \mathcal{A}(\mathcal{O}_1) \odot \mathcal{A}(\mathcal{O}_2)'$ extends to a normal isomorphism $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)' \to \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)'$.*

c) Finite μ-index. *Let $E = \mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{M}$ be the union of two double cones $\mathcal{O}_1, \mathcal{O}_2$ such that $\bar{\mathcal{O}}_1$ and $\bar{\mathcal{O}}_2$ are spacelike separated. Then the Jones index $[\mathcal{A}(E')' : \mathcal{A}(E)]$ is finite. This index is denoted by $\mu_\mathcal{A}$,* the μ-index of $\mathcal{A}$.

Proof. One immediately checks that the three properties $a), b), c)$ are satisfied for the two-dimensional net $\mathcal{A}_0 = \mathcal{A}_+ \otimes \mathcal{A}_-$ which is completely rational. Then $\mathcal{A}$ is an irreducible extension of $\mathcal{A}_0$ (see [53]) that must be of finite-index, and this implies that $\mathcal{A}$ satisfies $a), b), c)$ too, by the same arguments as in the chiral case, cf. [52]. $\square$

With $\mathcal{A}$ a local conformal net as above, we consider the quasi-local C^*-algebra $\mathfrak{A} \equiv \overline{\cup_{\mathcal{O}\in\mathcal{K}}\mathcal{A}(\mathcal{O})}$ (norm closure) and the time translation one-parameter automorphism group τ of $\mathfrak{A}$. We have

Theorem 4.4.3. *If $\mathcal{A}$ is completely rational, there exists a unique KMS state φ of $\mathfrak{A}$ w.r.t. τ. φ is the lift by the conditional expectation of the geometric KMS state of $\mathcal{A}_0$.*

The proof of the theorem follows by the above discussion and Thm. 4.3.11. One can easily see that φ is a geometric state too. We need the following proposition.

Proposition 4.4.4. *Let $\mathcal{A}_+$ $\mathcal{A}_-$ be translation covariant nets of von Neumann algebras on $\mathbb{R}$ and $\mathcal{A}_0$ the associated net on the two-dimensional Minkowski spacetime: $\mathcal{A}_0(I_+ \times I_-) \equiv \mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-)$. If φ_0 is an extremal KMS state of $\mathcal{A}_0$ w.r.t. time translations, then $\varphi_0 = \varphi_+ \otimes \varphi_-$, where $\varphi_\pm$ is an extremal KMS state of $\mathcal{A}_\pm$ w.r.t. translations.*

Proof. Let π_{φ_0} be the GNS representation of $\mathfrak{A}_{\mathcal{A}_0}$ w.r.t. φ_0 and consider the von Neumann algebras $\mathcal{M}_0 \equiv \pi_{\varphi_0}(\mathfrak{A}_{\mathcal{A}_0})''$ and $\mathcal{M}_\pm \equiv \pi_{\varphi_0}(\mathfrak{A}_{\mathcal{A}_\pm})''$. As π_{φ_0} is extremal KMS, $\mathcal{M}_0$ is a factor, so $\mathcal{M}_+$ and $\mathcal{M}_-$ are commuting subfactors.

Now the translation one-parameter automorphism group of $\mathfrak{A}_{\mathcal{A}_0}$ extends to the modular group of $\mathcal{M}_0$ w.r.t. (the extension of) φ_0 and leaves the subfactors $\mathcal{M}_\pm$ globally invariant. By Takesaki theorem, there exists a normal φ_0-invariant conditional expectation $\varepsilon_\pm : \mathcal{M}_0 \to \mathcal{M}_\pm$. With $x_\pm \in \mathcal{M}_\pm$ we have

$$\varphi_0(x_-x_+) = \varphi_0(\varepsilon_-(x_-x_+)) = \varphi_0(x_-\varepsilon_-(x_+)) = \varphi_0(x_-)\varphi_0(x_+) = \varphi_-(x_-)\varphi_+(x_+) \ ,$$

because $\varepsilon_-(x_+)$ belongs to the center of $\mathcal{M}_-$, so $\varepsilon_-(x_+) = \varphi_0(x_+)$. This concludes the proof. □

As a consequence, if $\mathcal{A}_\pm$ are completely rational, then $\mathcal{A}_0$ admits a unique KMS state w.r.t. time translations and this state is given by the geometric construction.

4.5 Preliminaries for non-rational models

From now on, we study KMS states on non-rational conformal nets. For this purpose, we need to extend the Araki-Haag-Kaster-Takesaki theorem to locally normal systems. This can be done in a general context (not necessarily for nets on S^1), hence we put an extended notion of net.

4.5.1 Net of von Neumann algebras on a directed set

Axioms and further properties

Let $\mathcal{I}$ be a directed set. We always assume that there is a countable subset $\{I_i\}_{i\in\mathbb{N}} \subset \mathcal{I}$ with $I_i \prec I_{i+1}$ of indices such that for any index I there is some i such that $I \prec I_i$. A **net (of von Neumann algebras)** $\mathcal{A}$ on $\mathcal{I}$ assigns a von Neumann algebra $\mathcal{A}(I)$ to each element I of $\mathcal{I}$ and satisfies the following conditions:

- **(Isotony)** If $I \prec J$ then $\mathcal{A}(I) \subset \mathcal{A}(J)$.

- **(Covariance)** There is a strongly-continuous unitary representation U of $\mathbb{R}$ and an order-preserving action of $\mathbb{R}$ on $\mathcal{I}$ such that

$$U(t)\mathcal{A}(I)U(t)^* = \mathcal{A}(t \cdot I),$$

  and for any index I and for any compact set $C \Subset \mathbb{R}$, there is another index I_C such that $t \cdot I \prec I_C$ for $t \in C$.

Since the net $\mathcal{A}$ is directed, it is natural to consider the norm-closed union of $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$. We simply denote

$$\mathfrak{A} = \overline{\bigcup_{I\in\mathcal{I}} \mathcal{A}(I)}^{\|\cdot\|}$$

and call it the **quasilocal algebra**. Each algebra $\mathcal{A}(I)$ is referred to as a **local algebra**. The adjoint action $\mathrm{Ad}U(t)$ naturally extends to an automorphism of the quasilocal algebra $\mathfrak{A}$. We denote by τ_t this action of $\mathbb{R}$ and call it **translation** (note that in this Section τ_t is a one-parameter family of automorphisms, although in first Sections where we assumed diffeomorphism covariance, we denoted it by $\mathrm{Ad}U(\tau_t)$ to unify the notation).

An **automorphism of the net** $\mathcal{A}$ (not just of $\mathfrak{A}$) is a family $\{\gamma_I\}$ of automorphisms of local algebras $\{\mathcal{A}(I)\}$ such that if $I \prec J$ then $\gamma_J|_{\mathcal{A}(I)} = \gamma_I$. Such an automorphism extends by norm continuity to the quasilocal algebra $\mathfrak{A}$ which preserves all the local algebras. Conversely, any automorphism of $\mathfrak{A}$ which preserves each local algebra can be described as an automorphism of the net $\mathcal{A}$.

A net $\mathcal{A}$ is said to be **asymptotically γ-abelian** if there is an automorphism γ of the quasilocal C^*-algebra $\mathfrak{A}$ implemented by a unitary operator $U(\gamma)$ such that

- γ is normal on each local algebra $\mathcal{A}(I)$ and maps it into another local algebra $\mathcal{A}(J)$.
- for any pair of indices I, J there is sufficiently large n such that $\mathcal{A}(I)$ and $\mathcal{A}(\gamma^n \cdot J) = U(\gamma)^n\mathcal{A}(J)(U(\gamma)^*)^n$ commute,
- γ and τ_t commute.

It is also possible (and in many cases more natural) to consider a one-parameter group $\{\gamma_s\}$ of automorphisms for the notion of asymptotic γ-abelianness (and weakly γ-clustering, see below). In that case, we assume that $\{\gamma_s\}$ is implemented by a strongly-continuous family $\{U(\gamma_s)\}$ and the corresponding conditions above can be naturally translated.

We say that a net $\mathcal{A}$ is **split** if, for the countable set $\{I_i\}$ in the definition of the net, there are type I factors $\{\mathcal{F}_i\}$ such that $\mathcal{A}(I_i) \subset \mathcal{F}_i \subset \mathcal{A}(I_{i+1})$. Note that in this case the argument in the appendix of [54] applies.

Examples of net

The definition of nets looks quite general, but we have principally two types of examples in mind.

The first comes from the nets on the circle S^1 which we recalled in Section 1.1.1. If $\mathcal{A}$ is a conformal net on S^1, we consider the restriction $\mathcal{A}|_{\mathbb{R}}$ with the family of all finite intervals in $\mathbb{R}$ as the index set. The translations in the present setting are the ordinary translations. If we take a finite translation as γ, this system is asymptotically γ-abelian. To consider split property, we can take the sequence of intervals $I_n = (-n, n)$.

The second type is a net of observables on Minkowski space $\mathbb{R}^d$ (see Section 1.1.8 for two-dimensional case and [46] for a general account). In this case the index set is the family of bounded open sets in $\mathbb{R}^d$. The group of translations in some fixed timelike direction plays the role of "translations", while a fixed spacelike translation plays the role of γ. The net satisfies asymptotic γ-abelianness.

In both cases, it is natural to consider the continuous group γ_s of (space-)translations for the notion of γ-abelianness.

4.5.2 States on a net

For a C^*-algebra $\mathfrak{A}$ and a one-parameter automorphism group $\{\tau_t\}$, it is possible to consider KMS states on $\mathfrak{A}$ with respect to τ. Since our local algebras are von Neumann algebras, it is natural to consider locally normal objects. Let φ be a state on the quasilocal algebra $\mathfrak{A}$. It is said to be **locally normal** if each restriction of φ to a local algebra $\mathcal{A}(I)$ is normal. A β**-KMS state** φ on $\mathfrak{A}$ with respect to τ is a state with the following properties: for any $x, y \in \mathfrak{A}$, there is an analytic function f in the interior of $D_\beta := \{0 \le \Im z \le \beta\}$ where $\Im$ means the imaginary part, continuous on D_β, such that

$$f(t) = \varphi(x\tau_t(y)), f(t + i\beta) = \varphi(\tau_t(y)x).$$

The parameter $\frac{1}{\beta}$ is called the temperature. For completely raitonal nets we considered only the case $\beta = 1$ since our main subject were the conformal nets, in which case the phase structure is uniform with respect to β. Furthermore, we studied completely rational models and proved that they admit only one KMS state at each temperature. Also in the present Sections, the main examples are conformal, but these models admit continuously many different KMS states and it should be useful to give concrete formulae which involve also the temperature.

A KMS state φ is said to be **primary** if the GNS representation of $\mathfrak{A}$ with respect to φ is factorial, i.e., $\pi_\varphi(\mathfrak{A})''$ is a factor. Any KMS states can be decomposed into primary states [83, Theorem 4.5] in many practical situation, for example if the net is split or if each local algebra is a factor. Hence, to classify KMS states of a given system, it is enough to consider the primary ones.

If the net $\mathcal{A}$ comes from a conformal net on S^1, namely if we assume the diffeomorphism covariance, we saw that there is at least one KMS state, the geometric state φ_{geo} (Theorem 1.3.5). It is easy to obtain a formula for φ_{geo} with general temperature $\frac{1}{\beta}$. We exhibit it for later use: let $\omega := \langle \Omega, \cdot \Omega \rangle$ be the vacuum state, then $\varphi_{\mathrm{geo}} := \omega \circ \mathrm{Exp}_\beta$, where, for any $I \Subset \mathbb{R}$, $\mathrm{Exp}_\beta|_{\mathcal{A}(I)} = \mathrm{Ad}U(g_{\beta,I})|_{\mathcal{A}(I)}$ and $g_{\beta,I}$ is a diffeomorphism of $\mathbb{R}$ with compact support such that for $t \in I$ it holds that $g_{\beta,I}(t) = e^{\frac{2\pi t}{\beta}}$.

If φ is γ-invariant (invariant under an automorphism γ or a one-parameter group $\{\gamma_s\}$) and cannot be written as a linear combination of different locally normal γ-invariant states, then it is said to be **extremal γ-invariant**.

We denote the GNS representation of $\mathfrak{A}$ with respect to φ by π_φ, the Hilbert space by $\mathcal{H}_\varphi$ and the vector which implements the state φ by Ω_φ. If φ is invariant under the action of an automorphism τ_t (respectively γ, γ_s), we denote by $U_\varphi(t)$ (resp. $U_\varphi(\gamma)$, $U_\varphi(\gamma_s)$) the canonical unitary operator which implements τ_t (resp. γ, γ_s) and leaves Ω_φ invariant. If φ is locally normal, the GNS representation π_φ is locally normal as well, namely the restriction of π_φ to each $\mathcal{A}(I)$ is normal. Indeed, let us denote the restriction $\varphi_i := \varphi|_{\mathcal{A}(I_i)}$. The representation π_{φ_i} is normal on $\mathcal{A}(I_i)$. The Hilbert space is the increasing union of $\mathcal{H}_{\varphi_i}$ and the restriction of π_φ to $\mathcal{A}(I_i)$ on $\mathcal{H}_{\varphi_j}$ $(i \leq j)$ is π_{φ_j}, hence is normal. Then $\pi_\varphi|_{\mathcal{A}(I_i)}$ is normal.

Furthermore, the map $t \mapsto U_\varphi(t)$ is weakly (and hence strongly) continuous, since the one-parameter automorphism τ_t is weakly (or even *-strongly) continuous and $U_\varphi(t)$ is defined as the closure of the map

$$\pi_\varphi(x)\Omega_\varphi \longmapsto \pi_\varphi(\tau_t(x))\Omega_\varphi.$$

Thus the weak continuity of $t \mapsto U_\varphi(t)$ follows from the local normality of π_φ and boundedness of $U_\varphi(t)$, which follows from the invariance of φ. By the same reasoning, if there is a one-parameter family γ_s, the GNS implementation $U_\varphi(\gamma_s)$ is weakly continuous.

If for any locally normal γ-invariant state φ the algebra $E_0\pi_\varphi(\mathfrak{A})E_0$ is abelian, where E_0 is the projection onto the space of $U_\varphi(\gamma)$-invariant (resp. $\{U_\varphi(\gamma_s)\}$) vectors, then the net $\mathcal{A}$ is said to be **γ-abelian**.

A locally normal state φ on $\mathfrak{A}$ is said to be **weakly γ-clustering** if it is γ-invariant and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varphi(\gamma^n(x)y) = \varphi(x)\varphi(y).$$

for any pair of $x, y \in \mathfrak{A}$. For a one parameter group $\{\gamma_s\}$, we define γ-clustering by

$$\lim_{N\to\infty} \frac{1}{N}\int_0^N \varphi(\gamma_s(x)y)ds = \varphi(x)\varphi(y).$$

At the end of this subsection, we remark that, in our principal examples coming from conformal nets on S^1, KMS states are automatically locally normal by the following general result [83, Theorem 1].

Theorem 4.5.1 (Takesaki-Winnink). *Let $\mathcal{A}$ be a net such that $\mathcal{A}(I_i)$ are σ-finite properly infinite von Neumann algebras. Then any KMS-state on $\mathcal{A}$ is locally normal.*

If $\mathcal{A}$ is a conformal net on S^1 defined on a separable Hilbert space, then each local algebra $\mathcal{A}(I)$ is a type III_1 factor, in particular it is properly infinite, and obviously σ-finite, hence Theorem 4.5.1 applies.

4.5.3 Subnets and group actions

Let $\mathcal{A}$ and $\mathcal{B}$ be two nets with the same index set $\mathcal{I}$ acting on the same Hilbert space. If for each index I it holds $\mathcal{A}(I) \subset \mathcal{B}(I)$, then we say that $\mathcal{A}$ is a **subnet** of $\mathcal{B}$ and write simply $\mathcal{A} \subset \mathcal{B}$. We always assume that each inclusion of algebras has a normal conditional expectation $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ such that

- (Compatibility) For $I \prec J$ it holds that $E_J|_{\mathcal{B}(I)} = E_I$.
- (Covariance) $\tau_t \circ E_I = E_{t \cdot I} \circ \tau_t$, and

See [60] for a general theory on nets with a conditional expectation.

Principal examples come again from nets of observables on S^1. As remarked in Section 1.1.3, if we have an inclusion of nets on S^1 there is always a compatible and covariant family of expectations.

Another case has a direct relation with one of our main results. Let $\mathcal{A}$ be a net on $\mathcal{I}$ and assume that there is a family of *-strongly continuous actions $\alpha_{I,g}$ of a compact Lie group G on $\mathcal{A}(I)$ such that if $I \subset J$ then $\alpha_{J,g}|_{\mathcal{A}(I)} = \alpha_{I,g}$ and $\tau_t \circ \alpha_{I,g} = \alpha_{t \cdot I,g} \circ \tau_t$. By the first condition (compatibility of α) we can extend α to an automorphism of the quasilocal C^*-algebra $\mathfrak{A}$, and by the second condition (covariance of α) α and τ commute. Then for each index I we can consider the fixed point subalgebra $\mathcal{A}(I)^{\alpha_I} =: \mathcal{A}^\alpha(I)$. Then $\mathcal{A}^\alpha$ is again a net on $\mathcal{I}$. Furthermore, since the group is compact, there is a unique normalized invariant mean dg on G. Then it is easy to see that the map $E(x) := \int_G \alpha_g(x)dg$ is a locally normal conditional expectation $\mathcal{A} \to \mathcal{A}^\alpha$. The group G is referred to as the **gauge group** of the inclusion $\mathcal{A}^\alpha \subset \mathcal{A}$.

The *-strong continuity of the group action is valid, for example, when the group action is implemented by weakly (hence strongly) continuous unitary representation of G. In fact, if $g_n \to g$, then $U_{g_n} \to U_g$ strongly, hence $\alpha_{g_n}(x) = \mathrm{Ad}U_{g_n}(x) \to \mathrm{Ad}U_g(x)$ and $\alpha_{g_n}(x^*) = \mathrm{Ad}U_{g_n}(x^*) \to \mathrm{Ad}U_g(x^*)$ strongly since $\{U_{g_n}\}$ is bounded. This is the case, as are our principal examples, when the net is defined in the vacuum representation and the vacuum state is invariant under the action of G.

If the net $\mathcal{A}$ is asymptotically γ-abelian, then we always assume that γ commutes with α_g.

4.5.4 C^*-dynamical systems

A pair of a C^*-algebra $\mathfrak{A}$ and a pointwise norm-continuous one-parameter automorphism group α_t is called a **C^*-dynamical system**. The requirement of pointwise norm-continuity is strong enough to allow extensive general results. Although our main objects are not C^*-dynamical systems, we recall here a standard result.

All notions defined for nets, namely asymptotic γ-abelianness, γ-abelianness, weakly γ-clustering of states and inclusion of systems, and corresponding results in Section 4.6, except Corollary 4.6.6, have variations for C^*-dynamical system ([50], see also [9]). Among them, we will prove a counterpart (Theorem 4.6.9) for nets of the following [1, Theorem II.4].

Theorem 4.5.2 (Araki-Haag-Kastler-Takesaki)**.** *Let $\mathfrak{A} \subset \mathfrak{B}$ be an inclusion of asymptotically γ-abelian C^*-dynamical systems and $\mathfrak{A}$ be the fixed point of $\mathfrak{B}$ with respect to a pointwise norm-continuous action α of a compact group G. Then, for any weakly γ-clustering extension ψ of a primary τ-KMS state φ on $\mathcal{A}$, there is a one-parameter subgroup χ in G such that ψ is a primary $\{\tau_t \circ \alpha_{\chi(t)}\}$-KMS state.*

4.5.5 Regularization

To classify the KMS states on Vir_1, we need to extend a KMS state on Vir_1 to $\mathcal{A}_{SU(2)_1}$ (explained below). Since Vir_1 is the fixed point subnet of $\mathcal{A}_{SU(2)_1}$ with respect to the action of $SU(2)$ [78], one would like to apply Theorem 4.5.2. The trouble is, however, that the theorem applies only to C^*-dynamical systems where the actions of the translation group and the gauge group are pointwise continuous in norm. The pointwise norm-continuity seems essential in the proof and it is not straightforward to modify it for locally normal systems; we instead aim to reduce our cases to C^*-dynamical systems.

More precisely, we assume that the net $\mathcal{A}$ has a locally *-strongly continuous action τ of translations (covariance, in subsection 4.5.1) and α of a gauge group G (subsection 4.5.3), has an automorphism γ (subsection 4.5.1) and they commute, then we construct a C^*-dynamical system $(\mathfrak{A}_\mathrm{r}, \tau)$ with the **regular subalgebra** $\mathfrak{A}_\mathrm{r}$ *-strongly dense in $\mathfrak{A}$.

Proposition 4.5.3. *For any net $\mathcal{A}$ with locally *-strongly continuous commuting actions τ of $\mathbb{R}$ and α of G and an automorphism γ commuting with both, there is a (τ, γ, G)-globally invariant *-strongly dense C^*-subalgebra $\mathfrak{A}_\mathrm{r}$ of the quasilocal algebra $\mathfrak{A}$ on which $\mathbb{R}$ and G act pointwise continuously in norm; any local element $x \in \mathcal{A}(I)$ can be approximated *-strongly by a bounded sequence from $\mathfrak{A}_\mathrm{r} \cap \mathcal{A}(I_C)$ for some $I_C \succ I$.*

If we consider a continuous action γ_s, then we can take $\mathfrak{A}_\mathrm{r}$ such that $\mathfrak{A}_\mathrm{r}$ is $\{\gamma_s\}$-invariant and the action of γ is pointwise continuous in norm.

Proof. Let x be an element of some local algebra $\mathcal{A}(I)$. We consider the smearing of x with a smooth function f on $\mathbb{R} \times G$ with compact support

$$x_f := \int_{\mathbb{R}\times G} f(t,g)\alpha_g(\tau_t(x))dtdg.$$

By the definition of net and the compactness of the support of f, the integrand belongs to another local algebra $\mathcal{A}(I_C)$ and the actions α and τ are normal on $\mathcal{A}(I_C)$, hence the (Bochner) integral can be defined. Smoothness of the actions on x_f is easily seen from the smoothness of f.

Take a sequence of functions approximating the Dirac distribution, i.e. a sequence of f_n with $\int_{\mathbb{R}\times G} f_n(x,g)dxdg = 1$ and whose supports shrink to the unit element in the group $\mathbb{R} \times G$, then x_{f_n} converges *-strongly to x, since group actions α and τ are *-strongly continuous by assumption. Thus, any element x in a local algebra $\mathcal{A}(I)$ can be approximated by a bounded sequence in a slightly larger local algebra $\mathcal{A}(I_C)$.

We take $\mathfrak{A}_{\mathrm{r}}$ as the C^*-algebra generated by $\{x_f\}$. Global invariance follows since τ, γ and α commute. As any element in $\mathfrak{A}$ can be approximated in norm (and a fortiori *-strongly) by local elements, $\mathfrak{A}_{\mathrm{r}}$ is *-strongly dense in $\mathfrak{A}$.

For a continuous action γ_s, it is enough to consider a smearing on $\mathbb{R} \times G \times \mathbb{R}$ with respect to the action of $\tau \times \alpha \times \gamma$. □

Remark 4.5.4. If $\mathcal{A}$ is the fixed point subnet of $\mathcal{B}$ in the sense of Section 4.5.3, then the fixed point subalgebra $\mathfrak{B}_{\mathrm{r}}^{\alpha}$ of $\mathfrak{B}_{\mathrm{r}}$ is included in $\mathcal{A}$ and $\mathfrak{A}_{\mathrm{r}}$ is included in $\mathfrak{B}_{\mathrm{r}}^{\alpha}$. The action of τ on $\mathfrak{B}_{\mathrm{r}}^{\alpha}$ is pointwise continuous, we obtain an inclusion of C^*-dynamical systems $\mathfrak{B}_{\mathrm{r}}^{\alpha} \subset \mathfrak{B}_{\mathrm{r}}$. The smaller algebra $\mathfrak{A}_{\mathrm{r}}$ has the desired approximation for $\mathcal{A}$, so does $\mathfrak{B}_{\mathrm{r}}^{\alpha}$.

Lemma 4.5.5. *If a state φ on the net $\mathcal{A}$ is weakly γ-clustering, then the restriction of φ to the regular system $(\mathfrak{A}_{\mathrm{r}}, \tau)$ is again weakly γ-clustering.*

Proof. The definition of weakly γ-clustering of a smaller algebra $\mathfrak{A}_{\mathrm{r}}$ refers only to elements in $\mathfrak{A}_{\mathrm{r}}$, hence it is weaker than the counterpart for $\mathfrak{A}$. □

Lemma 4.5.6. *Let φ be a locally normal state on $\mathfrak{A}$ which is a KMS state on $\mathfrak{A}_{\mathrm{r}}$. Then φ is a KMS state on $\mathfrak{A}$.*

Proof. We only have to confirm the KMS condition for $\mathfrak{A}$. Let $x, y \in \mathfrak{A}$ and take bounded sequences $\{x_n\}, \{y_n\}$ from $\mathfrak{A}_{\mathrm{r}}$ which approximate x, y *-strongly. Since φ is a KMS state on $\mathfrak{A}_{\mathrm{r}}$, there is an analytic function f_n such that

$$\begin{aligned} f_n(t) &= \varphi(x_n \tau_t(y_n)), \\ f_n(t+i) &= \varphi(\tau_t(y_n) x_n). \end{aligned}$$

In terms of GNS representation with respect to φ, these functions can be written as

$$\begin{aligned} \varphi(x_n \tau_t(y_n)) &= \langle \pi_\varphi(x_n^*)\Omega_\varphi, U_\varphi(t)\pi_\varphi(y_n)\Omega_\varphi \rangle, \\ \varphi(\tau_t(y_n) x_n) &= \langle U_\varphi(t)\pi_\varphi(y_n^*)\Omega_\varphi, \pi_\xi(x_n)\Omega_\varphi \rangle. \end{aligned}$$

Note that $\pi_\varphi(x_n)$ (respectively $\pi_\varphi(y_n)$) is *-strongly convergent to $\pi_\varphi(x)$ (resp. $\pi_\varphi(y)$) since the sequence $\{x_n\}$ (resp. $\{y_n\}$) is bounded. Let us denote a common bound of norms by M. We can estimate the difference as follows:

$$\begin{aligned} |\varphi(x\tau_t(y)) - \varphi(x_n\tau_t(y_n))| &= |\langle \pi_\varphi(x^*)\Omega_\varphi, U_\varphi(t)\pi_\varphi(y)\Omega_\varphi \rangle - \langle \pi_\varphi(x_n^*)\Omega_\varphi, U_\varphi(t)\pi_\varphi(y_n)\Omega_\varphi \rangle| \\ &\leq M \|\pi_\varphi(x^*) - \pi_\varphi(x_n^*)\Omega_\varphi\| + M \|\pi_\varphi(y) - \pi_\varphi(y_n)\Omega_\varphi\| \end{aligned}$$

and this converges to 0 uniformly with respect to t. Analogously we see that $\varphi(\tau_t(y_n)x_n)$ converges to $\varphi(\tau_t(y_n)x_n)$ uniformly. Then by the three-line theorem $f_n(z)$ is uniformly convergent on the strip $0 \leq \Im z \leq 1$ and the limit f is an analytic function. Obviously f connects $\varphi(x\tau_t(y))$ and $\varphi(t_t(y)x)$, hence φ satisfies the KMS condition for $\mathfrak{A}$. □

4.6 Extension results

In this section we exhibit variations of standard results on C^*-dynamical systems. Parts of proofs of Proposition 4.6.2, Lemma 4.6.7 and Proposition 4.6.8 are adaptations of [50], although we need local normality at several points and we exhibit proofs. In particular, if we implement several notions with one-parameter group $\{\gamma_s\}$, we need local normality to assure the weak-continuity of the GNS implementation $\{U_\varphi(\gamma_s)\}$. For some propositions we need split property in the relevance with local normality.

Remark 4.6.1. If we treat one-parameter group $\{\gamma_s\}$, in the following propositions (except Proposition 4.6.5: The corresponding modification shall be explicitly indicated) it is enough just to take the von Neumann algebra $(\pi_\varphi(\mathfrak{A}) \cup \{U_\varphi(\gamma_s)\})''$ and to consider invariance under $\{\gamma_s\}$ or $\{U_\varphi(\gamma_s)\}$ and the corresponding notion of γ-clustering property of states. Since $\{U_\varphi(\gamma_s)\}$ is weakly continuous, we can utilize the mean ergodic theorem in this case as well.

Proposition 4.6.2. *A state φ is extremal γ-invariant if and only if $(\pi_\varphi(\mathfrak{A}) \cup \{U_\varphi(\gamma)\})'' = B(\mathcal{H}_\varphi)$. If φ is locally normal, γ-invariant and not extremal γ-invariant, then φ decomposes into a convex combination of locally normal γ-invariant states.*

Proof. First, let us assume that $(\pi_\varphi(\mathfrak{A}) \cup \{U_\varphi(\gamma)\})'' \neq B(\mathcal{H}_\varphi)$. Then, there is a nontrivial projection P in the commutant and $P\Omega_\varphi \neq 0$, as by definition Ω_φ is cyclic for $\pi_\varphi(\mathfrak{A})$ and separating for $\pi_\varphi(\mathfrak{A})'$. By applying the same argument to $\mathbb{1} - P$, we see that $(\mathbb{1} - P)\Omega_\varphi \neq 0$. Let us define the states

$$\begin{aligned}
\varphi_1(\cdot) &:= \frac{1}{q}\langle P\Omega_\varphi, \pi_\varphi(\cdot)P\Omega_\varphi\rangle, \\
\varphi_2(\cdot) &:= \frac{1}{1-q}\langle(\mathbb{1} - P)\Omega_\varphi, \pi_\varphi(\cdot)(\mathbb{1} - P)\Omega_\varphi\rangle,
\end{aligned}$$

where $q = \|P\Omega_\varphi\|^2$ (and $1 - q = \|(\mathbb{1} - P)\Omega_\varphi\|^2$). Since P commutes with the representative $U_\varphi(\gamma)$ of the automorphism γ, $P\Omega_\varphi$ and $(\mathbb{1} - P)\Omega_\varphi$ are invariant under $U_\varphi(\gamma)$, so φ_1 and φ_2 are γ-invariant.

It is obvious that $\varphi = q\varphi_1 + (1 - q)\varphi_2$. We will show that $\varphi_1 \neq \varphi \neq \varphi_2$, from which it follows that φ is not extremal γ-invariant. Let us assume the contrary, namely that $\varphi = \varphi_1$ (and thus $\varphi = \varphi_2$). The vector $\Psi := P\Omega_\varphi - q\Omega_\varphi \neq 0$ is orthogonal to $\pi_\varphi(\mathfrak{A})\Omega_\varphi$,

$$\begin{aligned}
\langle\Psi, \pi_\varphi(x)\Omega_\varphi\rangle &= \langle P\Omega_\varphi - q\Omega_\varphi, \pi_\varphi(x)\Omega_\varphi\rangle = \langle P\Omega_\varphi, \pi_\varphi(x)P\Omega_\varphi\rangle - q\langle\Omega_\varphi, \pi_\varphi(x)\Omega_\varphi\rangle = \\
&= q\varphi_1(x) - q\varphi(x) = 0,
\end{aligned}$$

but this contradicts the cyclicity of Ω_φ. It is obvious that φ_1 and φ_2 are locally normal if φ is locally normal, since π_φ is locally normal.

Next we show that, if $(\pi_\varphi(\mathfrak{A}) \cup \{U_\varphi(\gamma)\})'' = B(\mathcal{H}_\varphi)$, then φ is extremal γ-invariant. Let $\varphi = \varphi_1 + \varphi_2$, we will see that φ, φ_1 and φ_2 are proportional.

We can define a linear map from the GNS representation space of φ to that of φ_1 using the correspondence

$$W : \pi_\varphi(x)\Omega_\varphi \longmapsto \pi_{\varphi_1}(x)\Omega_{\varphi_1},$$

since $\varphi \geq \varphi_1$ as positive linear functionals, W is well-defined on $\mathcal{H}_\varphi$ and is a contraction which intertwines π_φ and π_{φ_1}. It follows that, if $W = VA$ is the polar decomposition, A is a positive contraction on $\mathcal{H}_\varphi$ commuting with $\pi_\varphi(\mathfrak{A})$ and V is a partial isometry intertwining π_φ and π_{φ_1}. We can check that $A\Omega_\varphi$ implements φ_1 in the representation π_φ and that $U_\varphi(\gamma)AU_\varphi(\gamma)^*\Omega_\varphi$ implements the same state (note that $U_\varphi(\gamma)AU_\varphi(\gamma)^* \in \pi_\varphi(\mathfrak{A})'$):

$$\begin{aligned}\langle U_\varphi(\gamma)AU_\varphi(\gamma)^*\Omega_\varphi, \pi_\varphi(x)U_\varphi(\gamma)AU_\varphi(\gamma)^*\Omega_\varphi\rangle &= \langle A\Omega_\varphi, \gamma^{-1}(x)A\Omega_\varphi\rangle \\ &= \varphi_1(\gamma^{-1}(x)) \\ &= \varphi_1(x).\end{aligned}$$

Hence the map

$$W' : \pi_\varphi(x)A\Omega_\varphi \longmapsto \pi_\varphi(x)U_\varphi(\gamma)AU_\varphi(\gamma)^*\Omega_\varphi,$$

is an isometry. In other words, we have $W'A = U_\varphi(\gamma)AU_\varphi(\gamma)^*$. By the uniqueness of the polar decomposition, we have that $A = U_\varphi(\gamma)AU_\varphi(\gamma)^*$, or that $A \in (\pi_\varphi(\mathfrak{A}) \cup \{U_\varphi(\gamma)\})'$, hence it is a scalar by assumption. This means that φ_1 is proportional to φ. □

The following is essential to our argument of extension for locally normal systems.

Theorem 4.6.3 ([30], A 86)**.** *Let $\mathcal{H} = \int_X^\oplus \mathcal{H}_\lambda d\mu(\lambda)$ be a direct integral Hilbert space, $T_i = \int_X^\oplus T_{i,\lambda}\mu(\lambda)$ a sequence of decomposable operators, $\mathcal{M}$ be the von Neumann algebra generated by $\{T_i\}$ and $\mathcal{M}_\lambda$ be the von Neumann algebra generated by $\{T_{i,\lambda}\}$. Then the algebra $\mathcal{Z}$ of diagonalizable operators is maximally commutative in $\mathcal{M}'$ if and only if $\mathcal{M}_\lambda = B(\mathcal{H}_\lambda)$ for almost all λ.*

Let K_i be the ideal of compact operators of the type I factor $\mathcal{F}_i$, and $\mathfrak{K}$ be the C^*-algebra generated by $\{K_i\}$. With a slight modification about the index set, the following applies to our situation.

Theorem 4.6.4 ([54], Proposition 56)**.** *Let π be a locally normal representation of a split net $\mathcal{A}$ on a separable Hilbert space and denote by $\pi_\mathfrak{K}$ the restriction to the algebra $\mathfrak{K}$. If we have a disintegration*

$$\pi_\mathfrak{K} = \int_X^\oplus \pi_\lambda d\mu(\lambda),$$

then π_λ extends to a locally normal representation $\widetilde{\pi}_\lambda$ of $\mathfrak{A}$ for almost all λ.

We need further a variation of a standard result. The next Proposition would follow from a general decomposition of an invariant state into extremal invariant state and [83, Corollary 5.3] which state any decomposition is locally normal. In the present thesis we take another way through decomposition of representation.

Proposition 4.6.5. *Let φ be a locally normal γ-invariant state of the C^*-algebra $\mathfrak{A}$ and π_φ be the corresponding GNS representation, then φ decomposes into an integral of locally normal extremal γ-invariant states.*

Proof. We take a separable subalgebra $\mathfrak{K}$ as above analogously as in [54]. We fix a maximally abelian subalgebra $\mathfrak{m}$ in the commutant $(\pi_{\mathfrak{K}}(\mathfrak{K}) \cup \{U_\varphi(\gamma)\})'$. Since $\mathfrak{K}$ is separable, we can apply [30, Section II.3.1 Corollary 1] to obtain a measurable space X, a standard measure μ on X, a field of Hilbert spaces $\mathcal{H}_\lambda$ and a field of representations π_λ such that the original restricted representation $\pi_{\mathfrak{K}}$ is unitarily equivalent to the integral representation:

$$\pi_{\mathfrak{K}} = \int_X^\oplus \pi_\lambda d\mu(\lambda)$$

and $\mathfrak{m} = L^\infty(X, \mu)$. Now, by Theorem 4.6.4, we may assume that π_λ is locally normal for almost all λ, hence it extends to a locally normal representation $\widetilde{\pi}_\lambda$ and the original representation π_φ decomposes into

$$\pi_\varphi = \int_X^\oplus \widetilde{\pi}_\lambda d\mu(\lambda).$$

Furthermore, the GNS vector Ω_φ decomposes into a direct integral

$$\Omega_\varphi = \int_X^\oplus \Omega_\lambda d\mu(\lambda).$$

The representative $U_\varphi(\gamma)$ decomposes into direct integrals as well, since $\mathfrak{m}$ commutes with $U_\varphi(\gamma)$:

$$U_\varphi(\gamma) = \int_X^\oplus U_\lambda(\gamma) d\mu(\lambda).$$

From this it holds that Ω_λ is invariant under $U_\lambda(\gamma)$, thus the state $\varphi_\lambda(\cdot) := \langle \Omega_\lambda, \pi_\lambda(\cdot)\Omega_\lambda \rangle$ is invariant under the action of γ, for almost all λ. By the definition of the direct integral it holds that

$$\varphi = \int_X^\oplus \varphi_\lambda d\mu(\lambda).$$

It is obvious that φ_λ is locally normal.

It remains to show that each φ_λ is extremal γ-invariant. By assumption, $\mathfrak{m}$ is maximally commutative in the commutant of $(\pi_{\mathfrak{K}}(\mathfrak{K}) \cup \{U_\varphi(\gamma)\})''$. This von Neumann algebra is generated by a countable dense subset $\{\pi_{\mathfrak{K}}(x_i)\}$ and a representative $U_\varphi(\gamma)$. Then, by Theorem 4.6.3, this is equivalent to the condition that $(\{\pi_\lambda(x_i)\} \cup \{U_\lambda(\gamma)\})'' = B(\mathcal{H}_\lambda)$, namely φ_λ is extremal γ-invariant.

If we consider a continuous family $\{\gamma_s\}$, we only have to take a countable family of operators $\{\pi_{\mathfrak{K}}(x_i)\} \cup \{U_\varphi(\gamma_s)\}_{s\in\mathbb{Q}}$.

□

Corollary 4.6.6. *Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of split nets with a locally normal conditional expectation which commutes with γ. If φ is an extremal γ-invariant state on $\mathfrak{A}$, then φ extends to an extremal γ-invariant state on the quasilocal algebra $\mathfrak{B}$ the net $\mathcal{B}$.*

Proof. The composition $\varphi \circ E$ is a γ-invariant state on $\mathfrak{B}$. By Proposition 4.6.5, $\varphi \circ E$ can be written as an integral of extremal γ-invariant states:

$$\varphi \circ E = \int_X^{\oplus} \psi_\lambda d\mu(\lambda).$$

By assumption, the restriction of $\varphi \circ E$ to $\mathfrak{A}$ is equal to φ, which is extremal γ-invariant, hence the restriction $\psi_\lambda|_{\mathcal{A}}$ coincides with φ for almost all λ. Hence, each of ψ_λ is an extremal γ-invariant extension of φ. □

Lemma 4.6.7. *If the net $\mathcal{A}$ is asymptotically γ-abelian, then it is γ-abelian.*

Proof. Let φ be a locally normal γ-invariant state on $\mathfrak{A}$. The action of γ is canonically unitarily implemented by $U_\varphi(\gamma)$. Let E_0 be the projection onto the space of $U_\varphi(\gamma)$-invariant vectors in $\mathcal{H}_\varphi$ and $\Psi_1, \Psi_2 \in E_0\mathcal{H}_\varphi$. Let us put $\psi(x) = \langle \Psi_1, \pi_\varphi(x)\Psi_2\rangle$.

By the assumption of asymptotically γ-abelianness, it is easy to see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \psi(\gamma^n(x)y) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \psi(y\gamma^n(x)).$$

On the other hand, by the mean ergodic theorem we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \psi(\gamma^n(x)y) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \langle \Psi_1, U_\varphi(\gamma)^n\pi_\varphi(x)(U_\varphi(\gamma)^*)^n\pi_\varphi(y)\Psi_2\rangle \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \langle \Psi_1, \pi_\varphi(x)(U_\varphi(\gamma)^*)^n\pi_\varphi(y)\Psi_2\rangle \\
&= \langle \Psi_1, \pi_\varphi(x)E_0\pi_\varphi(y)\Psi_2\rangle \\
&= \langle \Psi_1, E_0\pi_\varphi(x)E_0\pi_\varphi(y)E_0\Psi_2\rangle.
\end{aligned}$$

Similarly we have $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \psi(y\gamma^n(x)) = \langle \Psi_1, E_0\pi_\varphi(y)E_0\pi_\varphi(x)E_0\Psi_2\rangle$. Together with the above equality we see that $\langle \Psi_1, E_0\pi_\varphi(x)E_0\pi_\varphi(y)E_0\Psi_2\rangle = \langle \Psi_1, E_0\pi_\varphi(x)E_0\pi_\varphi(y)E_0\Psi_2\rangle$, which means that $E_0\pi_\varphi(x)E_0$ and $E_0\pi_\varphi(y)E_0$ commute. □

Proposition 4.6.8. *If φ is a locally normal γ-invariant state on the asymptotically γ-abelian net $\mathcal{A}$, then the following are equivalent:*

(a) in the GNS representation π_φ, the space of invariant vectors under $U_\varphi(\gamma)$ is one dimensional.

(b) φ is weakly γ-clustering.

(c) φ is extremal γ-invariant.

Proof. First we show the equivalence (a)⇔(b). By the asymptotic γ-abelianness we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\varphi(\gamma^n(x)y) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\varphi(y\gamma^n(x)),$$

and it holds by the mean ergodic theorem that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\varphi(\gamma^n(x)y) &= \langle\Omega_\varphi, E_0\pi_\varphi(x)E_0\pi_\varphi(y)E_0\Omega_\varphi\rangle,\\ \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\varphi(y\gamma^n(x)) &= \langle\Omega_\varphi, E_0\pi_\varphi(y)E_0\pi_\varphi(x)E_0\Omega_\varphi\rangle.\end{aligned}$$

Now if E_0 is one dimensional, then it holds that

$$\langle\Omega_\varphi, E_0\pi_\varphi(y)E_0\pi_\varphi(x)E_0\Omega_\varphi\rangle = \langle\Omega_\varphi, \pi_\varphi(y)\Omega_\varphi\rangle\langle\Omega_\varphi, \pi_\varphi(x)\Omega_\varphi\rangle = \langle\Omega_\varphi, E_0\pi_\varphi(x)E_0\pi_\varphi(y)E_0\Omega_\varphi\rangle,$$

and this is weakly γ-clustering.

Conversely, if $\mathcal{A}$ is weakly γ-clustering, the above equality holds and it implies that E_0 is one dimensional, since Ω_φ is cyclic for $\pi_\varphi(\mathcal{A})$.

Next we see the implication (a)⇒(c). Let us take a projection P in the commutant $(\pi_\varphi(\mathcal{A})\cup\{U_\varphi(\gamma)\})'$. Since P commutes with $U_\varphi(\gamma)$, $P\Omega_\varphi$ is again an invariant vector. By assumption the space of invariant vector is one dimensional, it holds that $P\Omega_\varphi = \Omega_\varphi$ or that $P\Omega_\varphi = 0$. We may assume that $P\Omega_\varphi = \Omega_\varphi$ (otherwise consider $\mathbb{1} - P$). By the cyclicity of Ω_φ we have that

$$\begin{aligned}\mathcal{H}_\varphi &= \overline{\pi_\varphi(\mathcal{A})\Omega_\varphi}\\ &= \overline{\pi_\varphi(\mathcal{A})P\Omega_\varphi}\\ &= \overline{P\pi_\varphi(\mathcal{A})\Omega_\varphi}\\ &= P\mathcal{H}_\varphi,\end{aligned}$$

in other words $P = \mathbb{1}$.

Finally, we prove the implication (c)⇒(a). By Lemma 4.6.7, the algebra $E_0\pi_\varphi(\mathcal{A})E_0$ is abelian, but by assumption (c), $\pi_\varphi(\mathcal{A})\cup\{U_\varphi(\gamma)\}$ act irreducibly and $U_\varphi(\gamma)$ acts trivially on E_0. Hence $E_0\pi_\varphi(\mathcal{A})E_0$ acts irreducibly on E_0. This is possible only if E_0 is one dimensional. □

Theorem 4.6.9. *Let $\mathcal{A}\subset\mathcal{B}$ be an inclusion of asymptotically γ-abelian split nets, and suppose that $\mathcal{A}$ is the fixed point subnet of a locally normal action α by a compact group G which commutes with γ and τ. Then, for any weakly γ-clustering primary τ-KMS state φ on $\mathcal{A}$, there are a one-parameter subgroup χ in G and a γ-clustering state ψ on $\mathcal{B}$ which extends φ such that ψ is a primary $\{\tau_t\circ\alpha_{\chi(t)}\}$-KMS state.*

Proof. By Corollary 4.6.6 φ extends to a weakly γ-clustering state ψ on $\mathcal{B}$. Then the restriction of ψ to the regular subalgebra $\mathfrak{B}_{\mathrm{r}}$ is still γ-clustering by Lemma 4.5.5. We claim that the restriction of ψ (hence of φ) to $\mathfrak{B}_{\mathrm{r}}^\alpha$ (see Remark 4.5.4) is still a primary KMS state. Indeed, the GNS representation of $\varphi|_{\mathfrak{B}_{\mathrm{r}}^\alpha}$ can be identified with a subspace of the representation π_φ of $\mathcal{A}$. By the local normality, this subspace for $\mathfrak{B}_{\mathrm{r}}^\alpha$ contains the subspace generated by $\mathcal{A}(I)$ for each fixed index set. The whole representation space of π_φ is the closed union of such subspaces, hence these spaces coincide. Furthermore, by the local normality, $\pi_\varphi(\mathfrak{B}_{\mathrm{r}}^\alpha)''$ contains $\pi_\varphi(\mathcal{A}(I))''$ for each I. Hence the von Neumann algebras generated by $\pi_\varphi(\mathcal{A})$ and $\pi_\varphi(\mathfrak{B}_{\mathrm{r}}^\alpha)$ coincide and $\varphi|_{\mathfrak{B}_{\mathrm{r}}^\alpha}$ is primary. Then Theorem 4.5.2 applies and we see that $\psi|_{\mathfrak{B}_{\mathrm{r}}}$ is a primary KMS state with respect to $\tau_t \circ \alpha_{\chi(t)}$. By Lemma 4.5.6 ψ is a KMS state on the full algebra $\mathfrak{B}$. Again by local normality, the primarity of $\psi|_{\mathfrak{B}_{\mathrm{r}}}$ and ψ are equivalent. □

4.7 The $U(1)$-current model

We recall some constructions regarding the $U(1)$-current and discuss its KMS states for two reasons: being a free field model, it is simple enough to allow a complete classification of the KMS states, showing an example of non completely rational model with multiple KMS states; it is useful in the classification of states for the Virasoro nets, whose restrictions to $\mathbb{R}$ are translation-covariant subnets of the $U(1)$-current net.

4.7.1 The $U(1)$-current model, from current approach

Recall that the $U(1)$-current algebra $\mathcal{A}_{U(1)}$ is the net generated by Weyl operators $W(f) = e^{iJ(f)}$, where J is calld the current. acting on the corresponding Fock space (if f is a real function, $J(f)$ is essentially self-adjoint on the finite particle-number subspace). The net structure is given by $\mathcal{A}_{U(1)}(I) := \{W(f) : \mathrm{supp}(f) \subset I\}''$. This defines a conformal net on S^1 in the sense of Section 1.1.1 (see [16] for detail). The current operators satisfy $[J(f), J(g)] = i\sigma(f, g)$.

Let us briefly discuss the split property of the $U(1)$-current net. A sufficient condition for the split property for a conformal net on S^1 is the **trace class condition**, namely the condition that the generator e^{-sL_0} is a trace class operator for each $s > 0$ [27, 14]. The Fock space is spanned by the vectors of the following form $J(e_{-n_1})J(e_{-n_2})\cdots J(e_{-n_k})\Omega$, where $e_n(\theta) = e^{i2\pi n\theta}$, $0 \le n_1 \le n_2 \le \cdots \le n_k$, $k \in \mathbb{N}$, and all these vectors are linearly independent and eigenvectors of L_0 with eigenvalue $\sum_{i=1}^k n_i$. Hence the dimension of the eigenspace with eigenvalue N is $p(N)$, the partition number of N. There is an asymptotic estimate of the partition function [47]: $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$. Hence with some constants C_s, D_s, we have

$$\mathrm{Tr}(e^{-sL_0}) = \sum_{n=0}^\infty p(n)e^{-sn} \le \sum_{n=0}^\infty C_s e^{-D_s n},$$

which is finite for a fixed $s > 0$. Namely we have the trace class condition, and the split property.

The Sugawara construction $T := \frac{1}{2} : J^2 :$, using normal ordering, gives the stress-energy tensor, satisfying the commutation relations:

$$[T(f), T(g)] = iT([f, g]) + i\frac{c}{12}\int_{\mathbb{R}} f'''g\, dx \tag{4.13}$$

with $c = 1$ and $[f, g] = fg' - gf'$. This is the relation of $\mathrm{Vect}(S^1)$, which is the Lie algebra of $\mathrm{Diff}(S^1)$. This (projective) representation T of $\mathrm{Vect}(S^1)$ integrates to a (projective) representation U of $\mathrm{Diff}(S^1)$ Furthermore, T and J satisfy the following commutation relations

$$[T(f), J(g)] = iJ(fg'). \tag{4.14}$$

Accordingly, U acts on J covariantly: if γ is a diffeomorphism of $\mathbb{R}$, then $U(\gamma)J(f)U(\gamma)^* = J(f \circ \gamma^{-1})$ (see [72] for details).

4.7.2 KMS states of the $U(1)$-current model

We give here the complete classification of the KMS states of the $U(1)$-current model, first appeared in [90, Theorem 3.4.11].

Proposition 4.7.1. *There is a one-parameter group $q \mapsto \gamma_q$ of automorphisms of $\mathcal{A}_{U(1)}|_{\mathbb{R}}$ commuting with translations, locally unitarily implementable, such that*

$$\gamma_q (W (f)) = e^{iq \int_{\mathbb{R}} f dx} W (f) . \tag{4.15}$$

Proof. For any $I \Subset \mathbb{R}$, let s_I be a function in $C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\forall x \in I \ s_I (x) = x$; then $\sigma(s_I, f) := \int_{\mathbb{R}} f dx$ if $\mathrm{supp} f \subset I$ and therefore

$$\mathrm{Ad}W (qs_I)\ W (f) = e^{-i\sigma(qs_I,f)}W (f) = e^{iq \int_{\mathbb{R}} f dx}W (f) .$$

Set $\gamma_q|_{\mathcal{A}(I)} = \mathrm{Ad}W (qs_I)$, this is a well-defined automorphism, since $\mathrm{Ad}W (qs_I)|_{\mathcal{A}(I)} = \mathrm{Ad}W (qs_J)|_{\mathcal{A}(I)}$ when $I \subset J$, which can be extended to the norm closure $\mathfrak{A}_{U(1)}$ satisfying (4.15) and commuting with translations because so is the integral. □

Lemma 4.7.2. *A state φ is a primary KMS state of the $U(1)$-current model iff so is $\varphi \circ \gamma_q$ for one value (and hence all) of $q \in \mathbb{R}$.*

Proof. By a direct application of the KMS condition and the fact that γ_q is an automorphism commuting with translations. □

Theorem 4.7.3. *The primary (locally normal) KMS states of the $U(1)$-current model at inverse temperature β are in one-to-one correspondence with real numbers $q \in \mathbb{R}$; each state φ^q is uniquely determined by its value on the Weyl operators*

$$\varphi^q (W (f)) = e^{iq \int f\, dx} \cdot e^{-\frac{1}{4}\|f\|^2_{S_\beta}} \tag{4.16}$$

where $\|f\|^2_{S_\beta} = (f, S_\beta f)$ *and the operator* S_β *is defined by* $\widehat{S_\beta f}(p) := \coth \frac{\beta p}{2} \widehat{f}(p)$*. The geometric KMS state is* $\varphi_{\mathrm{geo}} = \varphi^0$ *and any other primary KMS state is obtained by composition of the geometric one with the automorphisms* (4.15)*:*

$$\varphi^q = \varphi_{\mathrm{geo}} \circ \gamma_q.$$

Proof. Let φ be a KMS state and $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$. Recall the Weyl commutation relations

$$W(f)\,W(g) = W(f+g)\exp\left(-\frac{i}{2}\sigma(f,g)\right).$$

In other words, a product of Weyl operators is again a (scalar multiple of) Weyl operator, so that the quasilocal C^*-algebra is linearly generated by Weyl operators. Hence the state φ is uniquely determined by its values on $\{W(f)\}$. Furthermore, under the KMS condition, the function $t \mapsto F(t) = \varphi(W(f)\,W(g_t))$, where $g_t(x) := g(x-t)$, has analytic continuation in the interior of $D_\beta := \{0 \le \Im z \le \beta\}$, continuous on D_β, satisfying

$$F(t+i\beta) = e^{-i\sigma(f,g_t)}F(t). \tag{4.17}$$

We search for a solution F_0 of the form $F_0(z) = \exp K(z)$, where K is analytic in the interior of D_β and has to satisfy the logarithm of (4.17), $K(t+i\beta) = -i\sigma(f,g_t) + K(t)$. The Fourier transform of $t \mapsto -i\sigma(f,g_t)$ is $p \mapsto -p\overline{\widehat{f}(p)}\widehat{g}(p)$, thus we have a simple equation for the Fourier transform w.r.t. t: $\exp(-\beta p)\widehat{K}(p) = \widehat{K}(p) - p\overline{\widehat{f}(p)}\widehat{g}(p)$, from which $\widehat{K}(p) = -\frac{p\overline{\widehat{f}(p)}\widehat{g}(p)}{\exp(-\beta p)-1}$. It can be explicitly checked that F_0 is a solution of (4.17); any other solution, divided by the never vanishing function F_0, has to be constant (w.r.t. t) by analyticity. The general solution can therefore be written as $F(t) = c(f,g_t)\cdot F_0(t)$, with $c(f,g_t)$ independent of t.

To obtain (4.16), notice that

$$\varphi(W(f+g_t)) = F(t)\,e^{-\frac{i}{2}\sigma(f,g_t)} = c(f,g_t)\cdot\exp\left[K(t) - \frac{i}{2}\sigma(f,g_t)\right],$$

and $K(t) - \frac{i}{2}\sigma(f_t,g)$ is the Fourier antitransform of

$$p\overline{\widehat{f}(p)}\left(-\frac{1}{e^{-\beta p}-1} - \frac{1}{2}\right)\widehat{g}(p) = -p\overline{\widehat{f}(p)}\coth\frac{\beta p}{2}\widehat{g}(p) = -\frac{1}{2}p\overline{\widehat{f}(p)}\widehat{S_\beta g}(p)$$

which is given by

$$-\frac{1}{2}\int e^{itp}p\overline{\widehat{f}(p)}\widehat{S_\beta g}(p)\,dp = -\frac{1}{2}(f, S_\beta g_t) = -\frac{1}{4}\left(\|f+g_t\|^2_{S_\beta} - \|f\|^2_{S_\beta} - \|g_t\|^2_{S_\beta}\right),$$

since $(f, S_\beta g_t)$ is a real form. Note that $\|g_t\|^2_{S_\beta}$ is independent of t. We finally have the general solution in the form

$$\varphi(W(f+g_t)) = c(f,g_t)\cdot e^{\frac{1}{4}\left(\|f\|^2_{S_\beta}+\|g_t\|^2_{S_\beta}\right)}\cdot e^{-\frac{1}{4}\|f+g_t\|^2_{S_\beta}}.$$

Note that factors $\varphi(W(f+g_t))$ and $e^{-\frac{1}{4}\|f+g_t\|^2}$ depend only on the sum $f+g_t$, hence so does the remaining factor: we define $c(f+g_t) := c(f,g_t)\cdot e^{\frac{1}{4}\left(\|f\|^2_{S_\beta}+\|g_t\|^2_{S_\beta}\right)}$. Since $c(f,g_t)$ and $\|g_t\|_{S_\beta}$ are independent of t, so is $c(f+g_t)$. As $\overline{\varphi(W(f))} = \varphi(W(-f))$, $\overline{c(f)} = c(-f)$. Now we have

$$\varphi(W(f+g_t)) = c(f+g_t)\cdot e^{-\frac{1}{4}\|f+g_t\|^2_{S_\beta}},$$

and we only have to determine $c(f+g_t)$.

Concerning the continuity, we notice that $\|f\|_{S_\beta} \geq \|f\|$, because $\coth p \geq 1$ for any $p\in\mathbb{R}_+$; the map $f\mapsto W(f)$ is weakly continuous when $C_c^\infty(\mathbb{R},\mathbb{R})$ is given the topology of the (one-particle space) norm $\|\cdot\|$ and a fortiori of the norm $\|\cdot\|_{S_\beta}$; φ, being a KMS state, has to be locally normal, therefore $f\mapsto\varphi(W(f))$ is continuous w.r.t. both norms and $f\mapsto c(f) = \varphi(W(f))\cdot\exp(-\frac{1}{4}\|f\|^2_{S_\beta})$ is continuous w.r.t. the norm $\|\cdot\|_{S_\beta}$; finally, both $\lambda\mapsto\lambda f$ and $t\mapsto f_t$ are continuous w.r.t. the $\|\cdot\|_{S_\beta}$ norm, thus in particular $\lambda\mapsto c(\lambda f)$ (and trivially the constant function $t\mapsto c(f+g_t)$) is continuous.

If we require φ to be primary, it satisfies the clustering property: for $t\to\infty$

$$\varphi(W(f+g_t)) = \varphi(W(f)W(g_t))\exp\left(\frac{i}{2}\sigma(f,g_t)\right)\to\varphi(W(f))\varphi(W(g))$$

and thus

$$c(f+g) = c(f)\cdot c(g), \tag{4.18}$$

because both $\sigma(f,g_t)$ and $(f,S_\beta g_t)$ go to 0. It follows that $c(0)=1$, $c(-f)=c(f)^{-1}=\overline{c(f)}$ and $|c(f)|=1$. As $\mathbb{R}\ni\lambda\mapsto c(\lambda f)$ is a continuous curve in $\{z\in\mathbb{C}:|z|=1\}$, there is a unique functional $\rho: C_c^\infty(\mathbb{R},\mathbb{R})\to\mathbb{R}$ s.t. $c(f)=\exp(i\rho(f))$, $\rho(0)=0$ and $\lambda\mapsto\rho(\lambda f)$ is continuous.

Clearly, (4.18) implies $\rho(f+g)-\rho(f)-\rho(g)\in 2\pi\mathbb{Z}$; by continuity of $\lambda\mapsto\rho(\lambda f+\lambda g)-\rho(\lambda f)-\rho(\lambda g)$ and $\rho(0)=0$, we get $\rho(f+g)=\rho(f)+\rho(g)$. Similarly, from [68, Proposition 6.1.2] we know that ρ has the same continuity property of c, i.e. w.r.t. the $\|\cdot\|_{S_\beta}$ norm; $c(f_t)=c(f)$ implies $\rho(f_t)-\rho(f)\in 2\pi\mathbb{Z}$, but this difference vanishes because $t\mapsto\rho(f_t)$ is continuous. Therefore, ρ is a real, translation invariant and linear functional. Any translation invariant linear functional ρ on $C_c^\infty(\mathbb{R},\mathbb{R})$ is of the form $\rho(f)=q\int f(x)dx$. So, if φ is a primary KMS state, it has to be of the form (4.16). Conversely, Lemma 4.7.2 implies that all these states are KMS.

These are regular states and the one point and two points functions are given by

$$\varphi^q(J(f)) = q\int f\,dx \tag{4.19}$$

$$\varphi^q(J(f)J(g)) = \frac{1}{2}\Re(f,S_\beta g)+\frac{i}{2}\sigma(f,g)+q^2\int f\,dx\int g\,dx, \tag{4.20}$$

where $\Re$ means the real part. The geometric KMS state has to coincide with one of those: it is φ^0. This can be proved by noticing that, if $\mathrm{supp} f\subset I$,

$$\varphi_{\mathrm{geo}}(W(\lambda f)) = (\Omega,\mathrm{Ad}U(\gamma_{I,\beta})W(\lambda f)\Omega) = \left(\Omega, W\left(\lambda f\circ\gamma^{-1}_{I,\beta}\right)\Omega\right) = e^{-\frac{1}{4}\lambda^2\left\|f\circ\gamma^{-1}_{I,\beta}\right\|^2}$$

where the exponent is a quadratic form in f, therefore the state is regular and taking the derivative w.r.t. λ we get $\varphi_{\text{geo}}(J(f)) = 0$, which implies $q = 0$ by comparison with (4.19). $\square$

Remark 4.7.4. The gauge automorphism defined by the map $J(f) \mapsto -J(f)$ acts as a change in the sign of q: $\varphi^q \circ \gamma_z = \varphi^{-q}$.

The 'energy density' of a state can be read off the expectation value of the stress-energy tensor as the constant c in the formula $\varphi(T(f)) = c \int f\, dx$. Beside its physical interpretation, this formula is also useful to classify the states on the Virasoro net (see Section 4.8.1). In order to evaluate $\varphi^q(T(f))$, we need a technical lemma.

Lemma 4.7.5. *$D_\infty := \cap_{n\in\mathbb{N}} D(L_0^n)$ is invariant for the Weyl operator $W(f) = e^{iJ(f)}$, $\forall f \in C^\infty(S^1)$, and the unitary $U(g)$, $\forall g \in \mathrm{Diff}(S^1)$, implementing the conformal symmetry.*

Proof. The operators $J(f)$ and $T(f)$ satisfy similar commutation relations $[L_0, J(f)] = iJ(\partial_\theta f)$, $[L_0, T(f)] = iT(\partial_\theta f)$ and can be estimated as $\|J(f)\psi\| \le c_f \|(\mathbb{1} + L_0)\psi\|$ and $\|T(f)\psi\| \le c_f \|(\mathbb{1} + L_0)\psi\|$ for any $\psi \in D_\infty$ with c_f independent of ψ [18, ineqalities (2.21) and (2.23)]: Precisely, these commutation relations are well-defined on the subspace generated by the polynomials of smoothed out current J from the vacuum vector Ω. The bound holds for D_∞ since both $J(f)$, $T(f)$ and L_0 are closed, hence these commutators hold on D_∞ by this estimate. One sees also that D_∞ is invariant under $J(f)$ and $T(f)$ by the closedness.

We can generalize these bounds to the following form:

$$\|P_n(J, L_0)\psi\| \le r_n \|(\mathbb{1} + L_0)^n \psi\|, \tag{4.21}$$

where $P_n(J, L_0)$ is a (noncommutative) polynomial in L_0 and some $J(f_i)$ of total degree n, with an appropriate r_n (depending on $\{f_i\}$ but not on ψ). Indeed, induction and commutation relations show that

$$(\mathbb{1} + L_0)^n J(f) = \sum_{0\le k\le n} \begin{pmatrix} n \\ k \end{pmatrix} i^k J(\partial_\theta^k f)(\mathbb{1} + L_0)^{n-k}. \tag{4.22}$$

Then, we use induction in the degree of the polynomial to prove (4.21): it is trivial for degree 0. Suppose (4.21) holds for degree n. First let us consider $J(f)$. Then $\|P_n(J, L_0) J(f)\psi\| \le r_n \|(1 + L_0)^n J(f)\psi\|$ and, applying (4.22) (notice that $\mathbb{1} + L_0 \ge L_0, \mathbb{1}$), the last norm is smaller than $\sum_{0\le k\le n} c_k \|J(\partial_\theta^k f_n)(\mathbb{1} + L_0)^{n-k}\psi\|$ where each term is estimated by constants times $\|(\mathbb{1} + L_0)^{n+1}\psi\|$. On the other hand, $\|P_n(J, L_0) L_0 \psi\| \le r_n \|(1 + L_0)^n L_0 \psi\| \le r_n \|(1 + L_0)^{n+1}\psi\|$ and thus (4.21) holds for degree $n + 1$.

The same argument applies with T in place of J.

The space of finite number of particles $D_{fin} := \operatorname{span}\{\psi = J(f_1)...J(f_n)\Omega : n \in \mathbb{N}\}$, which is included in D_∞ by (4.21), is invariant under L_0, as $[L_0, J(f)] = iJ(\partial_\theta f)$. Using also the commutator $[J(f), J(g)^k] = ik\sigma(f, g)J(g)^{k-1}$ (easy consequence of $[J(f), J(g)] = i\sigma(f, g)$), we compute $\forall \psi \in D_{fin}$

$$[L_0, J(f)^n]\psi = \left(inJ(f)^{n-1} J(\partial_\theta f) - \frac{n(n-1)}{2} J(f)^{n-2} \sigma(\partial_\theta f, f) \right) \psi. \tag{4.23}$$

We apply it to the expansion of Weyl operators $W(f) = \sum_k \frac{i^k}{k!} J(f)^k$, which is absolutely convergent on D_{fin} (it is well known that finite particle vectors are analytic for the free field, see e.g. the proof of [76, Thm. X.41], with the estimate $\left\|J(f)^k \psi\right\| \leq 2^{k/2}\sqrt{(n+k)!}\|f\|^k\|\psi\|$, where n is the number of particles of ψ). By the closedness of L_0 and the absolute convergence of $L_0 \sum_k \frac{i^k}{k!} J(f)^k \psi$, thanks to (4.23), we conclude that $W(f)D_{fin}$ is in the domain of L_0. We then easily compute, using the convergent series, the commutation relations $W(f)^* L_0 W(f) = L_0 - J(\partial_\theta f) + \frac{1}{2}\sigma(\partial_\theta f, f)$ and their powers

$$W(f)^* L_0^n W(f)\psi = \left(L_0 - J(\partial_\theta f) + \frac{1}{2}\sigma(\partial_\theta f, f)\right)^n \psi. \tag{4.24}$$

Finally, (4.21) applied to the r.h.s., which is a polynomial of degree n in $J(\partial_\theta f)$ and L_0, gives

$$\|L_0^n W(f)\psi\| \leq r\|(\mathbb{1} + L_0)^n \psi\| \tag{4.25}$$

$\forall \psi \in D_{fin}$. As D_{fin} contains the space of finite energy vectors, it is dense in D_∞ and is a core for L_0^n; any $\psi \in D_\infty$ is the limit of a sequence $\{\psi_i : i \in \mathbb{N}\}$ such that $(\mathbb{1} + L_0)^n \psi_i$ is convergent, thus, by (4.25) and the closedness of $L_0^n W(f)$, $W(f)D_\infty$ is in the domain of L_0^n. We have proved that $W(f)D^\infty \subset D^\infty$; the same is true for $W(f)^{-1} = W(-f)$, thus $W(f)D^\infty = D^\infty$.

A similar argument apply to $U(g)$: First one consider the case where $g = \exp T(f)$ is contained in a one-parameter group. We replace J with T in (4.21) and replace (4.24) with the known transformation property of the stress-energy tensor ($L_0 = T(1)$, where 1 has to be understood as the generator of rotations, the constant vector field on the circle; in the real line picture, it would be the smooth vector field $x \mapsto 1 + x^2$):

$$U(g)L_0^n U(g)^* = (T(g_* 1) + r_g \mathbb{1})^n . \tag{4.26}$$

For a general diffeomorphism g, it is possible to write g as a finite product of diffeomorphisms contained in one-parameter groups, since $\mathrm{Diff}(S^1)$ is algebraically simple [35, 69] and the subgroup generated by one-parameter groups is normal, hence $\mathrm{Diff}(S^1)$ itself. Thus we obtained the claimed invariance for any element g. □

Theorem 4.7.6. *Any primary KMS state φ^q (cf. (4.16)) is C^∞ w.r.t the one parameter group $t \mapsto e^{iT(f)}$, $\forall f \in \mathcal{D}$, and its value on the stress-energy tensor is given by*

$$\varphi^q\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f\, dx. \tag{4.27}$$

Moreover, in the GNS representation $(\pi_{\varphi^q}, \mathcal{H}_{\varphi^q}, \Omega_{\varphi^q})$, Ω_{φ^q} is in the domain of any (non commutative) polynomial of the stress-energy tensors $\pi_{\varphi^q}(T(f_k)) := -i\frac{d}{dt}\pi_{\varphi^q}(e^{iT(f_k)})$, with $f_k \in \mathcal{D}$, $k = 1, \ldots, n$.

Proof. Fix $f \in \mathcal{D}$ with $\mathrm{supp} f \subset I \Subset \mathbb{R}$.

We first consider the case $q = 0$. According to the proof of Proposition 1.3.4 and Theorem 1.3.5, the GNS representation of φ_{geo} is $\left(\pi_{\varphi_{\mathrm{geo}}} = \mathrm{Exp}_\beta, \mathcal{H}_\Omega, \Omega\right)$ and there is a $g_{\beta,I} \in$

$\mathcal{D}$ s.t. $\mathrm{Exp}_\beta|_{\mathcal{A}(I)} = \mathrm{Ad}U\left(g_{\beta,I}\right)$. It follows that the one parameter group $t \mapsto \pi_{\varphi_{\mathrm{geo}}}\left(e^{itT(f)}\right) = \mathrm{Ad}U\left(g_{\beta,I}\right)\left(e^{itT(f)}\right)$ has a generator $\mathrm{Ad}U\left(g_{\beta,I}\right)\left(T(f)\right)$ which can be computed: Indeed, [37, Proposition 3.1] proves that, in general diffeomorphism covariant nets, if $g \in \mathrm{Diff}(S^1)$ fixes the point ∞, $\mathrm{Ad}U\left(g\right)T\left(f\right) = T\left(g_* f\right) + r^{\mathbb{R}}\left(g, f\right)$, with $g_* f\left(x\right) = g' \cdot f\left(g^{-1}\left(x\right)\right)$ and $r^{\mathbb{R}}\left(g, f\right) = \frac{c}{12\pi}\int \sqrt{g'\left(x\right)}\frac{d^2}{dx^2}\frac{f(x)}{\sqrt{g'(x)}}dx$ with the central charge c set equal to 1 for the $U(1)$ case. Therefore, with $g_{\beta,I}$ in place of g, recalling that $g_{\beta,I}(t) = e^{\frac{2\pi t}{\beta}}$ on the support of f, we get

$$\pi_{\varphi_{\mathrm{geo}}}\left(T(f)\right) = \mathrm{Ad}U\left(g_I\right)T\left(f\right) = T\left(g_{I*}f\right) + \frac{\pi c}{12\beta^2}\int f\,dx. \tag{4.28}$$

The vacuum vector Ω is in the domain of the operator (4.28) and any product of such operators; from $(\Omega, T(h)\Omega) = 0$ for any $h \in \mathcal{D}$, we easily compute (4.27). The case $q = 0$ is proved.

We now consider the general case for q. In this case the GNS representation is $(\pi_{\varphi^q} = \mathrm{Exp} \circ \gamma_q, \mathcal{H}_\Omega, \Omega)$ with $\gamma_q|_{\mathcal{A}(I)} = \mathrm{Ad}W\left(qs_I\right)$ defined in Proposition 4.7.1. The one parameter group $t \mapsto \mathrm{Ad}U\left(g_I\right) \circ \mathrm{Ad}W\left(qs_I\right)\left(e^{itT(f)}\right)$ has a self-adjoint generator $\mathrm{Ad}U\left(g_I\right) \circ \mathrm{Ad}W\left(qs_I\right)\left(T(f)\right)$ which has to be computed. According to Lemma 4.7.5, for any $\psi \in D_{fin} \subset D_\infty$ with a finite number of particles, $\mathrm{Ad}W\left(qs_I\right)\left(T(f)\right)\psi$ is well-defined because D_∞ is in the domain of $T(f)$. Using, as for eq. (4.23), $[J(f), J(g)^k] = ik\sigma(f,g)J(g)^{k-1}$ and $[T(f), J(g)] = iJ(fg')$, we compute $\forall\psi \in D_{fin}$ a generalization of (4.23):

$$[T(f), J(g)^n]\,\psi = \left(inJ(g)^{n-1}J(fg') - \frac{n(n-1)}{2}J(g)^{n-2}\sigma(fg', g)\right)\psi.$$

We use a similar argument to that following eq. (4.23). The expansion of Weyl operators $W(g) = \sum_k \frac{i^k}{k!}J(g)^k$ is absolutely convergent on D_{fin}; using the absolute convergence of $L_0 \sum_k \frac{i^k}{k!}J(g)^k\psi$ and the estimate $\|T(f)\psi\| \leq c_f\,\|(\mathbb{1} + L_0)\psi\|$, we conclude that also $T(f)\sum_k \frac{i^k}{k!}J(g)^k\psi$ is absolutely convergent and therefore, by the closedness of $T(f)$, $W(g)D_{fin}$ is in the domain of $T(f)$. The convergent series let us compute (cf. (4.24)) $W(g)^*T(f)W(g)\psi = \left(T(f) - J(fg') + \frac{1}{2}\sigma(fg', g)\right)\psi$. In the particular case in which $g = -qs_I$, and thus $fg' = -qf$ (recall that $\mathrm{supp}f \subset I$), we obtain

$$\mathrm{Ad}W\left(qs_I\right)\left(T(f)\right) = T(f) + qJ(f) + \frac{q^2}{2}\int f dx$$

on the dense set D_{fin} and also on D_∞, where both sides are defined. We can apply $\mathrm{Ad}U\left(g_I\right)$ to this operator, as D_∞ is invariant for $U\left(g_I\right)$, and taking into account its action on $J(f)$ and $T(f)$, we get

$$\pi_{\varphi^q}\left(T(f)\right) = T\left(g_{I*}f\right) + \frac{\pi}{12\beta^2}\int f dx + qJ\left(f \circ g_I^{-1}\right) + \frac{q^2}{2}\int f dx. \tag{4.29}$$

Ω is in the domain of the operator (4.29) and any power of such operators; as before, using also $(\Omega, J(h)\Omega) = 0$ for any $h \in \mathcal{D}$, we easily compute (4.27). $\square$

We finally observe that the thermal completion, in the case of the $U(1)$-current model, does not give any new net.

Theorem 4.7.7. *The thermal completion of the $U(1)$-current net w.r.t. any of its primary (locally normal) KMS states is unitarily equivalent to the original net.*

Proof. In the case of the geometric KMS state, this is the content of Theorem 1.3.5. The general case follows from the fact that any other primary KMS state of the $U(1)$-current model is obtained by composition of the geometric one with an automorphism, so that the local algebras $\hat{\mathcal{A}}_{\varphi^q}(e^{2\pi t}, e^{2\pi s}) := \mathcal{A}_{\varphi^q}(t,\infty) \cap \mathcal{A}_{\varphi^q}(s,\infty)'$ do not depend on the value of q. □

4.8 The case of Virasoro nets

The Virasoro nets Vir_c with $c < 1$ are completely rational [52, Cor. 3.4], so our results in Section 4.3 apply and thus they have a unique KMS state: the geometric state φ_{geo}. This is not the case for $c \geq 1$. Before going to the classification of the KMS states of Vir_1 and a (possibly incomplete) list of KMS states for the Virasoro net with central charge $c > 1$, we characterize the geometric state for any c [90, Theorem 3.6.2].

Theorem 4.8.1. *The (primary locally normal) geometric KMS states of the* Vir_c *net w.r.t. translations assume the following value on the stress-energy tensor*

$$\varphi_{\mathrm{geo}}\left(T\left(f\right)\right) = \left(\frac{\pi c}{12\beta^2}\right)\int f\,dx. \tag{4.30}$$

Proof. The evaluation of the state on the stress-energy tensor (4.30) follows from (4.28) using the same argument of the proof of Theorem 4.7.6. □

4.8.1 KMS states of the Virasoro net Vir_1

Recall that the Virasoro net Vir_1 is defined as the net generated by the representatives of diffeomorphisms (see Section 1.5.3). In fact, it holds that $\mathrm{Vir}_1(I) = \{e^{iT(f)} : \mathrm{supp}(f) \subset I\}''$, since the latter contains the representatives of one-parameter diffeomorphisms, and this consists a normal subgroup in $\mathrm{Diff}(I)$ of the group of diffeomorphisms with support in I, then this turns out to be the full group because $\mathrm{Diff}(I)$ is algebraically simple [35, 69]. The net Vir_1 is realied as a subnet of the $U(1)$-current net which satisfies the trace class property, hence so does Vir_1 and it is split as well.

The primary (locally normal) KMS states of the $U(1)$-current, restricted to the Virasoro net, give primary (locally normal) KMS states. They are still primary because primarity for KMS states is equivalent to extremality in the set of τ-invariant states [9, Theorem 5.3.32], and this is in turn equivalent to the clustering property (Proposition 4.6.8) for asymptotically abelian nets; clustering property is obviously preserved under restriction. We denote these states $\varphi^{|q|}$. We know their values on the stress-energy tensor (4.27). Notice that the two different states φ^q and φ^{-q} coincide when restricted to Vir_1. We have thus a family of primary (locally normal) KMS states classified by a positive number $|q| \in \mathbb{R}^+$. We will show that these exhaust the KMS states on Vir_1.

An important observation for this purpose is that $U(1)$-current net, hence so does Vir_1, can be viewed as a subnet of an even larger net. Namely, let $\mathcal{A}_{SU(2)_1}$ be the net generated by the vacuum representation of the loop group $LSU(2)$ at level 1 (Section 1.5.2, [41]), or by the $SU(2)$-chiral current at level 1 [78], on which the compact group $SU(2)$ acts as inner symmetry (an automorphism of the net which preserves the vacuum state). This net satisfies the trace class condition by an analogous (in fact much simpler and better) estimate as in $U(1)$-current net in Section 4.7.1, hence it is split. It has been shown [78] that the Virasoro net Vir_1 can be realized as the fixed point subnet in $\mathcal{A}_{SU(2)_1}$ with respect to this inner symmetry and we can apply Theorem 4.6.9, since $\mathcal{A}$ comes from a local net on S^1 and asymptotically γ-abelian with respect to some finite translation γ.

Theorem 4.8.2. *The primary (locally normal) KMS states of the* Vir_1 *net w.r.t. translations are in one-to-one correspondence with positive real numbers* $|q| \in \mathbb{R}^+$*; each state* $\varphi^{|q|}$ *can be evaluated on the stress-energy tensor and it gives*

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f\, dx. \tag{4.31}$$

Proof. For any $q \in \mathbb{R}$, the restriction of the KMS state φ^q to the Vir_1 subnet gives a KMS state. The evaluation of the state on the stress-energy tensor (4.31), depending only on $|q|$, follows again from (4.28) using the same argument of the proof of Theorem 4.7.6.

We have to prove that any primary KMS state of Vir_1 arises in this way. To this end, we use Theorem 4.6.9 applied to the inclusion of Vir_1 in the $SU(2)_1$-current net. Let $\mathcal{B}$ be the $SU(2)_1$-current net $\mathcal{A}_{SU(2)_1}$. Then the Virasoro subnet Vir_1 is (isomorphic to) the fixed point of the $\mathcal{B}$ w.r.t. the action $\alpha : SU(2) \to \mathrm{Aut}\mathcal{B}$ of the gauge group $SU(2)$ [78]. Moreover, as shown in [22], all the subnets of $\mathcal{B}$ are classified as fixed points w.r.t. the actions of closed subgroups of $SU(2)$ (conjugate subgroups give rise to isomorphic fixed points); in particular, let $\mathcal{A}_{U(1)}$ be the $U(1)$-current net, it is the fixed point $\mathcal{B}^H$ of the net $\mathcal{B}$ w.r.t. the action of the subgroup $H \simeq S^1$ of rotations around a fixed axis.

Let φ be a primary KMS state of $\mathrm{Vir}_1 = \mathcal{B}^{SU(2)}$. By applying Theorem 4.6.9 we obtain a locally normal primary (i.e. extremal) τ-invariant extension $\tilde{\varphi}$ on $\mathcal{B}$, which is a KMS state w.r.t. the one parameter group $t \mapsto \tilde{\tau}_t = \tau_t \circ \alpha_{\chi(t)}$, with a suitable one parameter group $t \mapsto \chi(t) \in SU(2)$. The image of $t \mapsto \chi(t) \in SU(2)$ is a closed subgroup $H \simeq S^1$, therefore, if we consider the subnet $\mathcal{A} = \mathcal{B}^H$, it is $\tilde{\tau}$ invariant and, as $\tilde{\tau}_t|_{\mathcal{A}} = \tau_t|_{\mathcal{A}}$, the state $\tilde{\varphi}$ is a primary KMS state of $\mathcal{A}$ w.r.t. τ. It then follows that the KMS state φ of Vir_1 is the restriction of a KMS state $\tilde{\varphi}|_{\mathcal{A}}$ of $\mathcal{A}$, isomorphic to the $U(1)$-current net $\mathcal{A}_{U(1)}$. $\square$

Remark 4.8.3. The geometric KMS state corresponds to $q = 0$, because it is the restriction of the geometric KMS state on the $U(1)$-current net, and the corresponding value of the 'energy density' $\frac{\pi}{12\beta^2} + \frac{q^2}{2}$ is the lowest in the set of the KMS states.

Remark 4.8.4. In contrast to the case of the $U(1)$-current net (Theorem 4.7.3), here the different primary KMS states are not obtained through composition of the geometric one with automorphisms of the net.

By contradiction, suppose that there were an automorphism α of the net such that $\varphi^{|q|} = \varphi \circ \alpha$ with $q \neq 0$. The KMS condition for $\varphi \circ \alpha$ w.r.t. the one parameter group $t \mapsto \tau_t$ is equivalent to the KMS condition for φ w.r.t. the one parameter group $t \mapsto \alpha \circ \tau_t \circ \alpha^{-1}$ and, by the uniqueness of the modular group, τ_t has to coincide with $\alpha \circ \tau_t \circ \alpha^{-1}$, i.e. the automorphism of the net commutes with translations. By Proposition 4.3.2 α cannot preserve the vacuum state and, by Lemma 4.3.5 there is a continuous family of pairwise non unitarily equivalent automorphisms of $\mathcal{A}|_{\mathbb{R}}$ commuting with translations. By Proposition 4.3.6, there is a continuous family of automorphic sectors of $\mathcal{A}$, which contradicts the fact, proved in [23], that Vir_1 can have at most countable sectors with finite statistical dimension.

Recall that the thermal completion net played a crucial role for the uniqueness results. Let $\mathcal{A}_\varphi(t,s) := \pi_\varphi(\mathcal{A}(t,s))$ and $\mathcal{A}^d_\varphi(t,s) := \mathcal{A}_\varphi(t,\infty) \cap \mathcal{A}_\varphi(s,\infty)'$. Putting $\mathcal{A} \equiv \mathrm{Vir}_1$ and $\varphi \equiv \varphi^{|q|}$ with $q \neq 0$, we have examples for which

$$\mathcal{A}_\varphi(t,s) \neq \mathcal{A}^d_\varphi(t,s).$$

Indeed, if the inclusion $\mathcal{A}_\varphi(t,s) \subset \mathcal{A}^d_\varphi(t,s)$ were an equality, as $\mathcal{A} = \mathrm{Vir}_1$ has the split property, Theorem 4.2.1 tells that φ would have to be $\varphi_{\mathrm{geo}} \circ \alpha$ the observation in the previous paragraph would give a contradiction.

4.8.2 KMS states of the Virasoro net Vir_c with $c > 1$

Here we show a (possibly incomplete) list of KMS states of the net Vir_c with $c > 1$.

The restriction of Vir_1 to the real line $\mathbb{R}$ can be embedded as a subnet of the restriction to $\mathbb{R}$ of the $U(1)$-current net. One can simply define a new stress-energy tensor [18, equation (4.6)], with $k \in \mathbb{R}$ and $f \in \mathcal{D}$

$$\widetilde{T}(f) := T(f) + kJ(f')$$

and, using the commutation relations (4.13), calculate that

$$\left[\widetilde{T}(f), \widetilde{T}(g)\right] = i\widetilde{T}([f,g]) + i\frac{1+k^2}{12}\int_{\mathbb{R}} f''' g \, dx.$$

It follows that the net generated by $\widetilde{T}(f)$ as $\mathrm{Vir}_c(I) := \left\{ e^{i\widetilde{T}(f)} : \mathrm{supp} f \subset I \right\}''$ with $I \Subset \mathbb{R}$, is the restriction to $\mathbb{R}$ of the Virasoro net with $c = 1 + k^2 > 1$ [18]. We observe that $\mathrm{Vir}_c(I) \subset \mathcal{A}_{U(1)}(I)$ for $I \Subset \mathbb{R}$. Indeed, we know the locality of J and T, hence if $\mathrm{supp}(f) \subset I$, then $e^{i\widetilde{T}}(f)$ commutes with $W(g)$ with $\mathrm{supp}(g) \subset I'$ by the Trotter formula. By the Haag duality it holds that $e^{i\widetilde{T}(f)} \in \mathcal{A}_{U(1)}(I)$.

The primary (locally normal) KMS states of the $U(1)$-current, restricted again to this Virasoro net, give primary locally normal KMS states, noticing that $\varphi^q(J(f')) = q\int f'\,dx = 0$:

$$\varphi^{|q|}\left(\widetilde{T}(f)\right) = \varphi^{|q|}(T(f)) = \left(\frac{\pi}{12\beta^2}+\frac{q^2}{2}\right)\int f\,dx;$$

as in the $c=1$ case, the restrictions of φ^q and $\varphi^{-q}$are equal. We have thus the following

Theorem 4.8.5. *There is a set of primary (locally normal) KMS states of the* Vir_c *net with* $c>1$ *w.r.t. translations in one-to-one correspondence with positive real numbers* $|q|\in\mathbb{R}^+$*; each state* $\varphi^{|q|}$ *can be evaluated on polynomials of stress-energy tensor* $T(f)$ *and on a single* $T(f)$ *it gives:*

$$\varphi^{|q|}(T(f)) = \left(\frac{\pi}{12\beta^2}+\frac{q^2}{2}\right)\int f\,dx. \tag{4.32}$$

and the geometric KMS state corresponds to $q=\frac{1}{\beta}\sqrt{\frac{\pi(c-1)}{6}}$ and energy density $\frac{\pi c}{12\beta^2}$.

Proof. As in the case of Vir_1, the restriction of a primary KMS state of the $U(1)$-current net is a primary KMS state and $\varphi^q=\varphi^p$ iff $q=\pm p$.

The last statement on the geometric KMS state follows by comparison of (4.32) with (4.30). □

Remark 4.8.6. Unlike the Vir_1 case, here the geometric KMS state does not correspond either to $q=0$ or the lowest possible value $\frac{\pi}{12\beta^2}$ of the energy density.

4.9 The free fermion model

In this section we consider the free fermion net and the KMS states on its quasilocal C^*-algebra. For an algebraic treatment of this model, see [5, 67]. In contrast to the $U(1)$-current model, the free boson model, it turns out to admit a unique KMS state (for each temperature). The model is not local, but rather graded local. It is still possible to define a (fermionic) net [25].

The free fermion field ψ defined on S^1 satisfies the following Canonical Anticommutation Relation (CAR):

$$\{\psi(z),\psi(w)\} = 2\pi i z\cdot\delta(z-w),$$

and the Hermitian condition $\psi(z)^* = z\psi(z)$, or if we consider the smeared field, we have

$$\{\psi(f),\psi(g)\} = \oint_{S^1}\frac{dz}{2\pi i z}\overline{f(z)}g(z).$$

We put the Neveu-Schwarz boundary condition: $\psi(ze^{2\pi i}) = \psi(z)$. Then it is possible to expand $\psi(z)$ in terms of Fourier modes as follows.

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r z^{-r-\frac{1}{2}}.$$

The Fourier components satisfy the commutation relation $\{b_s, b_r\} = \delta_{s,-r}\mathbb{1}$, $s, r \in \mathbb{Z} + \frac{1}{2}$.

There is a faithful *-representation of this algebra which contains the lowest weight vector Ω, i.e., $b_s\Omega = 0$ for $s > 0$ (we omit the symbol for the representation since it is faithful). This representation is Möbius covariant [5], namely there is a unitary representation U of $SL(2,\mathbb{R}) \cong SU(1,1)$ such that $U(g)\psi(z)U(g) = \psi(g \cdot z)$ and $U(g)\Omega = \Omega$.

Let P be the orthogonal projection onto the space generated by even polynomials of $\{b_s\}$. It commutes with $U(g)$ and the unitary operator $\Gamma = I - P$ defines an inner symmetry (an automorphism which preserves the vacuum state $\langle\Omega, \cdot\Omega\rangle$).

For an interval I, we put $\mathcal{A}(I) := \{\psi(f) : \mathrm{supp}(f) \subset I\}''$. Then $\mathcal{A}$ is a Möbius covariant fermi net in the sense of [25], and graded locality is implemented by Z. As a consequence, we have twisted Haag duality: It holds that $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$, where $Z := \frac{\mathbb{1}-i\Gamma}{\mathbb{1}-i}$. In addition, we have Bisognano-Wichmann property: $\Delta^{it} = U(\Lambda(-2\pi t))$, where Δ^{it} is the modular group of $\mathcal{A}(\mathbb{R}_+)$ with respect to Ω under the identification of S^1 and $\mathbb{R}\cup\{\infty\}$, and Λ is the unique one-parameter group of $SL(2,\mathbb{R})$ which projects to the dilation subgroup in $PSL(2,\mathbb{R})$ under the quotient by $\{\mathbb{1}, -\mathbb{1}\}$ [27].

With $\{b_s\}$ we can construct a representation of the Virasoro algebra with $c = \frac{1}{2}$ as follows (see [67]):

$$L_n := \frac{1}{2}\sum_{s>\frac{n}{2}} \left(s - \frac{n}{2}\right) b_{-s}b_{n+s}, \text{ for } n \geq 0,$$

and $L_{-n} = L_n^*$. For a smooth function f on S^1, we can define the smeared stress-energy tensor $T(f) := \sum_n f_n L_n$, where $f_n = \oint_{S^1} \frac{dz}{2\pi i} z^{-n-1} f(z)$. Two fields ψ and T are relatively local, namely if f and g have disjoint supports, then $[\psi(f), T(g)] = 0$ ($\psi(f)$ is a bounded operator and this holds on a core of $T(g)$).

By the graded locality, we have $e^{iT(g)} \in \mathcal{A}(I)$ if $\mathrm{supp}(g) \subset I$ (since $\psi(f)$ is bounded for a smooth function f, there is no problem of domains). Let us define $\mathrm{Vir}_{\frac{1}{2}}(I) := \{e^{iT(g)} : \mathrm{supp}(g) \subset I\}$. This Virasoro net $\mathrm{Vir}_{\frac{1}{2}}$ has been studied in [52] and it has been shown that $\mathrm{Vir}_{\frac{1}{2}}$ admits a unique nonlocal, relatively local extension with index 2. Hence the fermi net $\mathcal{A}$ is the extension. Furthermore, by the relative locality, $\mathcal{A}$ is diffeomorphism covariant by an analogous argument as in [24, Theorem 3.7].

We consider the restricted net $\mathcal{A}|_\mathbb{R}$ on $\mathbb{R}$ as in Section 4.5.1, the quasilocal C^*-algebra $\mathfrak{A}$ and translation.

Theorem 4.9.1. *The free fermion net $\mathcal{A}$ admits one and only KMS state at each temperature.*

Proof. By the diffeomorphism covariance and Bisognano-Wichmann property, we can construct the geometric KMS state as in Section 1.3.2 (locality is not necessary). On the other hand, $\mathrm{Vir}_{\frac{1}{2}}$ is completely rational [52], hence it admits a unique KMS state. In this case, we have proved without locality (Theorem 4.3.11) that also the finite index extension $\mathcal{A}$ admits only the geometric KMS state. $\square$

4.10 Open problems

Classification and further properties

The classification problem of KMS states on Vir_c with $c > 1$ remains open. Furthermore, we have shown that the thermal completion of such a net with respect to a (non-geometric) KMS state is different from the original net. It is interesting to study these thermal completions, or even express them in terms of well-known nets. We investigated ground states in Chapter 3. Ground states are considered to be the states with temperature zero. More direct relation between KMS states and ground states on the specific nets we studied is desired.

KMS states on more two-dimensional nets

We will construct in Chapter 5 several new families of two-dimensional nets out of conformal nets. If they are really strictly local, then it would be interesting to consider their structure of thermal states. There are even interacting massive models on two-dimensional spacetime [59]. The existence of KMS states on such models is open. There models are not dilation-covariant, hence a more rich structre of thermal states is expected.

Chapter 5

Scattering theory of two-dimensional massless nets

Chapter Introduction

Quantum field theory is designed to describe interactions between elementary particles and can successfully account for a wide range of physical phenomena. However, its mathematical foundations are still unsettled and constitute an active area of research in mathematical physics. While the most important open problem in QFT is the existence of interacting models in physical four-dimensional spacetime, theories in lower dimensional spacetime have also attracted considerable interest. The purpose of this Chapter is twofold: to show that a part of two-dimensional Conformal field theory which admits a simple particle description is not interacting, and to construct a family of interacting two-dimensional net of observables with a weak localization property. For this purpose, we extend the theory of waves by Buchholz [11] to weakly localized theory.

In view of a large body of highly non-trivial results concerning two-dimensional CFT, both on the sides of physics and mathematics [29, 41, 39], our assertion that these theories have trivial scattering theory may seem surprising. In this connection we emphasise that the presence of interaction in scattering theory *cannot* be inferred solely from the fact that a particular expression for the Hamiltonian or the correlation functions differ from those familiar from free field theory. In fact, the Ising model, the most fundamental "interacting" model, can be considered as a subtheory of "free" fermionic field [67], hence the conventional term of "interaction" seems ill-defined. Instead, a conclusive argument should rely on a scattering theory which implements, in the theoretical setting, the quantum mechanical procedure of state preparation at asymptotic times. Such an intrinsic scattering theory was developed by Buchholz [11] in the framework of Algebraic QFT, which we also adopt in this thesis.

As the classical results on the absence of interaction in dilation-covariant theories in physical spacetime require the existence of irreducible representations of the Poincaré group with finite multiplicity [28, 15], they cannot be applied to two-dimensional CFT directly.

We combine essential ideas from [15] with the representation theory of the Möbius group to overcome this difficulty and obtain triviality of the scattering matrix. Exploiting again the Möbius symmetry, we construct chiral observables following Rehren [77] which live on the positive or negative lightrays and show that they generate all the collision states of waves from the vacuum. In examples of non-chiral two-dimensional CFT, the profile of chiral observables is well-known [53], hence this result gives an explicit description of the subspace of collision states. As a by-product we obtain an alternative proof of the noninteraction of waves and the insight that asymptotic completeness of a conformal field theory (in the sense of waves) is equivalent to chirality. This suggest that chiral Möbius covariant theories are generic examples of noninteracting massless theories in two-dimensional spacetime. Indeed, it turns out that asymptotically complete Poincaré covariant net satisfying Bisognano-Wichmann property and Haag duality can be recovered from the asymptotic chiral net and the scattering operator. This contains a strenghthened converse of the sufficient condition for noninteraction by Buchholz [11].

Then we turn to the problem of construction of interacting QFT. Recently, operator-algebraic methods have been applied to construct models with weaker localization property [44, 45, 19, 12, 58]. A remarkable feature of these new constructions is that they first consider a single von Neumann algebra (instead of a family of von Neumann algebras) which is acted on by the spacetime symmetry group in an appropriate way. The construction procedure relying on a single von Neumann algebra has been proposed in [6] and resulted in some intermediate constructions [44, 45, 12, 58] and even in a complete construction of local nets [59]. This von Neumann algebra is interpreted as the algebra of observables localized in a wedge-shaped region. There is a prescription to recover the strictly localized observables [6]. However, the algebras of strictly localized observables are not necessarily large enough and it can be even trivial [12].

Among above constructions, the deformation by Buchholz, Lechner and Summers starts with an arbitrary wedge-local net. When one applies the BLS deformation to chiral conformal theories in two dimensions, things get considerably simplified. We have seen that the theory remains to be asymptotically complete in the sense of waves [11] even after the deformation and the full S-matrix has been computed. Then we construct several families of wedge-local nets based on chiral conformal nets with a new scheme. It turns out that all these construction are related with endomorphisms of the half-line algebra in the chiral components recently studied by Longo and Witten [64]. Among such endomorphisms, the simplest ones are translations and inner symmetries. We show that the construction related to translations coincides with the BLS deformation of chiral CFT. The construction related to inner symmetries is new and we completely determine the strictly localized observables under some technical conditions. Furthermore, by using the family of endomorphisms on the $U(1)$-current net considered in [64], we construct a large family of wedge-local nets parametrized by inner symmetric functions. All these wedge-local nets have nontrivial S-matrix, but the strictly local part of the wedge-local nets constructed through inner symmetries has trivial S-matrix. The strict locality of the other constructions remains open. Hence, to our opinion, the true difficulty lies in strict locality.

This Chapter is organized as follows. In Section 5.1 we demonstrate that waves in two-dimensional CFT have always trivial scattering matrix. In Section 5.2 the chiral components are defined following [77]. They turn out to generate all the waves from the vacuum. In Section 5.3.1, under Bisognano-Wichmann property and Haag duality, we show that asymptotic fields are conditional expectations and that a Poincaré covariant net is asymptotically complete and noninteracting if and only if it is isomorphic to a chiral Möbius net. In Section 5.3.2 we show that in- and out-asymptotic fields coincide in Möbius covariant nets. In 5.4 remarks about various definitions of chiral component are given.

Then we turn to the problem of construction of interacting models. We summarize some variations of the scattering theory for wedge-local nets in Section 5.5. In Section 5.6 we show that the pair of S-matrix and the asymptotic algebra is a complete invariant of a massless asymptotically complete net. In Section 5.7 we construct wedge-local nets using one-parameter endomorphisms of Longo-Witten. It is shown that the case of translations coincides with the BLS deformation of chiral CFT and the strictly local elements are completely determined for the case of inner symmetry. A common argument is summarized in Section 5.7.1. Section 5.8 is devoted to the construction of wedge-local nets based on a specific example, the $U(1)$-current net. A similar construction is obtained also for the free fermionic net. Section 5.9 summarizes open problems and our perspectives.

5.1 Noninteraction of waves in conformal nets

5.1.1 Representations of the spacetime symmetry group

As a preliminary for the proof of the main result, we need to examine the structure of representations of the group generated by translations and dilations.

Recall that we denote by $\mathbf{P}$ the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by (one-dimensional) translations and dilations. The group $\mathbf{P}$ is simply connected, hence it can be considered as a subgroup of $\overline{\mathrm{PSL}(2,\mathbb{R})}$. The direct product $\mathbf{P}\times\mathbf{P} \subset \mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ is the group of (two-dimensional) translations, Lorentz boosts and dilations. For the later use, we only have to consider representations of $\mathbf{P}\times\mathbf{P}$ which extend to positive-energy representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$.

Recall further that irreducible positive-energy representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}$ are classified by a nonnegative number l, which is the lowest eigenvalue of the generator of (the universal covering of) the group of rotations (see [63]). We claim that irreducible representations of $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ are classified by pairs of nonnegative numbers $l_{\mathrm{L}}, l_{\mathrm{R}}$. Indeed, we can take the Garding domain $\mathcal{D}$ since $\overline{\mathrm{PSL}(2,\mathbb{R})}\times\overline{\mathrm{PSL}(2,\mathbb{R})}$ is a finite dimensional Lie group. Furthermore, if a representation is irreducible, then the center of the group must act as scalars. From this it follows that the joint spectrum of generators of left and right rotations is discrete and each point must have positive components by the assumed positivity of energy. The same argument as in [63] shows that an eigenvector with minimal eigenvalues of rotations generates an irreducible representation, hence irreducible representations are classified by this pair of minimal eigenvalues. Conversely, all of these representations are realized by product representations. Let us sum up these observations:

Proposition 5.1.1. *All the irreducible representations of* $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ *are completely classified by pairs of nonnegative numbers* $(l_\mathrm{L}, l_\mathrm{R})$*. A representation with a given* $(l_\mathrm{L}, l_\mathrm{R})$ *is unitarily equivalent to the product of representations of* $\overline{\mathrm{PSL}(2,\mathbb{R})}$ *with lowest weights* $l_\mathrm{L}, l_\mathrm{R}$ *(*$l_\mathrm{L} = 0$ *or* $l_\mathrm{R} = 0$ *correspond to the trivial representation). A vector in any of these irreducible representations is invariant under the subgroup* $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \mathrm{id}$ *if and only if it is invariant under* $\tau_0 \times \mathrm{id}$*, where* τ_0 *is the translation subgroup of* $\overline{\mathrm{PSL}(2,\mathbb{R})}$ *(and the same holds for the right component).*

We know that if $l \neq 0$ then the restriction of the representation to $\mathbf{P}$ is the unique strictly positive-energy representation [63] (here "positive-energy" means that the generator of translations is positive). As a consequence of Proposition 5.1.1, we can classify positive-energy irreducible representations of $\mathbf{P} \times \mathbf{P}$ which appear in Möbius covariant nets.

Corollary 5.1.2. *Let* ι *and* ρ *be the trivial and the unique strictly positive-energy representation of* $\mathbf{P}$ *respectively. Any irreducible positive-energy representation of* $\mathbf{P} \times \mathbf{P}$ *which extends to* $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ *is one of the following four representations.*

- $\iota \otimes \iota$,
- $\rho \otimes \iota$,
- $\iota \otimes \rho$,
- $\rho \otimes \rho$.

Any (possibly reducible) representation of $\mathbf{P} \times \mathbf{P}$ *extending to* $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ *is a direct sum of copies of the above four representations.*

Proof. The first part of the statement follows directly from Proposition 5.1.1. The second part is a consequence of the general result (for example, see [30, Sections 8.5 and 18.7]) that any continuous unitary representation (on a separable Hilbert space) of a (separable) locally compact group is unitarily equivalent to a direct integral of irreducible representations. Since by assumption the given representation extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$, it decomposes into a direct integral, and the components have positive-energy almost everywhere. Hence they are classified by $(l_\mathrm{L}, l_\mathrm{R})$ and when restricted to $\mathbf{P} \times \mathbf{P}$ they fall into irreducible representations listed above. Since the integrand takes only four different values (up to unitary equivalence), the direct integral reduces to a direct sum. □

5.1.2 Proof of noninteraction

As waves are defined in terms of representations of translations, we need to analyse the representation U. We continue to use notations from the previous section. A net $\mathcal{A}$ in this section is always assumed to be Möbius covariant.

The representation ρ of $\mathbf{P}$ does not admit any nontrivial invariant vector with respect to (one-dimensional) translations. The subgroup of dilations is noncompact (isomorphic to

$\mathbb{R}$) and for any vector ξ in the representation space of ρ it holds that $\rho(\delta_s)\xi$ tends weakly 0 as $s \to \pm\infty$, where δ_s represents the group element of dilation by e^s.

Remark 5.1.3. At this point we use the assumed covariance under the action of the two dimensional Möbius group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$. If we assume only the dilation covariance (as in [15]), I am not able to exclude the possibility of occurrence of a representation of $\mathbf{P}$ which is trivial only on translations. As we will see, the absence of such representations is essential to identify all the waves in the relevant representation space.

Among the four irreducible positive-energy representations of $\mathbf{P} \times \mathbf{P}$ (Corollary 5.1.2), only $\iota \otimes \iota$ contains a nonzero invariant vector with respect to two-dimensional translations. The representation space of $\iota \otimes \rho$ consists of invariant vectors with respect to positive-lightlike translations but contains no nonzero invariant vectors with respect to negative-lightlike translations. An analogous statement holds for $\rho \otimes \iota$. The representation $\rho \otimes \rho$ contains no nonzero invariant vectors, neither with respect to negative- nor positive-lightlike translations.

Let us consider the representation U of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ associated with a Möbius covariant net $\mathcal{A}$. The restriction of U to $\mathbf{P} \times \mathbf{P}$ is a direct sum of copies of representations which appeared in Corollary 5.1.2. By the uniqueness of the vacuum, the representation $\iota \otimes \iota$ appears only once. Waves of positive (respectively negative) direction correspond precisely to $\rho \otimes \iota$ (respectively $\iota \otimes \rho$). From these observations, it is straightforward to see the following.

Lemma 5.1.4. *Let us denote by P the spectral measure of the representation $T = U|_{\mathbb{R}^2}$ of translations. Each of the following spectral subspaces of T carries the multiple of one of the irreducible representations in Corollary 5.1.2 (the correspondence is the order of appearance)*

- $Q_0 := P(\{(0,0)\})$,
- $Q_{\mathrm{L}} := P(\{(a_0,a_1) : a_0 = a_1, a_0 > -a_1\})$,
- $Q_{\mathrm{R}} := P(\{(a_0,a_1) : a_0 = -a_1, a_0 > a_1\})$,
- $Q_{\mathrm{L,R}} := P(\{(a_0,a_1) : a_0 > a_1, a_0 > -a_1\})$.

Let δ^{L} be the dilation in the left-component of $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$. Then for any vector $\xi \in \mathcal{H}$, $\text{w-}\lim_{s\to 0} U(\delta^{\mathrm{L}}_s)\xi = (Q_{\mathrm{R}}+Q_0)\xi$. Similarly for the dilation in the right component δ^{R} we have $\text{w-}\lim_{s\to 0} U(\delta^{\mathrm{R}}_s)\xi = (Q_{\mathrm{L}}+Q_0)\xi$. Furthermore, it holds that $Q_{\mathrm{L}}+Q_0 = P_+, Q_{\mathrm{R}}+Q_0 = P_-$ (see Section 1.4.1 for definitions)

After this preparation we proceed to our main result:

Theorem 5.1.5. *Let $\mathcal{A}$ be a Möbius covariant net. We have the equality $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$. In particular, such waves do not interact and we have $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}}$.*

Proof. We show the equality $\langle \xi_+ \overset{\text{in}}{\times} \xi_-, \eta_+ \overset{\text{out}}{\times} \eta_- \rangle = \langle \xi_+ \overset{\text{in}}{\times} \xi_-, \eta_+ \overset{\text{in}}{\times} \eta_- \rangle$ for any $\xi_+, \eta_+ \in \mathcal{H}_-$ and $\xi_-, \eta_- \in \mathcal{H}_-$. This is in fact enough for the first statement, since we know that $\|\eta_+ \overset{\text{out}}{\times} \eta_-\| = \|\eta_+ \overset{\text{in}}{\times} \eta_-\|$. As a particular case we have $\langle \eta_+ \overset{\text{in}}{\times} \eta_-, \eta_+ \overset{\text{out}}{\times} \eta_- \rangle = \langle \eta_+ \overset{\text{in}}{\times} \eta_-, \eta_+ \overset{\text{in}}{\times} \eta_- \rangle$, which is possible only if $\eta_+ \overset{\text{out}}{\times} \eta_- = \eta_+ \overset{\text{in}}{\times} \eta_-$.

Obviously it suffices to show the equality for a dense set of vectors in $\mathcal{H}_+$ and $\mathcal{H}_-$. Let us take three double cones O_+, O_0, O_- which are timelike separated in this order, more precisely O_0 stays in the future of O_- and in the past of O_+, and assume that O_0 is a neighborhood of the origin. We choose elements $x_+ \in \mathcal{A}(O_+)$ and $y_+, y_- \in \mathcal{A}(O_-)$. We take a self-adjoint element $b \in \mathcal{A}(O_0)$ and set $b_s := \mathrm{Ad}(U(\delta_s^{\mathrm{L}}))(b)$ for $s < 0$. Then $\{b_s\}$ are still contained in $\mathcal{A}(O_0)$. We set:

$$
\begin{aligned}
\xi_+ &:= \Phi_+^{\text{in}}(x_+)\Omega, & \xi_- &:= \operatorname*{w-lim}_{s\to 0} b_s\Omega = \operatorname*{w-lim}_{s\to 0} U(\delta_s^{\mathrm{L}}) b\Omega,\\
\eta_+ &:= \Phi_+^{\text{out}}(y_+)\Omega, & \eta_- &:= \Phi_-^{\text{out}}(y_-)\Omega,\\
\zeta_- &:= \Phi_-^{\text{out}}(y_-^*)\Omega = \Phi_-^{\text{out}}(y_-)^*\Omega.
\end{aligned}
$$

Note that b_s commutes with $\Phi_+^{\text{in}}(x_+)$, $\Phi_+^{\text{out}}(y_+)$ and $\Phi_-^{\text{out}}(y_-)$ since Φ^{in} and Φ^{out} are defined as strong limits of local operators and from some point they are spacelike separated (see Remark 1.4.2). We see that

$$
\begin{aligned}
\langle \xi_+ \overset{\text{in}}{\times} \xi_-, \eta_+ \overset{\text{out}}{\times} \eta_- \rangle &= \langle \Phi_+^{\text{in}}(x_+)(\operatorname*{w-lim}_{s\to 0} b_s\Omega), \Phi_+^{\text{out}}(y_+)\Phi_-^{\text{out}}(y_-)\Omega \rangle\\
&= \lim_s \langle \Phi_+^{\text{in}}(x_+) b_s\Omega, \Phi_+^{\text{out}}(y_+)\Phi_-^{\text{out}}(y_-)\Omega \rangle\\
&= \lim_s \langle \Phi_-^{\text{out}}(y_-^*)\Phi_+^{\text{in}}(x_+)\Omega, \Phi_+^{\text{out}}(y_+) b_s\Omega \rangle,
\end{aligned}
$$

where we used Remark 1.4.2 in the 3rd line. Continuing the calculation, with the help of the definition of asymptotic fields, this can be transformed as

$$
\begin{aligned}
\langle \xi_+ \overset{\text{in}}{\times} \xi_-, \eta_+ \overset{\text{out}}{\times} \eta_- \rangle &= \langle \Phi_-^{\text{out}}(y_-^*)\Phi_+^{\text{in}}(x_+)\Omega, \Phi_+^{\text{out}}(y_+)(\operatorname*{w-lim}_{s\to 0} b_s\Omega) \rangle\\
&= \langle \Phi_-^{\text{out}}(y_-^*)\xi_+, \Phi_+^{\text{out}}(y_+)\xi_- \rangle\\
&= \langle \xi_+ \overset{\text{out}}{\times} \zeta_-, \eta_+ \overset{\text{out}}{\times} \xi_- \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \zeta_-, \xi_- \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \Phi_-^{\text{out}}(y_-^*)\Omega, (\operatorname*{w-lim}_{s\to 0} b_s\Omega) \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle (\operatorname*{w-lim}_{s\to 0} b_s\Omega), \Phi_-^{\text{out}}(y_-)\Omega \rangle\\
&= \langle \xi_+, \eta_+ \rangle \cdot \langle \xi_-, \eta_- \rangle\\
&= \langle \xi_+ \overset{\text{in}}{\times} \xi_-, \eta_+ \overset{\text{in}}{\times} \eta_- \rangle,
\end{aligned}
$$

where the 6th equality follows from Remark 1.4.2 and the self-adjointness of b, the 4th and 8th equalities follow from Lemma 1.4.3. This equation is linear with respect to b (which is implicitly contained in ξ_-), hence it holds for any $b \in \mathcal{A}(O_0)$.

Note that $\Phi_+^{\mathrm{in}}(x_+)\Omega = P_+x_+\Omega, \Phi_+^{\mathrm{out}}(y_+)\Omega = P_+y_+\Omega, \Phi_-^{\mathrm{out}}(y_-)\Omega = P_-y_-\Omega$ and we have $\lim_s b_s\Omega = P_-b\Omega$ by Lemma 5.1.4. By the Reeh-Schlieder property, each set of vectors of these forms is dense in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. Thus the required equality is obtained for dense subspaces and this concludes the proof. □

5.2 Subspace of collision states of waves

It has been shown by Rehren that any Möbius covariant net contains the maximal chiral subnet, consisting of observables localized on the lightrays [77]. Here we show that the subspace generated by such observables from the vacuum exhausts the subspace of collision states. With this information at hand, we provide an alternative proof of noninteraction of waves and show that a Möbius covariant field theory is asymptotically complete if and only if it is chiral.

5.2.1 The maximal chiral subnet and collision states

As we have seen in Section 1.1.8, from a pair of Möbius covariant nets on S^1 we can construct a two-dimensional Möbius covariant net. In this section we explain a converse procedure: Namely, starting with a two-dimensional Möbius covariant net $\mathcal{A}$, we find a pair of Möbius covariant nets $\mathcal{A}_\pm$ on S^1 which are maximally contained in $\mathcal{A}$. In general such a chiral part is just a subnet of the original net. Moreover, we show that the subspace generated by this subnet from the vacuum coincides with the subspace of collision states of waves. It follows that a Möbius covaraint net is asymptotically complete if and only if it is chiral.

It is possible to define chiral components in several ways. We follow the definition by Rehren [77]. Recall that the two-dimensional Möbius group $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$ is a direct product of two copies of the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. We write this as $\widetilde{G}_{\mathrm{L}} \times \widetilde{G}_{\mathrm{R}}$, where $\widetilde{G}_{\mathrm{L}}$ and $\widetilde{G}_{\mathrm{R}}$ are copies of $\overline{\mathrm{PSL}(2,\mathbb{R})}$ [1].

Definition 5.2.1. *For a two-dimensional Möbius net $\mathcal{A}$ we define nets of von Neumann algebras on $\mathbb{R}$ by the following: For an interval $I \subset \mathbb{R}$ we set the von Neumann algebras*

$$\mathcal{A}_{\mathrm{L}}(I) := \mathcal{A}(I \times J) \cap U(\widetilde{G}_{\mathrm{R}})',$$
$$\mathcal{A}_{\mathrm{R}}(J) := \mathcal{A}(I \times J) \cap U(\widetilde{G}_{\mathrm{L}})'.$$

The definition of $\mathcal{A}_{\mathrm{L}}$ (respectively $\mathcal{A}_{\mathrm{R}}$) does not depend on the choice of J (respectively of I) since $\widetilde{G}_{\mathrm{R}}$ (respectively $\widetilde{G}_{\mathrm{R}}$) acts transitively on the family of intervals.

If the net $\mathcal{A}$ is conformal, then the components $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ are nontrivial (see Remark 5.4.3)

[1] Generally, the symbol $\widetilde{G}$ is used to indicate the universal covering group for a group G, but for $\mathrm{PSL}(2,\mathbb{R})$ it is customary to use the notation $\overline{\mathrm{PSL}(2,\mathbb{R})}$ for its universal cover.

Lemma 5.2.2 ([77])**.** *The nets $\mathcal{A}_\mathrm{L}, \mathcal{A}_\mathrm{R}$ extend to Möbius nets on S^1. For a fixed double cone $I \times J$, there holds*

$$\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J) \simeq \mathcal{A}_\mathrm{L}(I) \otimes \mathcal{A}_\mathrm{R}(J).$$

Then we determine $\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in}$ in terms of chiral components. The key is the following lemma.

Lemma 5.2.3 ([77], Lemma 2.3)**.** *Let $\mathcal{A}$ be a Möbius covariant net. The subspace $\overline{\mathcal{A}_\mathrm{L}(I)\Omega}$ coincides with the subspace of $\widetilde{G}_\mathrm{R}$-invariant vectors. A corresponding statement holds for $\mathcal{A}_\mathrm{R}(J)$.*

Remark 5.2.4. The proof of this lemma requires Möbius covariance. On the other hand, in Section 5.1.2, where we utilized the fact that the representation U of $\mathbf{P} \times \mathbf{P}$ extends to $\overline{\mathrm{PSL}(2,\mathbb{R})} \times \overline{\mathrm{PSL}(2,\mathbb{R})}$, what was really needed is that U decomposes into a direct sum of copies of the four irreducible representations in Corollary 5.1.2.

Theorem 5.2.5. *It holds that $\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in} = \overline{\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)\Omega}$.*

Proof. As we have seen in Proposition 5.1.1, the spaces of invariant vectors with respect to $\widetilde{G}_\mathrm{L}, \widetilde{G}_\mathrm{R}$ and to positive/negative lightlike translations coincide. Lemma 5.2.3 tells us that $\overline{\mathcal{A}_\mathrm{L}(I)\Omega} = \mathcal{H}_+$ and $\overline{\mathcal{A}_\mathrm{R}(J)\Omega} = \mathcal{H}_-$.

As elements in $\mathcal{A}_\mathrm{L}$ are fixed under the action of $\widetilde{G}_\mathrm{R}$, for $x \in \mathcal{A}_\mathrm{L}(I)$ it holds that $\Phi_+^\mathrm{in}(x) = x$. Similarly we have $\Phi_-^\mathrm{in}(y) = y$ for $y \in \mathcal{A}_\mathrm{R}(J)$. Thus we see that

$$x\Omega \overset{\mathrm{in}}{\times} y\Omega = \Phi_+^\mathrm{in}(x)\Phi_-^\mathrm{in}(y)\Omega = xy\Omega \in \overline{\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)\Omega}$$

Conversely, since $\mathcal{A}_\mathrm{L}(I)$ and $\mathcal{A}_\mathrm{R}(J)$ commute, any element in $\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)$ can be approximated strongly by linear combinations of elements of product form xy. This implies the required equality of subspaces. □

As a simple corollary, we have another proof of noninteraction of waves and a relation between asymptotic completeness and chirality:

Corollary 5.2.6. *Let $\mathcal{A}$ be a Möbius covariant net.*

(a) (same as Theorem 5.1.5) We have the equality $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+$ and $\xi_- \in \mathcal{H}_-$. In particular, such waves do not interact.

(b) $\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in} = \mathcal{H}$ if and only if $\mathcal{A}$ coincides with its maximal chiral subnet.

Proof. Theorem 5.2.5 tells us that the space of collision states of waves is generated by chiral observables $\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)$. Lemma 1.4.1 assures that to investigate the S-matrix it is enough to consider observables which generate the collision states. Then, on the space of waves $\mathcal{H}_0 = \overline{\mathcal{A}_\mathrm{L}(I) \vee \mathcal{A}_\mathrm{R}(J)\Omega}$ and regarding the chiral observables, it has been shown that a chiral net is asymptotically complete ($\mathcal{H}^\mathrm{out} = \mathcal{H}^\mathrm{in} = \mathcal{H}_0$) and the S-matrix is trivial [34, 33].

If $\mathcal{H}_0 \neq \mathcal{H}$, then by the Reeh-Schlieder property, the full net $\mathcal{A}$ must contain non-chiral observables, and $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}} \neq \mathcal{A}$. If $\mathcal{H}_0 = \mathcal{H}$, since both $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$ and $\mathcal{A}$ are Möbius covariant, there is a conditional expectation $E_O : \mathcal{A}(O) \to \mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}(O)$ which preserves $\langle \cdot \Omega, \Omega \rangle$, but E_O is in fact the identity map since Ω is cyclic for $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}(O)$ (see Theorem 1.1.3). □

5.2.2 How large is the space of collision states?

We have seen that a part $\overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$ of the Hilbert space $\mathcal{H}$ can be interpreted as the space of collision states of waves and that these waves do not interact. Then of course it is natural to investigate the particle aspects of the orthogonal complement of this space. We do not go into the detail of this problem here, but restrict ourselves to a few comments.

The full Hilbert space decomposes into a direct sum of subspaces invariant under the action of chiral observables $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$:

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\rho_i},$$

where $\{\rho_i\}$ are irreducible representations (see [60]) of $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$. When $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$ are completely rational [54], then the representations ρ_i are tensor products $\rho_i^{\mathrm{L}} \otimes \rho_i^{\mathrm{R}}$ of representations ρ_i^{L} of $\mathcal{A}_{\mathrm{L}}$ and ρ_i^{R} of $\mathcal{A}_{\mathrm{R}}$. As we consider the maximal chiral subnet introduced by Rehren, the vacuum representations $\rho_0^{\mathrm{L}}, \rho_0^{\mathrm{R}}$ appear only once, in the form $\rho_0^{\mathrm{L}} \otimes \rho_0^{\mathrm{R}}$ [77, Corollary 3.5]. Theorem 5.2.5 says that the waves are contained only in $\mathcal{H}_0$ where $\rho_0^{\mathrm{L}}, \rho_0^{\mathrm{R}}$ are the vacuum representations of $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$, respectively.

Hence, when $\mathcal{A}$ is not chiral, the space of collision states is at most a half of the full Hilbert space, if we simply count the number of representations which appear in the decomposition. A conceptually more satisfactory measure is the index of the inclusion $[\mathcal{A} : \mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}]$. The minimal value of the index of a nontrivial inclusion is 2, which would mean again that waves occupy half of the available space. This case indeed happens: Let $\mathcal{A}_0$ be a Möbius covariant net on S^1 with $\mathbb{Z}_2$ symmetry. If we define $\mathcal{A} = (\mathcal{A}_0 \otimes \mathcal{A}_0)^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts on $\mathcal{A}_0 \otimes \mathcal{A}_0$ by the diagonal action and $(\mathcal{A}_0 \otimes \mathcal{A}_0)^{\mathbb{Z}_2}$ is the fixed point subnet of this action, then $\mathcal{A}$ has $\mathcal{A}_0^{\mathbb{Z}_2} \otimes \mathcal{A}_0^{\mathbb{Z}_2}$ as the maximal chiral subnet and the index $[\mathcal{A} : \mathcal{A}_0^{\mathbb{Z}_2} \otimes \mathcal{A}_0^{\mathbb{Z}_2}]$ is 2. But in this case it is natural to say that the orthogonal complement can be interpreted as collision states in a bigger net $\mathcal{A}_0 \otimes \mathcal{A}_0$ which do not interact. In general, if a given net is not the fixed point, such a reinterpretation of the orthogonal complement as waves is impossible and the index is typically larger than 2. New ideas are needed to clarify this general case.

5.3 Asymptotic fields as conditional expectations

5.3.1 Characterization of noninteracting nets

In [11], in the general setting of Poincaré covariant nets, Buchholz has proved that timelike commutativity implies the absence of interaction. The purpose of this subsection is to show a strenghthened converse, namely that if a two-dimensional Poincaré covariant net is asymptotically complete and noninteracting, then under natural assumptions it is (unitarily equivalent to) a chiral Möbius covariant net.

Let $\mathcal{A}$ be a Poincaré covariant net satisfying the Bisognano-Wichmann property. We start with general remarks on asymptotic fields. Let $\mathcal{N}_+^{\mathrm{out}}$ be the von Neumann algebra generated by $\Phi_+^{\mathrm{out}}(x)$ where $x \in \mathcal{A}(O)$, $O \subset W_{\mathrm{R}}$ and O is bounded [2].

Lemma 5.3.1. *The asymptotic field Φ_+^{out} is a conditional expectation (cf. 1.1.9) from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_+^{\mathrm{out}}$ which preserves the vacuum state $\omega := \langle\Omega, \cdot\Omega\rangle$.*

Proof. By construction, $\Phi_+^{\mathrm{in}}(x) \in \mathcal{A}(W_{\mathrm{R}})$ for such $x \in \mathcal{A}(O), O \subset W_{\mathrm{R}}$ as above. Recall that if g is a Poincaré transformation, it holds that $\mathrm{Ad}U(g)\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{out}}(\mathrm{Ad}U(g)(x))$ (see Lemma 1.4.1). Hence $\mathcal{N}_+^{\mathrm{out}}$ is invariant under Lorentz boosts $\mathrm{Ad}U(-\Lambda(2\pi t)), t \in \mathbb{R}$. Since we assume the Bisognano-Wichmann property, $\mathcal{N}_+^{\mathrm{out}}$ is invariant under the modular group of $\mathcal{A}(W_{\mathrm{R}})$ with respect to ω.

By Takesaki's Theorem 1.1.3, there is a conditional expectation E from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_+^{\mathrm{out}}$ and this is implemented by the projection P_+^{out} onto $\overline{\mathcal{N}_+^{\mathrm{out}}\Omega}$. By Lemma 1.4.1, we know that $P_+^{\mathrm{out}} = P_+$. Two operators $E(x)$ and $\Phi_+^{\mathrm{out}}(x)$ in $\mathcal{A}(W_{\mathrm{R}})$ satisfy $E(x)\Omega = P_+^{\mathrm{out}}x\Omega = P_+x\Omega = \Phi_+^{\mathrm{in}}(x)\Omega$. The vacuum vector Ω is separating for $\mathcal{A}(W_{\mathrm{R}})$, hence they coincide. $\square$

Analogously, we consider $\mathcal{N}_-^{\mathrm{in}}$ generated by $\{\Phi_-^{\mathrm{in}}(x) : x \in \mathcal{A}(O), O \subset W_{\mathrm{R}}, O \text{ bounded}\}$. The asymptotic field Φ_-^{in} is the conditional expectation from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_-^{\mathrm{in}}$.

Proposition 5.3.2. *Let us assume that $\mathcal{A}$ is asymptotically complete. The wedge algebra $\mathcal{A}(W_{\mathrm{R}})$ is generated by $\mathcal{N}_-^{\mathrm{out}}$ and $\mathcal{N}_+^{\mathrm{in}}$.*

Proof. As we observed before Lemma 5.3.1, $\mathcal{N}_+^{\mathrm{out}}$ and $\mathcal{N}_-^{\mathrm{in}}$ are invariant under Lorentz boosts. Hence the same holds for $\mathcal{N}_{\mathrm{R}} := \mathcal{N}_+^{\mathrm{out}} \vee \mathcal{N}_-^{\mathrm{in}}$. Again by Theorem 1.1.3, there is a conditional expectation E from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_{\mathrm{R}}$. The wedge algebra $\mathcal{A}(W_{\mathrm{R}})$ is already in the GNS representation of the vacuum ω since Ω is cyclic and separating for $\mathcal{A}(W_{\mathrm{R}})$. $\mathcal{N}_R\Omega$ contains all the collision states, since $\mathcal{N}_{\mathrm{R}}\Omega \supset \{\Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{in}}(y)\Omega\}$ and the assumption of asymptotic completeness tells us that $\mathcal{N}_{\mathrm{R}}\Omega$ is dense in $\mathcal{H}$, hence the projection $P_{\mathcal{N}_{\mathrm{R}}}$ onto $\overline{\mathcal{N}_{\mathrm{R}}\Omega}$ is equal to $\mathbb{1}$. Therefore the conditional expectation E is in fact the identity map and $\mathcal{N}_{\mathrm{R}} = \mathcal{A}(W_{\mathrm{R}})$. $\square$

Lemma 5.3.3. *Let us assume that $\mathcal{A}$ is asymptotically complete and noninteracting. Then it holds that $\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{in}}(x)$ and $\Phi_-^{\mathrm{in}}(x) = \Phi_-^{\mathrm{out}}(x)$ for $x \in \mathcal{A}(O)$.*

Proof. We present the proof for "+" objects only, since the other assertion is analogous. By the assumption that $S = \mathbb{1}$, it follows that $\xi_+ \overset{\mathrm{in}}{\times} \xi_- = \xi_+ \overset{\mathrm{out}}{\times} \xi_-$ for any pair $\xi_+ \in \mathcal{H}_+, \xi_- \in \mathcal{H}_-$.

[2] From Lemma 5.3.1 it is immediate that Φ_+^{out} naturally extends to $\mathcal{A}(W_{\mathrm{R}})$, but it is convenient to define $\mathcal{N}_+^{\mathrm{in}}$ with bounded regions since we see the relation between Φ_+^{in} and Φ_-^{out} in Lemma 5.3.3.

Then we have

$$\begin{aligned}
\Phi_+^{\rm out}(x)\cdot \xi_+ \overset{\rm out}{\times} \xi_- &= (\Phi_+^{\rm in}(x)\xi_+)\overset{\rm out}{\times}\xi_- \\
&= P_+ x\xi_+ \overset{\rm out}{\times} \xi_- \\
&= P_+ x\xi_+ \overset{\rm in}{\times} \xi_- \\
&= (\Phi_+^{\rm in}(x)\xi_+)\overset{\rm in}{\times}\xi_- \\
&= \Phi_+^{\rm in}(x)\cdot \xi_+ \overset{\rm in}{\times} \xi_- \\
&= \Phi_+^{\rm in}(x)\cdot \xi_+ \overset{\rm out}{\times} \xi_-,
\end{aligned}$$

where, in the 1st and 5th lines we used the fact that right- and left- moving asymptotic fields commute, the 2nd and 4th equalities come from Lemma 1.4.1 and the rest is particular cases of the equivalence between "$\overset{\rm in}{\times}$" and "$\overset{\rm out}{\times}$". By the assumption of asymptotic completeness, $\xi_+ \overset{\rm in}{\times} \xi_- = \xi_+ \overset{\rm out}{\times} \xi_-$ span the whole space, hence we have the equiality of operators $\Phi_+^{\rm out}(x) = \Phi_+^{\rm in}(x)$. □

Lemma 5.3.4. *Let us assume that $\mathcal{A}$ is asymptotically complete and noninteracting. The map*

$$W : \xi_+ \otimes \xi_- \mapsto \xi_+ \overset{\rm in}{\times} \xi_- = \xi_+ \overset{\rm out}{\times} \xi_-$$

gives a natural unitary equivalence $(P_+\mathcal{N}_+^{\rm out}) \otimes (P_-\mathcal{N}_-^{\rm in}) \simeq \mathcal{A}(W_{\rm R})$, which is elementwise expressed as $P_+\Phi_+^{\rm out}(x) \otimes P_-\Phi_-^{\rm in}(y) \mapsto \Phi_+^{\rm out}(x)\Phi_-^{\rm in}(y)$. Furthermore, this decomposition is compatible with the action of the Poincaré group $\mathcal{P}_+^\uparrow$: $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under $\mathcal{P}_+^\uparrow$, hence there is a tensor product representation on $\mathcal{H}_+ \otimes \mathcal{H}_-$ and it holds that $W \cdot (U(g)P_+\Phi_+^{\rm out}(x)\Omega \otimes U(g)P_-\Phi_-^{\rm in}(y)\Omega) = U(g)W \cdot (P_+\Phi_+^{\rm out}(x)\Omega \otimes P_-\Phi_-^{\rm in}(y)\Omega)$.

Proof. The unitarity of the map W in the statement is clear from Lemma 1.4.3 and it follows that W intertwines the actions of asymptotic fields by Lemma 1.4.1: Namely, $\Phi_+^{\rm out}$ and $\Phi_-^{\rm out}$ act as in a tensor product (Lemma 1.4.1, 1.4.3) but we know that $\Phi_-^{\rm out}(x) = \Phi_-^{\rm in}(x)$ from noninteraction (Lemma 5.3.3). As for the action of the Poincaré group, we see from Lemma 1.4.1, for x and y as in Lemma 5.3.3, that

$$\begin{aligned}
W\cdot U(g)\Phi_+^{\rm out}(x)\Phi_-^{\rm in}(y)\Omega &= W\cdot {\rm Ad}U(g)(\Phi_+^{\rm out}(x)){\rm Ad}U(g)(\Phi_-^{\rm in}(y))\Omega \\
&= W\cdot \Phi_+^{\rm out}({\rm Ad}U(g)(x))\Phi_-^{\rm in}({\rm Ad}U(g)(y))\Omega \\
&= P_+\Phi_+^{\rm out}({\rm Ad}U(g)(x))\Omega \otimes P_-\Phi_-^{\rm in}({\rm Ad}U(g)(y))\Omega \\
&= P_+U(g)\Phi_+^{\rm out}(x)\Omega \otimes P_-U(g)\Phi_-^{\rm in}(y)\Omega \\
&= U(g)P_+\Phi_+^{\rm out}(x)\Omega \otimes U(g)P_-\Phi_-^{\rm in}(y)\Omega,
\end{aligned}$$

where in the last step we used the fact that $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under $U(g)$. This completes the proof. □

For a von Neumann algebra $\mathcal{N}$ on the Hilbert space $\mathcal{H}$ (on which the net $\mathcal{A}$ is defined), we denote $\mathcal{N}(a) = \mathrm{Ad}T(a)(\mathcal{N})$ for $a \in \mathbb{R}^2$, where T is the representation of the translation group for the net $\mathcal{A}$ (see Section 1.4.1). We put $a_1 := (1,1), a_{-1} := (-1,1) \in \mathbb{R}^2$.

Lemma 5.3.5. *The inclusion $P_+\mathcal{N}_+^{\mathrm{out}}(a_{-1}) \subset P_+\mathcal{N}_+^{\mathrm{out}}$ is a standard $+$half-sided modular inclusion with respect to Ω on $\mathcal{H}_+$. Analogously, $P_-\mathcal{N}_-^{\mathrm{in}}(a_1) \subset P_-\mathcal{N}_-^{\mathrm{in}}$ is a standard $-$half-sided modular inclusion with respect to Ω on $\mathcal{H}_-$.*

Proof. We prove only the former claim, since the latter is analogous. Recall that the conditional expectation Φ_+^{out} commutes with translations (Lemma 1.4.1), hence $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$ is generated by $\{\Phi_+^{\mathrm{out}}(x) : x \in \mathcal{A}(O), O \subset W_{\mathrm{R}} + a_{-1}, O \text{ bounded}\}$. The region $W_{\mathrm{R}} + a_{-1}$ is mapped into itself by Lorentz boosts $\Lambda(-t), t \geq 0$. Lemma 5.3.1 tells us that Φ_+^{out} is a conditional expectation which preserves $\omega := \langle \Omega, \cdot\Omega\rangle$, hence the modular automorphism of $\mathcal{N}_+^{\mathrm{out}}$ with respect to ω is the restriction of the modular automorphism of $\mathcal{A}(W_{\mathrm{R}})$. Thus Bisognano-Wichmann property shows that $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$ is invariant under the modular automorphism σ_t^{Ω} of $\mathcal{N}_+^{\mathrm{out}}$ for $t \geq 0$. The projection P_+ commutes with both of $\mathcal{N}_+^{\mathrm{out}}$ and $\mathcal{N}_+^{\mathrm{out}}(a_{-1})$, hence it is a $+$half-sided modular inclusion.

As for standardness, note that $\mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1)$ contains $\mathcal{A}(D)$ where $D = W_{\mathrm{R}} \cap (W_{\mathrm{L}} + a_{-1} + a_1)$ is a double cone. Recall that $\mathcal{A}(W_{\mathrm{R}}) \simeq P_+\mathcal{N}_+^{\mathrm{out}} \otimes P_-\mathcal{N}_-^{\mathrm{in}}$ and the action of the Poincaré group splits as well (Lemma 5.3.4). According to this unitary equivalence we have $\mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \simeq P_+\mathcal{N}_+^{\mathrm{out}}(a_{-1})' \otimes P_-\mathcal{N}_-^{\mathrm{in}}(a_1)'$ and $\mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \simeq P_+(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})') \otimes P_-(\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)')$, since we have wedge duality (Proposition 1.1.4). The vector $\Omega \simeq \Omega \otimes \Omega$ is cyclic for $\mathcal{A}(D)$ (Reeh-Schlieder property) and this is possible only if Ω is cyclic for both $P_+\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})'$ and $P_-\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)'$. The cyclicity for $P_+\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})'$ is the standardness. □

Theorem 5.3.6. *Let $\mathcal{A}$ be a Poincaré covariant net, asymptotically complete and noninteracting (satisfying Haag duality and Bisognano-Wichmann property). Then it is a chiral Möbius covariant net.*

Proof. First we have to prepare two Möbius covariant nets on S^1: This has been done in Lemma 5.3.5. Namely, putting $a_{\pm t} = (s, \pm t) \in RR^2$ for $t \in \mathbb{R}$, we have two nets

$$\begin{aligned}
\mathcal{A}_{\mathrm{L}}((s,t)) &= P_+\left(\mathcal{N}_+^{\mathrm{out}}(a_{-s})' \cap \mathcal{N}_+^{\mathrm{out}}(a_{-t})\right), \\
\mathcal{A}_{\mathrm{R}}((s,t)) &= P_-\left(\mathcal{N}_-^{\mathrm{in}}(a_s) \cap \mathcal{N}_-^{\mathrm{in}}(a_t)'\right).
\end{aligned}$$

Under the unitary equivalence between $\mathcal{H}$ and $\mathcal{H}_+ \otimes \mathcal{H}_-$ from Lemma 5.3.4, Haag duality implies that, for the double cone $D = W_{\mathrm{R}} \cap (W_{\mathrm{L}} + a_{-1} + a_1)$, we have

$$\begin{aligned}
\mathcal{A}(D) &= \mathcal{A}(W_{\mathrm{R}}) \cap \mathcal{A}(W_{\mathrm{L}} + a_{-1} + a_1) \\
&\simeq P_+(\mathcal{N}_+^{\mathrm{out}} \cap \mathcal{N}_+^{\mathrm{out}}(a_{-1})') \otimes P_-(\mathcal{N}_-^{\mathrm{in}} \cap \mathcal{N}_-^{\mathrm{in}}(a_1)') \\
&= \mathcal{A}_{\mathrm{L}}((-1,0)) \otimes \mathcal{A}_{\mathrm{R}}((0,1)).
\end{aligned}$$

The corresponding equality for general intervals $(s_{\mathrm{L}}, t_{\mathrm{L}}), (s_{\mathrm{R}}, t_{\mathrm{R}})$ follows from the above definition of nets $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$. In Lemma 5.3.4 we saw that the actions of the Poincaré group are compatible with this unitary equivalence. □

Remark 5.3.7. Haag duality is used only in Theorem 5.3.6. Since a Poincaré covariant net $\mathcal{A}$ with Bisognano-Wichmann property is wedge dual (Propositions 1.1.4, 1.1.5), we can infer that the dual net $\mathcal{A}^{\mathrm{d}}$ (see [3]) is a chiral Möbius net even if we do not assume Haag duality.

5.3.2 Asymptotic fields in Möbius covariant nets

Finally, as a further consequece of the considerations on conditional expectations, we show that in- and out-asymptotic fields coincide in Möbius covariant nets even without assuming the asymptotic completeness. Lemma 5.3.1 has been proved for general Poincaré covariant nets with Bisognano-Wichmann property, hence it applies to Möbius covariant net as well (see Section 1.1.8). We use the same notations as in Section 5.2.

Let $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$ be the maximal chiral subnet. Since both nets $\mathcal{A}$ and $\mathcal{A}_{\mathrm{L}} \otimes \mathcal{A}_{\mathrm{R}}$ are Möbius covariant, they satisfy Bisognano-Wichmann property in $\mathcal{E}$ (see Section 1.1.8). Theorem 1.1.3 of Takesaki implies that there is a conditional expectation E_D from $\mathcal{A}(D)$ onto $\mathcal{A}_{\mathrm{L}}(I) \otimes \mathcal{A}_{\mathrm{R}}(J)$, where $D = I \times J$ is a double cone in $\mathcal{E}$, which is implemented by the projection P onto $\mathcal{H}^{\mathrm{in}} = \mathcal{H}^{\mathrm{out}} = \overline{\mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)\Omega}$ (see Theorem 5.2.5). Since the projection P does not depend on D, the conditional expectations $\{E_D\}_{D \subset \mathcal{E}}$ are compatible, namely, if $D_1 \subset D_2$ then it holds that $E_{D_2}|_{\mathcal{A}(D_1)} = E_{D_1}$. Indeed, it holds that $E_{D_1}(x)\Omega = Px\Omega = E_{D_2}(x)\Omega$ and Ω is separating for $\mathcal{A}(D_2)$.

In addition, there is a conditional expectation $\mathrm{id} \otimes \omega$ from $\mathcal{A}_{\mathrm{L}}(I) \otimes \mathcal{A}_{\mathrm{R}}(J) \simeq \mathcal{A}_{\mathrm{L}}(I) \vee \mathcal{A}_{\mathrm{R}}(J)$ onto $\mathcal{A}_{\mathrm{L}}(I)$ which obviously preserves ω and is implemented by P_+ (see Theorem 1.1.3). If we take intervals $I_1 \subset I_2$, then the corresponding expectations are obviously compatible. By composing this expectation and E_D, we find an expectation E_{L} from $\mathcal{A}(D)$ onto $\mathcal{A}_{\mathrm{L}}(I)$ which preserves ω and is implemented by P_+ (we omit the dependence on D since this family of expectations is compatible). Analogous statements hold for $\mathcal{A}_{\mathrm{R}}(J)$.

Theorem 5.3.8. *If $\mathcal{A}$ is a Möbius covariant net, then for $x \in \mathcal{A}(D)$ with some bounded double cone $D = I \times J$, it holds that $\Phi_+^{\mathrm{out}}(x) = \Phi_+^{\mathrm{in}}(x)$ and $\Phi_-^{\mathrm{out}}(x) = \Phi_-^{\mathrm{in}}(x)$.*

Proof. We exhibit the proof only for "+" objects since the other is analogous. As we have seen in Lemma 5.3.1, Φ_+^{out} is a conditional expectation from $\mathcal{A}(W_{\mathrm{R}})$ onto $\mathcal{N}_+^{\mathrm{out}}$ which preserves ω.

We claim that $\Phi_+^{\mathrm{out}}(x) = E_+(x)$. We may assume that $D \subset W_{\mathrm{R}}$ since Φ_+^{out} commutes with translations, and E_+ is compatible and the translated expectation $\mathrm{Ad}T(a) \circ E_+ \circ \mathrm{Ad}T(-a)$ still preserves ω (hence E_+ commutes with translation $\mathrm{Ad}T(a)$ as well). It holds that $\Phi_+^{\mathrm{out}}(x) \in \mathcal{A}_{\mathrm{L}}(\mathbb{R}_-) \subset \mathcal{A}(W_{\mathrm{R}})$ and $E_+(x) \in \mathcal{A}_{\mathrm{L}}(I) \subset \mathcal{A}(D) \subset \mathcal{A}(W_{\mathrm{R}})$. In addition we have $\Phi_+^{\mathrm{out}}(x)\Omega = P_+x\Omega = E_+(x)\Omega$, hence by the separating property of Ω for $\mathcal{A}(W_{\mathrm{R}})$ we obtain the claimed equality.

Similarly one sees $\Phi_+^{\mathrm{in}}(x) = E_+(x)$, hence two asymptotic fields Φ_+^{out} and Φ_+^{in} coincide. □

5.4 Chiral components of conformal nets

In this Section, we consider various definitions of chiral components when a net $\mathcal{A}$ is conformal. These observations are not needed in the proof of noninteraction at the technical level but justify the interpretation of chiral observables as observables localized on lightlines.

We use the notations from Section 5.2.

Proposition 5.4.1. *For distant intervals $J_1, J_2 \subset \mathbb{R}$ (i.e. they are disjoint with nonzero distance), it holds that*

$$\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2).$$

Proof. Since the Möbius group $\mathrm{PSL}(2,\mathbb{R}) = G_{\mathrm{R}}$ acts transitively on the family of intervals in $\mathbb{R} \subset S^1$, the inclusion $\mathcal{A}_{\mathrm{L}}(I) \subset \mathcal{A}(I \times J)$ holds for any interval J by the covariance of the net $\mathcal{A}$. Thus inclusions in one direction is proven.

To see the converse inclusion, we consider the commutants. By the Haag duality on $\mathcal{E}$, we have

$$\mathcal{A}_{\mathrm{L}}(I)' = (\mathcal{A}(I \times J) \cap U(\widetilde{G}_{\mathrm{R}})')' = \mathcal{A}(I^+ \times J^-) \vee U(\widetilde{G}_{\mathrm{R}}) \left(= \mathcal{A}(I^- \times J^+) \vee U(\widetilde{G}_{\mathrm{R}})\right),$$

where $I^\pm, J^\pm$ are defined in Section 1.1.8, and

$$(\mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2))' = \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

Recall that we can choose an arbitrary J. Let J be an interval which includes both J_1 and J_2. In this case we have $J^- \subset J_1^-$ and $J^- \subset J_2^-$, hence

$$\mathcal{A}(I^+ \times J^-) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

Furthermore, the fact that J_1 and J_2 are distant implies that the family of (two) intervals $\{J_1^-, J_2^-\}$ is an open cover of a closed interval of length 2π. The algebra $\mathcal{A}(I^+ \times J_1^-)$ (respectively $\mathcal{A}(I^+ \times J_2^-)$) contains the representatives of diffeomorphisms of the form $\mathrm{id} \times g_{\mathrm{R}}$ with $\mathrm{supp}(g_{\mathrm{R}}) \subset J_1$ (respectively $\mathrm{supp}(g_{\mathrm{R}}) \subset J_2$) in the sense that $\mathrm{Conf}(\mathcal{E})$ is a quotient group of $\overline{\mathrm{Diff}(S^1)} \times \overline{\mathrm{Diff}(S^1)}$ (see Section 1.1.8).

We claim that the algebra $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ contains any representative of the form $\mathrm{id} \times g_{\mathrm{R}}$ where g_- is an arbitrary element in $\overline{\mathrm{Diff}(S^1)}$. Note that $\overline{\mathrm{Diff}(S^1)}$ can be identified with the group of diffeomorphisms of $\mathbb{R}$ commuting with the translation by 2π and an element of the form $\mathrm{id} \times g_{\mathrm{R}}$ where $\mathrm{supp}(g_{\mathrm{R}}) \subset J_1$ or $\mathrm{supp}(g_{\mathrm{R}}) \subset J_2$ can be viewed as a diffeomorphism with a periodic support. The group $\overline{\mathrm{Diff}(S^1)}$ is generated by such elements, hence we obtain the claim. In particular it contains the representatives of the universal cover $\widetilde{G}_{\mathrm{R}}$ of the Möbius group. Summing up, we have shown the inclusion

$$\mathcal{A}(I^+ \times J^-) \vee U(\widetilde{G}_{\mathrm{R}}) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-).$$

The commutant of this relation gives the required inclusion. □

In [53], the intersection $\bigcap_J \mathcal{A}(I \times J)$ is taken as the definition of the chiral component. In fact, this and Definition 5.2.1 coincide under the diffeomorphism covariance.

Corollary 5.4.2. *We have $\mathcal{A}_{\mathrm{L}}(I) = \bigcap_J \mathcal{A}(I \times J)$. Here, the intersection can be taken over the set of finite length intervals contained in $\mathbb{R}$ or even all intervals in $\mathbb{R}$ as the covering space of S^1 by considering $\mathcal{A}$ as a net on $\mathcal{E}$.*

Remark 5.4.3. From Proposition 1.1.2, it follows that, for a conformal net $\mathcal{A}$, $\mathcal{A}_{\mathrm{L}}(I)$ contains the representatives of diffeomorphisms $g_{\mathrm{L}} \times \mathrm{id}$ with g_{L} supported in I and hence it is nontrivial, although the intersection of regions $\bigcap_J I \times J$ is empty. A similar statement holds for $\mathcal{A}_{\mathrm{R}}$.

If the chiral components $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$ satisfy strong additivity, another (potentially useful) definition is possible. This should support an intuitive view that $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\mathrm{R}}$ live on lightrays.

Proposition 5.4.4. *Assume that $\mathcal{A}_{\mathrm{R}}$ is strongly additive. If $\{J_n\}$ is a sequence of intervals shrinking to a point, then it holds that $\mathcal{A}_{\mathrm{L}}(I) = \bigcap_n \mathcal{A}(I \times J_n)$.*

Proof. First we claim that $\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2)$ if J_1 and J_2 are obtained from an interval J by removing an interior point. One sees that the proof of Proposition 5.4.1 works except the part concerning the diffeomorphisms. Namely, it holds that $\mathcal{A}_{\mathrm{R}}(J_1 \cup J_2) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$

This time, the union $J_1^- \cup J_2^-$ is of length 2π. By the assumed strong additivity of the component $\mathcal{A}_{\mathrm{R}}$, this implies that $\mathcal{A}_{\mathrm{R}}(S^1) \subset \mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$. In fact, if J is an interval with length less than 2π which contains a boundary point of $J_1^- \cup J_2^-$, then $\mathcal{A}_{\mathrm{R}}(J)$ is contained in $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ by strong additivity (note that the restriction of $\mathcal{A}_{\mathrm{R}}(I)$ to its vacuum representation is injective if I is a bounded interval). By the additivity of the chiral component, $\mathcal{A}(I^+ \times J_1^-) \vee \mathcal{A}(I^+ \times J_2^-)$ contains all the representatives of diffeomorphisms of the form $\mathrm{id} \times g$, $g \in \overline{\mathrm{Diff}(S^1)}$, in particular representatives of $\mathrm{id} \times \widetilde{G}_{\mathrm{R}}$. The rest of the argument is the same as Proposition 5.4.1.

Let $\{J_n\}$ be a sequence of intervals shrinking to a point, where $J_n = (a_n, b_n)$. Let $\{J_{1,n}\}$ and $\{J_{2,n}\}$ be sequences of intervals which are obtained by $J_{1,n} = (a_0, b_n)$ and $J_{2,n} = (a_n, b_0)$. Let us denote $J_1 := \mathrm{int}(\bigcap_n J_{1,n}) = (a_0, c)$, $J_2 := \mathrm{int}(\bigcap_n J_{2,n}) = (c, b_0)$, where $c = \lim_n a_n = \lim_n b_n$ and $\mathrm{int}(\cdot)$ means the interior. It is clear that

$$\mathcal{A}_{\mathrm{L}}(I) \subset \mathcal{A}(I \times J_n) \subset \mathcal{A}(I \times J_{1,n}) \cap \mathcal{A}(I \times J_{2,n}),$$

but the last expression tends to

$$\begin{aligned} \bigcap_n \mathcal{A}(I \times J_{1,n}) \cap \mathcal{A}(I \times J_{2,n}) &= \left(\bigcap_n \mathcal{A}(I \times J_{1,n})\right) \cap \left(\bigcap_n \mathcal{A}(I \times J_{2,n})\right) \\ &= \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2), \end{aligned}$$

where the last equality follows from the Haag duality on $\mathcal{E}$ and additivity. We have seen that this is equal to $\mathcal{A}_{\mathrm{L}}(I)$, hence the intersection of the middle terms above is equal to this as well. $\square$

Remark 5.4.5. Rehren defined the "generating property" of the net by

$$U(\widetilde{G}_{\mathrm{L}}) \subset \mathcal{A}(I \times J) \vee \mathcal{A}(I' \times J)$$
$$U(\widetilde{G}_{\mathrm{R}}) \subset \mathcal{A}(I \times J) \vee \mathcal{A}(I \times J'),$$

for any I, J. We proved Proposition 5.4.4 by showing the generating property for $\mathcal{A}$ with the strongly additive conformal components. It has been shown in [77] that the generating property implies that $\mathcal{A}_{\mathrm{L}}(I) = \mathcal{A}(I \times J_1) \cap \mathcal{A}(I \times J_2)$ where J_1 and J_2 are obtained by removing an interior point from an interval.

5.5 Scattering theory for wedge-local nets

We then turn to the consideration on wedge-local nets. A **wedge-local net** or a **Borchers triple** on a Hilbert space $\mathcal{H}$ is a triple $(\mathcal{M}, T, \Omega)$ of a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$, a unitary representation T of $\mathbb{R}^2$ on $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$ such that

- $\mathrm{Ad}T(t_0, t_1)(\mathcal{M}) \subset \mathcal{M}$ for $(t_0, t_1) \in W_{\mathrm{R}} := \{(t_0, t_1) \in \mathbb{R}^2 : t_1 > |t_0|\}$.
- The joint spectrum $\mathrm{sp}\, T$ is contained in the forward lightcone $V_+ = \{(p_0, p_1) \in \mathbb{R}_2 : p_0 \geq |p_1|\}$.
- Ω is a unique (up to scalar) invariant vector under T.

By the theorem of Borchers [6, 38], the representation T extends to the Poincaré group $\mathcal{P}_+^\uparrow$, with Lorentz boosts represented by the modular group of $\mathcal{M}$ with respect to Ω. With this extension U, $\mathcal{M}$ is Poincaré covariant in the sense that if $gW_{\mathrm{R}} \subset W_{\mathrm{R}}$ for $g \in \mathcal{P}_+^\uparrow$, then $U(g)\mathcal{M}U(g)^* \subset \mathcal{M}$.

We denote by $\mathcal{H}_+$ (respectively by $\mathcal{H}_-$) the space of the single excitations with positive momentum, (respectively with negative momentum) i.e., $\mathcal{H}_+ = \{\xi \in \mathcal{H} : T(t,t)\xi = \xi \text{ for } t \in \mathbb{R}\}$ (respectively $\mathcal{H}_- = \{\xi \in \mathcal{H} : T(t,-t)\xi = \xi \text{ for } t \in \mathbb{R}\}$).

Our fundamental examples come from Poincaré covariant nets. For a Poincaré covariant net $\mathcal{A}$, we can construct a wedge-local net as follows:

- $\mathcal{M} = \mathcal{A}(W_{\mathrm{R}})$
- $T := U|_{\mathbb{R}^2}$, the restriction of U to the translation subgroup.
- Ω: the vacuum vector.

Indeed, the first condition follows from the Poincaré (in particular, translation) covariance of the nets and the other conditions are assumed properties of U and Ω of the net. If $(\mathcal{M}, T, \Omega)$ comes from a chiral conformal net $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, then we say this triple is chiral, as well. The simple construction by tensor product of chiral nets is considered to be the "undeformed net". We will exhibit later different constructions.

Given a wedge-local net $(\mathcal{M}, T, \Omega)$, we can consider the scattering theory with respect to massless particles [34], which is an extension of the theory explained in Section 1.4: For a bounded operator $x \in B(\mathcal{H})$ we write $x(a) = \mathrm{Ad}T(a)(x)$ for $a \in \mathbb{R}^2$. Furthermore we define a family of operators parametrized by $\mathcal{T}$:

$$x_\pm(h_\mathcal{T}) := \int dt\, h_\mathcal{T}(t) x((t, \pm t)),$$

where $h_\mathcal{T}(t) = |\mathcal{T}|^{-\varepsilon} h(|\mathcal{T}|^{-\varepsilon}(t - \mathcal{T}))$, $0 < \varepsilon < 1$ is a constant, $\mathcal{T} \in \mathbb{R}$ and h is a nonnegative symmetric smooth function on $\mathbb{R}$ such that $\int dt\, h(t) = 1$.

Lemma 5.5.1 ([11] Lemma 2(b), [34] Lemma 2.1)**.** *Let $x \in \mathcal{M}$, then the limits $\Phi_+^{\mathrm{out}}(x) := \operatorname*{s-lim}_{\mathcal{T}\to+\infty} x_+(h_\mathcal{T})$ and $\Phi_-^{\mathrm{in}}(x) := \operatorname*{s-lim}_{\mathcal{T}\to-\infty} x_-(h_\mathcal{T})$ exist and it holds that*

- $\Phi_+^{\mathrm{out}}(x)\Omega = P_+ x\Omega$ *and* $\Phi_-^{\mathrm{in}}(x)\Omega = P_- x\Omega$
- $\Phi_+^{\mathrm{out}}(x)\mathcal{H}_+ \subset \mathcal{H}_+$ *and* $\Phi_-^{\mathrm{in}}(x)\mathcal{H}_- \subset \mathcal{H}_-$.
- $\mathrm{Ad}U(g)(\Phi_+^{\mathrm{out}}(x)) = \Phi_+^{\mathrm{out}}(\mathrm{Ad}U(g)(x))$ *and* $\mathrm{Ad}U(g)(\Phi_-^{\mathrm{in}}(x)) = \Phi_-^{\mathrm{in}}(\mathrm{Ad}U(g)(x))$ *for* $g \in \mathcal{P}_+^\uparrow$ *such that* $gW_\mathrm{R} \subset W_\mathrm{R}$.

Furthermore, the limits $\Phi_+^{\mathrm{out}}(x)$ (respectively $\Phi_-^{\mathrm{in}}(x)$) depends only on $P_+x\Omega$ (respectively on $P_-x\Omega$).

Proof. We show the limit only for Φ_+^{out}, since the other is analogous. Since there holds the estimate $\|x_+(h_T)\| \le \|x\| \int dt\, |h(t)|$, it suffices to show the convergence on the dense set of vectors $\mathcal{M}'\Omega$. First, using the mean ergodic theorem, one verifies that

$$\operatorname*{s-lim}_{T\to\infty} x_+(h_T)\Omega = P_+ x\Omega.$$

It is easy to see that $x_t(h_T) \in \mathcal{M}$ for a sufficiently large t since h_T is compactly supported. Hence, for any $y \in R'$,

$$\operatorname*{s-lim}_{T\to\infty} x_+(h_T) y\Omega = y P_+ F\Omega,$$

which proves the convergence. Since $\mathcal{M}$ is a von Neumann algebra, the limit $\Phi_+^{\mathrm{out}}(x)$ is an element of $\mathcal{M}$. Since Ω is separating for $\mathcal{M}$, this operator depends only on $\Phi_+^{\mathrm{out}}(x)\Omega = P_+ x\Omega$.

The rest follows immediately from the definitions of limits of Φ_+^{out}, Φ_-^{in}. □

Similarly we define asymptotic objects for the left wedge W_L. Since $J\mathcal{M}'J = \mathcal{M}$, where J is the modular conjugation for $\mathcal{M}$ with respect to Ω, we can define for any $y \in \mathcal{M}'$

$$\Phi_+^{\mathrm{in}}(y) := J\Phi_+^{\mathrm{out}}(JyJ)J,\ \Phi_-^{\mathrm{out}}(y) := J\Phi_-^{\mathrm{in}}(JyJ)J.$$

Then we have the following.

Lemma 5.5.2 ([34], Lemma 2.2). *Let $y \in \mathcal{M}'$. Then*

$$\Phi_+^{\mathrm{in}}(y) = \operatorname*{s-lim}_{\mathcal{T}\to-\infty} y_+(h_\mathcal{T}),\ \Phi_-^{\mathrm{out}}(y) = \operatorname*{s-lim}_{\mathcal{T}\to\infty} y_-(h_\mathcal{T}).$$

These operators depend only on the respective vectors $\Phi_+^{\mathrm{in}}(y)\Omega = P_+y\Omega$, $\Phi_-^{\mathrm{out}}(y)\Omega = P_-y\Omega$ and we have

(a) $\Phi_+^{\mathrm{in}}(y)\mathcal{H}_+ \subset \mathcal{H}_+,\ \Phi_-^{\mathrm{out}}(y)\mathcal{H}_- \subset \mathcal{H}_-$,

(b) $\mathrm{Ad}U(g)(\Phi_+^{\mathrm{in}}(y)) = \Phi_+^{\mathrm{in}}(\mathrm{Ad}U(g)(y))$, $\mathrm{Ad}U(g)(\Phi_-^{\mathrm{out}}(y)) = \Phi_-^{\mathrm{out}}(\mathrm{Ad}U(g)(y))$ *for* $g \in \mathcal{P}_+^\uparrow$ *such that* $gW_\mathrm{L} \subset W_\mathrm{L}$.

For $\xi_+ \in \mathcal{H}_+$, $\xi_- \in \mathcal{H}_-$ there are sequences of local operators $\{x_n\} \subset \mathcal{M}$ and $\{y_n\} \subset \mathcal{M}'$ such that $\operatorname*{s-lim}_{n\to\infty} P_+x_n\Omega = \xi_+$ and $\operatorname*{s-lim}_{n\to\infty} P_-y_n\Omega = \xi_-$. With these sequences we define collision states as in [34]:

$$\begin{aligned} \xi_+ \overset{\mathrm{in}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{in}}(x_n)\Phi_-^{\mathrm{in}}(y_n)\Omega \\ \xi_+ \overset{\mathrm{out}}{\times} \xi_- &= \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{out}}(x_n)\Phi_-^{\mathrm{out}}(y_n)\Omega. \end{aligned}$$

We interpret $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$) as the incoming (respectively outgoing) state which describes two non-interacting waves ξ_+ and ξ_-. These asymptotic states have the following natural properties.

Lemma 5.5.3 ([34], Lemma 2.3). *For the collision states $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ and $\eta_+ \overset{\mathrm{in}}{\times} \eta_-$ it holds that*

1. $\langle \xi_+ \overset{\mathrm{in}}{\times} \xi_-, \eta_+ \overset{\mathrm{in}}{\times} \eta_- \rangle = \langle \xi_+, \eta_+ \rangle \cdot \langle \xi_-, \eta_- \rangle$.

2. $U(g)(\xi_+ \overset{\mathrm{in}}{\times} \xi_-) = (U(g)\xi_+) \overset{\mathrm{in}}{\times} (U(g)\xi_-)$ *for all* $g \in \mathcal{P}_+^\uparrow$ *such that* $gW_\mathrm{R} \subset W_\mathrm{R}$.

And analogous formulae hold for outgoing collision states.

The rest of the scattering theory is similar as in Section 1.4. Here we repeat it for the convenience of the reader. We set the spaces of collision states: Namely, we let $\mathcal{H}^{\mathrm{in}}$ (respectively $\mathcal{H}^{\mathrm{out}}$) be the subspace generated by $\xi_+ \overset{\mathrm{in}}{\times} \xi_-$ (respectively $\xi_+ \overset{\mathrm{out}}{\times} \xi_-$). From Lemma 5.5.3, we see that the following map

$$S : \xi_+ \overset{\mathrm{out}}{\times} \xi_- \longmapsto \xi_+ \overset{\mathrm{in}}{\times} \xi_-$$

is an isometry. The operator $S : \mathcal{H}^{\mathrm{out}} \to \mathcal{H}^{\mathrm{in}}$ is called the **scattering operator** or the **S-matrix** of the wedge-local net $(\mathcal{M}, U, \Omega)$. We say the waves in the triple are **interacting** if S is not a constant multiple of the identity operator on $\mathcal{H}^{\mathrm{out}}$. We say that the wedge-local net is **asymptotically complete (and massless)** if it holds that $\mathcal{H}^{\mathrm{in}} = \mathcal{H}^{\mathrm{out}} = \mathcal{H}$. We have seen that a chiral net and its BLS deformations (see Section 5.7.2) are asymptotically complete [34]. If the wedge-local net $(\mathcal{M}, T, \Omega)$ is constructed from a Poincaré covariant net $\mathcal{A}$, then we refer to these objects as $S, \mathcal{H}_\pm$, asymptotic completeness of $\mathcal{A}$, etc. This notion concerns only massless excitations. Indeed, if one considers the massive free model for example, then it is easy to see that all the asymptotic fields are just the vacuum expectation (mapping to $\mathbb{C}\mathbb{1}$).

To conclude this section, we put a remark on the importance of Borchers triples. If $(\mathcal{M}, T, \Omega)$ comes from a Haag dual Poincaré covariant net $\mathcal{A}$, then the local algebras are recovered by the formula $\mathcal{A}(D) = T(a)\mathcal{M}T(a)^* \cap T(b)\mathcal{M}'T(b)^*$, where $D = (W_\mathrm{R}+a) \cap (W_\mathrm{L}+b)$ is a double cone. Furthermore, if $\mathcal{A}$ satisfies Bisognano-Wichmann property, then the Lorentz boost is obtained from the modular group, hence all the components of the net are regained from the triple. Conversely, for a given wedge-local net, one can define a "local net" by the same formula above. In general, this "net" satisfies isotony, locality, Poincaré covariance and positivity of energy, but not necessarily satisfies additivity and cyclicity of vacuum [6]. Addivity is usually used only in the proof of Reeh-Schlieder property, thus we do not consider it here. If the "local net" constructed from a wedge-local net satisfies cyclicity of vacuum, we say that the original wedge-local net is **strictly local**. In this respect, a wedge-local net is considered to have a weaker localization property. Hence the search for Poincaré covariant nets reduces to the search for strictly local nets. Indeed, by this approach a family of (massive) interacting Poincaré nets has been obtained [59].

5.6 Asymptotic chiral algebra and S-matrix

5.6.1 Complete invariant of nets

Here we observe that asymptotically complete (massless) net $\mathcal{A}$ is completely determined by its behaviour at asymptotic times. This is particularly nice, since the search for Poincaré covariant nets is reduced to the search for appropriate S-matrices. Having seen the classification of a class of chiral components [52], one would hope even for a similar classification result for massless asymptotically complete nets.

Specifically, we construct a complete invariant of a net consisting of two elements. We already know the first element, the S-matrix. Let us construct the second element, the asymptotic algebra. An essential tool is half-sided modular inclusion (see Section 1.1.7, and [95, 2] for the original references). Indeed, we use an analogous argument as in [86, Lemma 5.5].

We put $\tilde{\mathcal{A}}_+^{\mathrm{out}}(O) := \{\Phi_+^{\mathrm{out}}(x), x \in \mathcal{A}(O)\}$. We will show that $\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R} + (-1,1)) \subset \tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R})$ is a +half-sided modular inclusion. Indeed, Φ_+^{out} commutes with $\mathrm{Ad}U(g_t)$ where $g_t = \Lambda(-2\pi t)$ is a Lorentz boost (Lemma 5.5.1), and $\tilde{\mathcal{A}}(W_\mathrm{R} + (-1,1))$ is sent into itself under $\mathrm{Ad}U(g_t)$ for $t \geq 0$. Hence by Bisognano-Wichmann property, $\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R} + (-1,1)) \subset \tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R})$ is a +half-sided modular inclusion. In addition, when restricted to $\mathcal{H}_+$, this inclusion is standard. To see this, note that $\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R} + (-1,1)) = \tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R} + (-1,1) + (1,1)) = \tilde{\mathcal{A}}_-^{\mathrm{in}}(W_\mathrm{R} + (0,2))$ because Φ_+^{out} is invariant under $V((1,1))$, and hence $\tilde{\mathcal{A}}_+^{\mathrm{out}}(D) \subset$

$\left(\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R}+(-1,1))'\cap\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R})\right)$, where $D = W_\mathrm{R}\cap(W_\mathrm{L}+(0,2))$. It follows that

$$\overline{\left(\tilde{\mathcal{A}}_+^{\mathrm{out}}\left(W_\mathrm{R}+(-1,1)\right)'\cap\tilde{\mathcal{A}}_+^{\mathrm{out}}(W_\mathrm{R})\right)\Omega}\supset\overline{\tilde{\mathcal{A}}_+^{\mathrm{out}}(D)\Omega}=\overline{P_+\mathcal{A}(D)\Omega}=\mathcal{H}_+,$$

which is the standardness on $\mathcal{H}_+$. Then we obtain a Möbius covariant net on S^1 acting on $\mathcal{H}_+$, which we denote by $\mathcal{A}_+^{\mathrm{out}}$. Similarly we get a Möbius covariant net $\mathcal{A}_-^{\mathrm{out}}$ on $\mathcal{H}_-$. Two nets $\mathcal{A}_+^{\mathrm{out}}$ and $\mathcal{A}_-^{\mathrm{out}}$ act like tensor product by Lemma 5.5.3, and span the whole space $\mathcal{H}$ from the vacuum Ω by asymptotic completeness. In other words, $\mathcal{A}_+^{\mathrm{out}}\otimes\mathcal{A}_-^{\mathrm{out}}$ is a chiral conformal net on $\mathbb{R}^2$ acting on $\mathcal{H}$. We call this chiral net $\mathcal{A}_+^{\mathrm{out}}\otimes\mathcal{A}_-^{\mathrm{out}}$ the **(out-)asymptotic algebra** of the given net $\mathcal{A}$. Similarly one defines $\mathcal{A}_+^{\mathrm{in}}$ and $\mathcal{A}_-^{\mathrm{in}}$.

Let $(\mathcal{M},T,\Omega)$ be the wedge-local net associated to an asymptotically complete Poincare covariant net $\mathcal{A}$ which satisfies Bisognano-Wichmann property and Haag duality. Our next observation is that $\mathcal{M}$ can be recovered from asymptotic fields.

Proposition 5.6.1. *It holds that* $\mathcal{M}=\{\Phi_+^{\mathrm{out}}(x),\Phi_-^{\mathrm{in}}(y): x,y\in\mathcal{M}\}''=\tilde{\mathcal{A}}_+^{\mathrm{out}}(\mathbb{R}_-)\vee\tilde{\mathcal{A}}_-^{\mathrm{in}}(\mathbb{R}_+)$.

Proof. The inclusion $\mathcal{M}\supset\{\Phi_+^{\mathrm{out}}(x),\Phi_-^{\mathrm{in}}(y): x,y\in\mathcal{M}\}''$ is obvious. The converse inclusion is established by the modular theory: From the assumption of Bisognano-Wichmann property, the modular automorphism of $\mathcal{M}$ with respect to Ω is the Lorentz boosts $U(\Lambda(-2\pi t))$. Furthermore, it holds that $\mathrm{Ad}U(\Lambda(-2\pi t))(\Phi_+^{\mathrm{out}}(x))=\Phi_+^{\mathrm{out}}(\mathrm{Ad}U(\Lambda(-2\pi t))(x))$ 5.5.1. An analogous formula holds for Φ^{in}. Namely, the algebra in the middle term is invariant under the modular group.

By the assumed asymptotic completeness, the algebra in the middle term spans the whole space $\mathcal{H}$ from the vacuum Ω as well. Hence by a simple consequence of Takesaki's theorem [82, Theorem IX.4.2] [86, Theorem A.1], these two algebras coincide.

The last equation follows by the definition of asymptotic algebra and their invariance under translations in respective directions. □

Proposition 5.6.2. *It holds that* $S\cdot\Phi_\pm^{\mathrm{out}}(x)\cdot S^*=\Phi_\pm^{\mathrm{in}}(x)$ *and* $S\cdot\tilde{\mathcal{A}}_\pm^{\mathrm{out}}(\mathbb{R}_\mp)S^*=\tilde{\mathcal{A}}_\pm^{\mathrm{in}}(\mathbb{R}_\mp)$.

Proof. This follows from the calculation, using Lemmata 5.5.1, 5.5.2 and 5.5.3,

$$\begin{aligned}
\Phi_+^{\mathrm{in}}(x)(\xi\overset{\mathrm{in}}{\times}\eta) &= (P_+x\xi)\overset{\mathrm{in}}{\times}\eta\\
&= S\left((P_+x\xi)\overset{\mathrm{out}}{\times}\eta\right)\\
&= S\cdot\Phi_+^{\mathrm{out}}(x)(\xi\overset{\mathrm{out}}{\times}\eta)\\
&= S\cdot\Phi_+^{\mathrm{out}}(x)\cdot S^*(\xi\overset{\mathrm{in}}{\times}\eta),
\end{aligned}$$

and asymptotic completeness. The equation for "−" fields is proved analogously. The last equalities are simple consequences of the formulae for asymptotic fields. □

Theorem 5.6.3. *The out-asymptotic algebra* $\mathcal{A}_+^{\mathrm{out}},\mathcal{A}_-^{\mathrm{out}}$ *and the S-matrix* S *completely characterizes the original net* $\mathcal{A}$ *if it satisfies Bisognano-Wichmann property, Haag duality and asymptotic completeness.*

Proof. The wedge algebra is recovered by $\mathcal{A}(W_\mathrm{R}) = \{\Phi_+^{\mathrm{out}}(x), \Phi_-^{\mathrm{in}}(y) : x, y \in \mathcal{M}\}''$ by Proposition 5.6.1. In the right-hand side, Φ_-^{in} is recovered from Φ_-^{out} and S by Proposition 5.6.2. Hence the wedge algebra is completely recovered from the data $\Phi_\pm^{\mathrm{out}}$ and S, or $\mathcal{A}_\pm^{\mathrm{in}}$ and S by Proposition 5.6.1. By Haag duality, the data of wedge algebras are enough to recover the local algebras. By Bisognano-Wichmann property, the representation U of the whole Poincaré group is recovered from the modular data. □

Remark 5.6.4. Among the conditions on $\mathcal{A}$, Bisognano-Wichmann property is satisfied in almost all known examples. Haag duality can be satisfied by extending the net [3] without changing the S-matrix. Hence we consider them as standard assumptions. On the other hand, asymptotic completeness is in fact a very strong condition. For example, a conformal net is asymptotically complete if and only if it is chiral [86]. Hence the class of asymptotically complete nets could be very small even among Poincaré covariant nets. But a clear-cut scattering theory is available only for asymptotically complete cases. The general case is under investigation [32].

5.6.2 Recovery of interacting net

We can express the modular objects of the interacting net in terms of the ones of the asymptotic chiral net.

Proposition 5.6.5. *Let Δ^{out} and J^{out} be the modular operator and the modular conjugation of $\mathcal{A}_+^{\mathrm{out}}(\mathbb{R}_-) \otimes \mathcal{A}_-^{\mathrm{out}}(\mathbb{R}_+)$ with respect to Ω. Then it holds that $\Delta = \Delta^{\mathrm{out}}$ and $J = SJ^{\mathrm{out}}$.*

Proof. First we note that the modular objects of $\mathcal{A}(W_\mathrm{R})$ restrict to $\mathcal{H}_+$ and $\mathcal{H}_-$ by Takesaki's theorem [82, Theorem IX.4.2]. Indeed, $\mathcal{A}_+^{\mathrm{out}}(\mathbb{R}_+)$ and $\mathcal{A}_-^{\mathrm{out}}(\mathbb{R}_-)$ are subalgebras of $\mathcal{A}(W_\mathrm{R})$ and invariant under $\mathrm{Ad}\Delta^{it}$, or equivalently under the Lorentz boosts $\mathrm{Ad}U(-2\pi t)$ by Bisognano-Wichmann property, as we saw in the proof of Proposition 5.6.1, then the projections onto the respective subspaces commute with the modular objects. Let us denote these restrictions by $\Delta_+^{it}, J_+, \Delta_-^{it}$ and J_-, respectively.

We identify $\mathcal{H}_+ \otimes \mathcal{H}_-$ and the full Hilbert space $\mathcal{H}$ by the action of $\mathcal{A}_+^{\mathrm{out}} \otimes \mathcal{A}_-^{\mathrm{out}}$. By Bisognano-Wichmann property and Lemma 5.5.3, we have

$$\begin{aligned} \Delta^{it} \cdot \xi \overset{\mathrm{out}}{\times} \eta &= (U(-2\pi t)\xi) \overset{\mathrm{out}}{\times} (U(-2\pi t)\eta) \\ &= \Delta_+^{it}\xi \otimes \Delta_-^{it}\eta \\ &= (\Delta_+ \otimes \Delta_-)^{it} \cdot \xi \otimes \eta, \end{aligned}$$

which implies that $\Delta = \Delta_+ \otimes \Delta_- = \Delta^{\mathrm{out}}$.

As for modular conjugations, we take $x \in \mathcal{M}$ and $y \in \mathcal{M}'$ and set $\xi = \Phi_+^{\mathrm{out}}(x)\Omega$ and

$\eta = \Phi_-^{\mathrm{out}}(y)\Omega$. Then we use Lemma 5.5.2 to see

$$\begin{aligned} J \cdot \xi \overset{\mathrm{out}}{\times} \eta &= J \cdot \Phi_+^{\mathrm{out}}(x)\Phi_-^{\mathrm{out}}(y)\Omega \\ &= \Phi_+^{\mathrm{in}}(JxJ)\Phi_-^{\mathrm{in}}(JyJ)\Omega \\ &= (J\xi) \overset{\mathrm{in}}{\times} (J\eta) \\ &= S \cdot (J_+\xi) \overset{\mathrm{out}}{\times} (J_-\eta) \\ &= S \cdot (J_+ \otimes J_-) \cdot \xi \otimes \eta \end{aligned}$$

from which one infers that $J = S \cdot (J_+ \otimes J_-) = S \cdot J^{\mathrm{out}}$. □

Theorem 5.6.3 tells us that chiral conformal nets can be viewed as free field nets for massless two-dimensional theory (cf. [86]). Let us formulate the situation the other way around. Let $\mathcal{A}_+ \otimes \mathcal{A}_-$ be a chiral CFT, then it is an interesting open problem to characterize unitary operators which can be interpreted as a S-matrix of a net whose asymptotic net is the given $\mathcal{A}_+ \otimes \mathcal{A}_-$. We restrict ourselves to point out that there are several immediate necessary conditions: For example, S must commute with the Poincaré symmetry of the chiral net since it coincides with the one of the interacing net. Analogously it must hold that $(J_+ \otimes J_-)S(J_+ \otimes J_-) = S^*$. Furthermore, the algebra of the form as in Proposition 5.6.1 must be strictly local.

If one has an appropriate operator S, an interacting wedge-local net can be constructed by (cf. Propositions 5.6.1, 5.6.2)

- $\mathcal{M}_S := \{x \otimes \mathbb{1}, \mathrm{Ad}S(\mathbb{1} \otimes y) : x \in \mathcal{A}_+(\mathbb{R}_-), y \in \mathcal{A}_-(\mathbb{R}_+)\}''$,
- $U := U_+ \otimes U_-$,
- $\Omega := \Omega_+ \otimes \Omega_-$.

By the formula for the modular conjugation in Proposition 5.6.5, it is immediate to see that

$$\mathcal{M}_S' := \{\mathrm{Ad}S(x \otimes \mathbb{1}), \mathbb{1} \otimes y : x \in \mathcal{A}_+(\mathbb{R}_+), y \in \mathcal{A}_-(\mathbb{R}_-)\}''.$$

Then for $x \in \mathcal{A}_+(\mathbb{R}_-), y \in \mathcal{A}_-(\mathbb{R}_+)$ it holds that $\Phi_+^{\mathrm{out}}(x \otimes \mathbb{1}) = x \otimes \mathbb{1}$ and $\Phi_-^{\mathrm{in}}(\mathrm{Ad}S(\mathbb{1} \otimes y)) = \mathrm{Ad}S(\mathbb{1} \otimes y)$. Similarly, we have $\Phi_-^{\mathrm{out}}(x) = x$ and $\Phi_+^{\mathrm{in}}(\mathrm{Ad}S(\mathbb{1} \otimes y)) = \mathrm{Ad}S(\mathbb{1} \otimes y)$ for $x \in \mathcal{A}_+(\mathbb{R}_+)$ and $y \in \mathcal{A}_-(\mathbb{R}_-)$. From this it is easy to see that S is indeed the S-matrix of the constructed wedge-local net.

In the following Sections we will construct unitary operators which comply with these conditions except strict locality. To my opinion, however, the true difficulty is the strict locality, which has been so far established only for "regular" massive models [59]. But it is also true that the class of S-matrices constructed in the present Chapter can be seen rather small (see the discussion in Section 5.9).

5.7 Construction through one-parameter semigroup of endomorphisms

In this Section, we construct families of wedge-local nets using one-parameter semigroup of endomorphisms of Longo-Witten type. The formula to define the von Neumann algebra is very simple and the proofs use a common argument based on spectral decomposition.

Our construction is based on chiral conformal nets on S^1, and indeed one family can be identified as the deformation of chiral nets (see Section 5.7.2). But in our construction, the meaning of the term "deformation" is not clear and we refrain from using it. From now on, we consider only chiral net with the identical components $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_0$ for simplicity. It is not difficult to generalize it to "heterotic case" where $\mathcal{A}_1 \neq \mathcal{A}_2$.

5.7.1 The commutativity lemma

The following Lemma is the key of all the arguments and will be used later in this Section concerning one-parameter endomorphisms. Typical examples of the operator Q_0 in Lemma will be the generator of one-dimensional translations P_0 (Section 5.7.2), or of one-parameter inner symmetries of the chiral component (Section 5.7.4).

As a preliminary, we give a remark on tensor product. See [30] for a general account on spectral measure and measurable family. Let E_0 be a projection-valued measure on Z (typically, the spectral measure of some self-adjoint operator) and $V(\lambda)$ be a measurable family of operators (bounded or not). Then one can define an operator

$$\int_Z V(\lambda) \otimes dE_0(\lambda)(\xi \otimes \eta) := \int_Z V(\lambda)\xi \otimes dE_0(\lambda)\eta.$$

If $V(\lambda)$ is unbounded, the pair ξ should be in a common domain of $\{V(\lambda)\}$. As we will see, this will not matter in our cases. For two bounded measurable families V, V', it is easy to see that

$$\int_Z V(\lambda) \otimes dE_0(\lambda) \cdot \int_Z V'(\lambda) \otimes dE_0(\lambda) = \int_Z V(\lambda)V'(\lambda) \otimes dE_0(\lambda).$$

Lemma 5.7.1. *We fix a parameter $\kappa \in \mathbb{R}$. Let Q_0 be self-adjoint operators on $\mathcal{H}_0$ and Let $Z \subset \mathbb{R}$ be the spectral supports of Q_0. If it holds that $[x, \mathrm{Ad}e^{is\kappa Q_0}(x')] = 0$ for $x, x' \in B(\mathcal{H}_0)$ and $s \in Z$, then we have that*

$$[x \otimes \mathbb{1}, \mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(x' \otimes \mathbb{1})] = 0,$$
$$[\mathbb{1} \otimes x, \mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(\mathbb{1} \otimes x')] = 0.$$

Proof. We prove only the first commutation relation, since the other is analogous. Let $Q_0 = \int_Z s \cdot dE_0(s)$ be the spectral decomposition of Q_0. According to this spectral decomposition, we can decompose only the second component:

$$Q_0 \otimes Q_0 = Q_0 \otimes \int_Z s \cdot dE_0(x) = \int_Z sQ_0 \otimes dE_0(s).$$

Hence we can describe the adjoint action of $e^{i\kappa Q_0 \otimes Q_0}$ explicitly:

$$\begin{aligned}
\mathrm{Ad}e^{i\kappa Q_0\otimes Q_0}(x'\otimes \mathbb{1}) &= \int_Z e^{is\kappa Q_0}\otimes dE(s)\cdot (x'\otimes\mathbb{1})\cdot \int_Z e^{-is\kappa Q_0}\otimes dE_0(s)\\
&= \int_Z \left(\mathrm{Ad}e^{is\kappa Q_0}(x')\right)\otimes dE_0(s)
\end{aligned}$$

Then it is easy to see that this commutes with $x\otimes \mathbb{1}$ by the assumed commutativity. $\square$

5.7.2 Construction of wedge-local nets with respect to translation

The objective here is to apply the commutativity lemma in Section 5.7.1 to the endomorphism of translation. Then it turns out that the wedge-local nets obtained by the BLS deformation of a chiral net coincides with this construction. A new feature is that our construction involves only von Neumann algebras.

Construction of wedge-local nets

Let $(\mathcal{M}, T, \Omega)$ be a chiral wedge-local net with chiral component $\mathcal{A}_0$ and $T_0(t) = e^{itP_0}$ the chiral translation: Namely, $\mathcal{M} = \mathcal{A}_0(\mathbb{R}_-)\otimes \mathcal{A}_0(\mathbb{R}_+)$, $T(t_0,t_1) = T_0\left(\frac{t_0-t_1}{\sqrt{2}}\right)\otimes T_0\left(\frac{t_0+t_1}{\sqrt{2}}\right)$ and $\Omega = \Omega_0\otimes\Omega_0$.

Note that $T_0(t)$ implements a Longo-Witten endomorphism of $\mathcal{A}_0$ for $t\geq 0$. In this sense, the construction of this Section is considered to be based on the endomorphisms $\{\mathrm{Ad}T_0(t)\}$. A nontrivial family of endomorphisms will be featured in Section 1.5.1.

We construct a new wedge-local net on the same Hilbert space $\mathcal{H} = \mathcal{H}_0\otimes\mathcal{H}_0$ as follows. Let us fix $\kappa\in\mathbb{R}_+$.

- $\mathcal{M}_{P_0,\kappa} := \{x\otimes\mathbb{1}, \mathrm{Ad}e^{i\kappa P_0\otimes P_0}(\mathbb{1}\otimes y), : x\in\mathcal{A}_0(\mathbb{R}_-), y\in\mathcal{A}_0(\mathbb{R}_+)\}''$,
- the same T from the chiral net,
- the same Ω from the chiral net.

Theorem 5.7.2. *Let $\kappa\geq 0$. Then the triple $(\mathcal{M}_{P_0,\kappa}, T, \Omega)$ is a wedge-local net with the S-matrix $S_{P_0,\kappa} = e^{i\kappa P_0\otimes P_0}$.*

Proof. The vector $\Omega_0\otimes\Omega_0$ is obviously invariant under T and T has the spectrum contained in V_+. The generator P_0 of one-dimensional translations obviously commutes with one-dimensional translation T_0, hence $P_0\otimes P_0$ commutes with $T = T_0\otimes T_0$, so does $e^{i\kappa P_0\otimes P_0}$. We claim that $\mathcal{M}_{P_0,\kappa}$ is preserved under translations in the right wedge. Indeed, if $(t_0,t_1)\in W_{\mathrm{R}}$, then we have

$$\mathrm{Ad}T(t_0,t_1)\left(\mathrm{Ad}e^{i\kappa P_0\otimes P_0}(\mathbb{1}\otimes y)\right) = \mathrm{Ad}e^{i\kappa P_0\otimes P_0}\left(\mathrm{Ad}T(t_0,t_1)(\mathbb{1}\otimes y)\right).$$

and $\mathrm{Ad}T(t_0,t_1)(\mathbb{1}\otimes y) \in \mathbb{1}\otimes\mathcal{A}_0(\mathbb{R}_+)$ and it is obvious that $\mathrm{Ad}T(t_0,t_1)(x\otimes\mathbb{1}) \in \mathcal{A}_0(\mathbb{R}_-)\otimes\mathbb{1}$, hence the generators of the von Neumann algebra $\mathcal{M}_{P_0,\kappa}$ are preserved.

We have to show that Ω is cyclic and separating for $\mathcal{M}_{P_0,\kappa}$. Note that it holds that $e^{i\kappa P_0\otimes P_0}\cdot \xi\otimes\Omega_0 = \xi\otimes\Omega_0$ for any $\kappa\in\mathbb{R}, \xi\in\mathcal{H}_0$, by the spectral calculus. Now cyclicity is seen by noting that

$$\begin{aligned}(x\otimes\mathbb{1})\cdot \mathrm{Ad}e^{i\kappa P_0\otimes P_0}(\mathbb{1}\otimes y)\cdot\Omega &= (x\otimes\mathbb{1})\cdot e^{i\kappa P_0\otimes P_0}\cdot(x\Omega_0)\otimes\Omega_0 \\ &= (x\Omega_0)\otimes(y\Omega_0)\end{aligned}$$

and by the cyclicity of Ω for the original algebra $\mathcal{M} = \mathcal{A}_0(\mathbb{R}_-)\otimes\mathcal{A}_0(\mathbb{R}_+)$.

Finally we show separating property as follows: we set

$$\mathcal{M}^1_{P_0,\kappa} = \{\mathrm{Ad}e^{i\kappa P_0\otimes P_0}(x'\otimes\mathbb{1}), \mathbb{1}\otimes y', x'\in\mathcal{A}_0(\mathbb{R}_+), y'\in\mathcal{A}_0(\mathbb{R}_-)\}''.$$

Note that Ω is cyclic for $\mathcal{M}^1_{P_0,\kappa}$ by an analogous proof for $\mathcal{M}_{P_0,\kappa}$, thus for the separating property, it suffices to show that $\mathcal{M}_{P_0,\kappa}$ and $\mathcal{M}^1_{P_0,\kappa}$ commute. Let $x, y'\in\mathcal{A}_0(\mathbb{R}_-), x'\in\mathcal{A}_0(\mathbb{R}_+)$. First, $x\otimes\mathbb{1}$ and $\mathbb{1}\otimes y'$ obviously commute. Next, we apply Lemma 5.7.1 to x, x' and $Q_0 = P_0$ to see that $x\otimes\mathbb{1}$ and $\mathrm{Ad}e^{i\kappa P_0\otimes P_0}(x'\otimes\mathbb{1})$ commute: Indeed, the spectral support of P_0 is $\mathbb{R}_+$, and for $s\in\mathbb{R}_+$, x and $\mathrm{Ad}e^{isP_0}(x')$ commute since P_0 is the generator of one-dimensional translations and since $x\in\mathcal{A}_0(\mathbb{R}_-), x\in\mathcal{A}_0(\mathbb{R}_+)$. Similarly, for $y\in\mathcal{A}_0(\mathbb{R}_-)$, $\mathbb{1}\otimes y$ and $\mathcal{M}^1_{P_0,\kappa}$ commute. This implies that $\mathcal{M}_{P_0,\kappa}$ and $\mathcal{M}^1_{P_0,\kappa}$ commute.

The S-matrix corresponds to the unitary used to twist the chiral net as we saw in the discussion at the end of Section 5.6.2. □

Now we have constructed a wedge-local net, it is possible to express its modular objects in terms of the ones of the chiral net by an analogous argument as Proposition 5.6.5. Then one sees that $\mathcal{M}^1_{P_0,\kappa}$ is indeed the commutant $\mathcal{M}'_{P_0,\kappa}$.

BLS deformation

Let us recall briefly the deformation procedure of [12]. Let $(\mathcal{M}, T, \Omega)$ be a wedge-local net. We denote by $\mathcal{M}^\infty$ the subset of elements of $\mathcal{M}$ which are smooth under the action of α in the norm topology. It is easy to see that $\mathcal{M}^\infty$ is a dense subalgebra of $\mathcal{M}$ in the strong operator topology. Let $\mathcal{D}$ be the dense domain of vectors which are smooth with respect to the action of T. Then one can define for any $x\in\mathcal{M}^\infty$, and a matrix $\Theta_\kappa = \begin{pmatrix}0 & \kappa\\ \kappa & 0\end{pmatrix}$, the warped convolution

$$x_\kappa = \int dE(a)\,\alpha_{\Theta_\kappa a}(F) := \lim_{\varepsilon\searrow 0}(2\pi)^{-2}\int d^2a\,d^2b\,f(\varepsilon a,\varepsilon y)e^{-ia\cdot b}\alpha_{\Theta_\kappa a}(x)T(b)$$

on a suitable domain, where dE is the spectral measure of T and $f\in\mathscr{S}(\mathbb{R}^2\times\mathbb{R}^2)$ satisfies $f(0,0)=1$. The limit exists in the strong sense on vectors from $\mathcal{D}$ and is independent of the function f within the above restrictions. We set

$$\mathcal{M}_\kappa := \{x_\kappa : x\in\mathcal{M}^\infty\}''.$$

For $\kappa > 0$, the following holds.

Theorem 5.7.3 ([12])**.** *If $(\mathcal{M}, T, \Omega)$ is a wedge-local net, then $(\mathcal{M}_\kappa, T, \Omega)$ is also a wedge-local net.*

We call the latter the **BLS deformation** of the original triple $(\mathcal{M}, T, \Omega)$. One of the main results of this Section is to obtain the BLS deformation by a simple procedure.

Let us consider the case where $(\mathcal{M}, T, \Omega)$ is a chiral wedge-local net. We can determine the collision states in terms of the original chiral structure.

Theorem 5.7.4. *For any $\xi \in \mathcal{H}_+$ and $\eta \in \mathcal{H}_-$, the following relations hold:*

$$\begin{array}{rcl} \xi \overset{\mathrm{out}}{\times}_\kappa \eta &=& e^{-\frac{i\kappa}{2}P_0 \otimes P_0}(\xi \otimes \eta), \\ \xi \overset{\mathrm{in}}{\times}_\kappa \eta &=& e^{\frac{i\kappa}{2}P_0 \otimes P_0}(\xi \otimes \eta), \end{array}$$

where on the left-hand sides there appear the collision states of the deformed theory.

Proof. Let us first prove the first relation. To this end, we pick $x \in \mathcal{M}^\infty$, $y \in (\mathcal{M}')^\infty$. We set $\xi = P_+ x\Omega = P_+ x_\kappa \Omega$ and $\eta = P_- y\Omega = P_- y_{(-\Theta_\kappa)}\Omega$, where we exploited the translational invariance of the state Ω. Since $x_{\Theta_\kappa} \in \mathcal{M}_\kappa$ and, by Theorem 5.7.3, $y_{(-\Theta_\kappa)} \in \mathcal{M}'_\kappa$, the outgoing state of the deformed theory is given by

$$\begin{array}{rcl} \xi \overset{\mathrm{out}}{\times}_\kappa \eta &=& \lim\limits_{\mathcal{T}\to\infty} x_{\Theta_\kappa,+}(h_\mathcal{T}) y_{(-\Theta_\kappa),-}(h_\mathcal{T})\Omega \\ &=& \lim\limits_{\mathcal{T}\to\infty} x_{\Theta_\kappa,+}(h_\mathcal{T}) y_-(h_\mathcal{T})\Omega \\ &=& \lim\limits_{\mathcal{T}\to\infty} \lim\limits_{\epsilon \searrow 0} (2\pi)^{-2} \displaystyle\int d^2s\, d^2t\, f(\epsilon s, \epsilon t) e^{-ist} \mathrm{Ad}T(\Theta_\kappa s)(x_+(h_\mathcal{T})) y_-(h_\mathcal{T})(t)\Omega, \end{array}$$

where in the last step we made use of the fact that $y_-(h_\mathcal{T})\Omega \in \mathcal{D}$, and that Ω is invariant under translations. To exchange the order of the limits, we use methods from the proof of Lemma 2.1 of [12]: We note that for each polynomial $(s,t) \to L(s,t)$, there exists a polynomial $(s,t) \to K(s,t)$ such that

$$L(s,t)e^{-ist} = K(-\partial_s, -\partial_t)e^{-ist}.$$

We choose L so that L^{-1} and its derivatives are absolutely integrable. Denoting temporarily $\zeta_\mathcal{T}(s,t) := \mathrm{Ad}T(\Theta_\kappa s)(x_+(h_\mathcal{T})) y_-(h_\mathcal{T})(t)\Omega$, we obtain

$$\begin{array}{rl} & \lim\limits_{\epsilon\searrow 0} (2\pi)^{-2} \displaystyle\int d^2s\, d^2t, f(\epsilon s, \epsilon t) e^{-ist} \zeta_\mathcal{T}(s,t) \\ = & \lim\limits_{\epsilon\searrow 0} (2\pi)^{-2} \displaystyle\int d^2s\, d^2t\, e^{-ist} K(\partial_s, \partial_t) f(\epsilon s, \epsilon t) L(s,t)^{-1} \zeta_\mathcal{T}(s,t) \\ = & (2\pi)^{-2} \displaystyle\int d^2s\, d^2t\, e^{-ist} K(\partial_s, \partial_t) L(s,t)^{-1} \zeta_\mathcal{T}(s,t), \end{array}$$

where in the first step we integrated by parts and in the second step we applied the dominated convergence theorem. To obtain the last expression, we used the fact that derivatives of $(s,t) \to f(\epsilon s, \epsilon t)$ contain powers of ϵ and thus vanish in the limit. Substituting this expression to formula and making use again of the dominated convergence theorem, we arrive at

$$\xi \overset{\mathrm{out}}{\times}_\kappa \eta = (2\pi)^{-2} \int d^2s\, d^2t\, e^{-ist} K(\partial_s, \partial_t) L(s,t)^{-1} (T(\Theta_\kappa s)\xi) \otimes (T(t)\eta).$$

To interchange the limit $\mathcal{T} \to \infty$ with the action of the derivatives, we exploited the fact that for any $x_1 \in \mathcal{M}^\infty$, $\mu \in \{0,1\}$, the derivative $\partial_{s^\mu} x_1 := (\partial_{s^\mu} x_1(s))|_{s=0}$ is an element of $\mathcal{M}^\infty$ and $\Phi_+^{\mathrm{out}}(\partial_{s^\mu} x_1)(s) = \partial_{s^\mu} \Phi_+^{\mathrm{out}}(x_1)(s)$. This equality (as well as its counterpart for Φ_-^{out}) follows immediately from the norm continuity of the respective map.

We introduce a (standard) notation $H := \frac{1}{\sqrt{2}}(P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0)$, $P := \frac{1}{\sqrt{2}}(P_0 \otimes \mathbb{1} - \mathbb{1} \otimes P_0)$, namely H is the Hamiltonian and P is the generator of the spacelike translation. Then it holds that $(H-P)\cdot \xi \otimes \Omega = \sqrt{2}(\mathbb{1} \otimes P_0) \cdot \xi \otimes \Omega = 0$ and

$$T(\Theta_\kappa s)\xi \otimes \Omega = e^{i\kappa(Hs_1 - Ps_0)} \xi \otimes \Omega = e^{-\frac{i\kappa}{2}(H+P)(s_0 - s_1)} \xi \otimes \Omega.$$

Similarly, since $(H-P)\cdot \Omega \otimes \eta = 0$, we obtain

$$T(t)\Omega \otimes \eta = e^{\frac{i}{2}(H-P)(t^0+t^1)} \Omega \otimes \eta.$$

Hence, using Lemma 5.5.3, we get

$$\begin{aligned}(T(\Theta_\kappa s)\xi) \otimes (T(t)\eta) &= e^{-\frac{i\kappa}{2}(H+P)(s_0-s_1)} e^{\frac{i}{2}(H-P)(t_0+t_1)} (\xi \otimes \eta) \\ &= T(v(s,t))(\xi \otimes \eta),\end{aligned}$$

where $v(s,t) = (\frac{1}{2}(t_0 + t_1 - \kappa s_0 + \kappa s_1), \frac{1}{2}(t_0 + t_1 + \kappa s_0 - \kappa s_1))$. We substitute this expression to the formula above to obtain

$$\begin{aligned}\xi \overset{\mathrm{out}}{\times}_\kappa \eta &= (2\pi)^{-2} \int d^2s\, d^2t\, e^{-ist} K(\partial_s, \partial_t) L(s,t)^{-1} T(v(s,t))(\xi \otimes \eta) \\ &= \int \left(\lim_{\epsilon \searrow 0} (2\pi)^{-2} \int d^2s\, d^2t\, e^{-ist} f(\epsilon s, \epsilon t) e^{ipv(s,t)} \right) dE(p)(\xi \otimes \eta).\end{aligned}$$

Here in the second step we expressed $T(v(\cdot,\cdot))$ as a spectral integral and used the Fubini theorem to exchange the order of integration. Now we analyze the function in the bracket above. Setting $p_\pm = \frac{1}{2}(p_0 \pm p_1)$, we get

$$\begin{aligned}&(2\pi)^{-2} \int d^2s\, d^2t\, e^{-ist} f(\epsilon s, \epsilon t) e^{ipv(s,t)} \\ =\ & (2\pi)^{-2} \int d^2s\, d^2t\, f(\epsilon(s_0, s_1), \epsilon(t_0, t_1)) e^{-i(\kappa p_+ + t_0)s_0} e^{i(\kappa p_+ + t^1)s^1} e^{ip_-(t_0+s_1)} \\ =\ & (2\pi)^{-1} \int d^2t\, \epsilon^{-2} \hat{f}(-\epsilon^{-1}(\kappa p_+ + t_0, \kappa p_+ + t_1), \epsilon(t_0, t_1)) e^{ip_-(t_0+t_1)} \\ =\ & (2\pi)^{-1} \int d^2t\, \hat{f}(-(t_0, t_1), \epsilon(\epsilon t_0 - \kappa p_+, \epsilon t_1 - \kappa p_+)) e^{ip_-((t_0+t_1)\epsilon - 2\kappa p_+)}.\end{aligned}$$

Here $\hat{f}$ denotes the Fourier transform of f w.r.t. the s variable and in the last step we made use of the change of variables: $(t_0, t_1) \to (\epsilon t_0 - \kappa p_+, \epsilon t_1 - \kappa p_+)$. Making use of the dominated convergence theorem, we perform the limit $\epsilon \searrow 0$, obtaining

$$\lim_{\epsilon \searrow 0} (2\pi)^{-2} \int d^2 s \, d^2 t \, e^{-ist} f(\epsilon s, \epsilon t) e^{ipv(s,t)} = e^{-\frac{i\kappa}{2}((p_0)^2 - (p_1)^2)}.$$

This completes the proof for dense sets of vectors $\xi \in \mathcal{H}_+, \eta \in \mathcal{H}_-$. For arbitrary $\xi \overset{\text{out}}{\times} \eta$, the statement follows by the limiting procedure. □

We immediately obtain the scattering matrix:

Corollary 5.7.5. *The wedge-local net* $(\mathcal{M}_\kappa, T, \Omega)$ *has the S-matrix*

$$S_\kappa = e^{i\kappa P_0 \otimes P_0}$$

Proof. Making use of Theorem 5.7.4, we obtain

$$\begin{aligned} S_\kappa(\xi \overset{\text{out}}{\times}_\kappa \eta) &= \xi \overset{\text{in}}{\times}_\kappa \eta \\ &= e^{\frac{i\kappa}{2} P_0 \otimes P_0} (\xi \otimes \eta) \\ &= e^{i\kappa P_0 \otimes P_0} (\xi \overset{\text{out}}{\times}_\kappa \eta). \end{aligned}$$

Asymptotic completeness is preserved under the deformation, since $e^{i\kappa P_0 \otimes P_0}$ is a unitary. □

Reproduction of BLS deformation

In this Section we show that the wedge-local net $(\mathcal{M}_{T_0,\kappa}, T, \Omega)$ obtained above is unitarily equivalent to the BLS deformation $(\mathcal{M}_\kappa, T, \Omega)$. Then we can calculate the asymptotic fields very simply. We use symbols $\overset{\text{out}}{\times}_\kappa, \overset{\text{in}}{\times}_\kappa$ to denote collision states with respect to the corresponding wedge-local nets with $\mathcal{M}_\kappa$.

Lemma 5.7.6. *It holds that* $(x \otimes \mathbb{1})_{\Theta_\kappa} \xi \otimes \Omega = x\xi \otimes \Omega$.

Proof. The equation (2.2) from [12] translates in our notation to

$$(x \otimes \mathbb{1})_{\Theta_\kappa} = \lim_{\substack{B \nearrow \mathbb{R}^2 \\ F \nearrow \mathbb{1}}} \int_B \mathrm{Ad} U(\kappa t_1, \kappa t_0)(x \otimes \mathbb{1}) F dE(t_0, t_1),$$

where B is a bounded subset in $\mathbb{R}^2$ and F is a finite dimensional subspace in $\mathcal{H}_0$. Now it is easy to see that $(x \otimes \mathbb{1})_{\Theta_\kappa}(\xi \otimes \Omega) = x\xi \otimes \Omega$. Indeed, we have $\xi \otimes \Omega \in E(\mathrm{L}_+)$, where $\mathrm{L}_+ = \{(p_0, p_1) \in \mathbb{R}^2 : p_0 + p_1 = 0\}$, hence the integral above is concentrated in L_+, and for $(t,t) \in \mathrm{L}_+$ it holds that $\mathrm{Ad}U(\kappa t, \kappa t)(x \otimes \mathbb{1}) = \mathrm{Ad}\mathbb{1} \otimes U_0(\kappa t)(x \otimes \mathbb{1}) = x \otimes \mathbb{1}$. Then the integral simplifies as follows:

$$\begin{aligned}(x \otimes \mathbb{1})_{\Theta_\kappa}(\xi \otimes \Omega) &= \lim_{\substack{B \nearrow \mathbb{R}^2 \\ F \nearrow \mathbb{1}}} \int_{B \cap \mathrm{L}_+} \mathrm{Ad}U(\kappa t, \kappa t)(x \otimes \mathbb{1}) \cdot F \cdot dE(t,t)(\xi \otimes \Omega) \\ &= \lim_{\substack{B \nearrow \mathbb{R}^2 \\ F \nearrow \mathbb{1}}} \int_{B \cap \mathrm{L}_+} x \otimes \mathbb{1} \cdot F \cdot dE(t,t)(\xi \otimes \Omega) \\ &= x\xi \otimes \Omega.\end{aligned}$$

This is what we had to prove. □

Theorem 5.7.7. *Let us put $\mathcal{N}_{T_0,\kappa} := \mathrm{Ad}e^{-\frac{i\kappa}{2}P_0 \otimes P_0}\mathcal{M}_{T_0,\kappa}$. Then it holds that $\mathcal{N}_{T_0,\kappa} = \mathcal{M}_\kappa$, hence we have the coincidence of two wedge-local nets $(\mathcal{N}_{T_0,\kappa}, T, \Omega) = (\mathcal{M}_\kappa, T, \Omega)$.*

Proof. In the previous Seciton, we have seen that the deformed BLS triple is asymptotically complete and we have

$$\xi \overset{\mathrm{out}}{\times}_\kappa \eta = e^{-\frac{i\kappa}{2}P_0 \otimes P_0} \xi \overset{\mathrm{out}}{\times} \eta.$$

As for observables, let $x \in A_0(\mathbb{R}_-)$ and we use the notation x_{Θ_κ} from [12] [3]. For the asymptotic field $\Phi^{\mathrm{out}}_{\kappa,+}$ of BLS deformation, we have

$$\begin{aligned}\Phi^{\mathrm{out}}_{\kappa,+}((x \otimes \mathbb{1})_{\Theta_\kappa})\xi \overset{\mathrm{out}}{\times}_\kappa \eta &= ((x \otimes \mathbb{1})_{\Theta_\kappa}\xi) \overset{\mathrm{out}}{\times}_\kappa \eta \\ &= (x\xi) \overset{\mathrm{out}}{\times}_\kappa \eta \\ &= e^{-\frac{i\kappa}{2}P_0 \otimes P_0} \cdot (x\xi) \otimes \eta \\ &= e^{-\frac{i\kappa}{2}P_0 \otimes P_0} \cdot x \otimes \mathbb{1} \cdot \xi \otimes \eta \\ &= \mathrm{Ad}e^{-\frac{i\kappa}{2}P_0 \otimes P_0}(x \otimes \mathbb{1}) \cdot e^{-\frac{i\kappa}{2}P_0 \otimes P_0} \cdot \xi \otimes \eta \\ &= \mathrm{Ad}e^{-\frac{i\kappa}{2}P_0 \otimes P_0}(x \otimes \mathbb{1}) \cdot \xi \overset{\mathrm{out}}{\times}_\kappa \eta,\end{aligned}$$

by Lemma 5.7.6 for the first equality, hence, we have $\Phi^{\mathrm{out}}_{\kappa,+}((x \otimes \mathbb{1})_{\Theta_\kappa}) = \mathrm{Ad}e^{-\frac{i\kappa}{2}P_0 \otimes P_0}(x \otimes \mathbb{1})$. Analogously we have $\Phi^{\mathrm{in}}_{\kappa,-}((\mathbb{1} \otimes y)_{\Theta_\kappa}) = \mathrm{Ad}e^{\frac{i\kappa}{2}P_0 \otimes P_0}(\mathbb{1} \otimes y)$ for $y \in \mathcal{A}_0(\mathbb{R}_+)$.

Note that by definition we have

$$\mathcal{N}_{\mathcal{P}_0,\kappa} = \{\mathrm{Ad}e^{-\frac{i\kappa}{2}P_0 \otimes P_0}(x \otimes \mathbb{1}), \mathrm{Ad}e^{\frac{i\kappa}{2}P_0 \otimes P_0}(\mathbb{1} \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''.$$

Since the image of the right-wedge algebra by Φ^{out}_+ and Φ^{in}_- remains in the right-wedge algebra, from the above observation, we see that $\mathcal{N}_{P_0,\kappa} \subset \mathcal{M}_\kappa$ [34]. To see the converse inclusion, recall that it has been proved that the modular group Δ^{it} of the right-wedge

[3] The reader is suggested to look at the notation F_Q in [12], where F is an observable in $\mathcal{M}$ and Q is a 2×2 matrix. We keep the symbol Q for a generator of one-parameter automorphisms, hence we changed the notation to avoid confusions.

algebra with respect to Ω remains unchanged under the BLS deformation. We have that $\mathrm{Ad}\Delta^{it}(e^{i\kappa P_0\otimes P_0}) = e^{i\kappa P_0\otimes P_0}$, hence it is easy to see that $\mathcal{N}_{P_0,\kappa}$ is invariant under $\mathrm{Ad}\Delta^{it}$. By the theorem of Takesaki [82, Theorem IX.4.2], there is a conditional expectation from $\mathcal{M}_\kappa$ onto $\mathcal{N}_{P_0,\kappa}$ which preserves the state $\langle\Omega,\cdot\Omega\rangle$ and in particular, $\mathcal{M}_\kappa = \mathcal{N}_{P_0,\kappa}$ if and only if Ω is cyclic for $\mathcal{N}_{P_0,\kappa}$. We have already seen the cyclicity in Theorem 5.7.2, thus we obtain the thesis.

The translation T and Ω remain unchanged under $e^{-\frac{i\kappa}{2}P_0\otimes P_0}$, which established the unitary equivalence between two wedge-local nets. $\square$

Remark 5.7.8. It is also possible to formulate Theorem 5.6.3 for wedge-local net, although the asymptotic algebra will be neither local nor conformal in general. From this point of view, Theorem 5.7.7 is just a corollary of the coincidence of S-matrix. Here we preferred a direct proof, instead of formulating non local net on $\mathbb{R}$.

5.7.3 Endomorphisms with asymmetric spectrum

Here we briefly describe a generalization of the construction in previous Sections. Let $\mathcal{A}_0$ be a local net on S^1, T_0 be the representation of the translation. We assume that there is a one-parameter family $V_0(t) = e^{iQ_0t}$ of unitary operators with a positive or negative generator Q_0 such that $V_0(t)$ and $T_0(s)$ commute and $\mathrm{Ad}V_0(t)(\mathcal{A}_0(\mathbb{R}_+)) \subset \mathcal{A}_0(\mathbb{R}_+)$ for $t \geq 0$. With these ingredients, we have the following:

Theorem 5.7.9. *The triple*

- $\mathcal{M}_{Q_0,\kappa} := \{x\otimes\mathbb{1}, \mathrm{Ad}e^{\pm i\kappa Q_0\otimes Q_0}(\mathbb{1}\otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}_0(\mathbb{R}_+)\}''$,
- $T := T_0 \otimes T_0$,
- $\Omega := \Omega_0 \otimes \Omega_0$,

where $\pm$ *corresponds to* $\mathrm{sp}\, Q_0 \subset \mathbb{R}_\pm$, *is a wedge-local net with the S-matrix* $e^{\pm i\kappa Q_0\otimes Q_0}$.

The proof is analogous to Theorem 5.7.2 and we refrain from repeating it here.

The construction looks very simple, but to our knowledge, there are very few examples. The one-parameter group of translation itself has been studied in the previous Sections. Another one-parameter family of unitaries with a negative generator has been found for the $U(1)$-current. Indeed, by Borchers's theorem [6, 38], such one-parameter group together with the modular group forms a representation of the "$ax + b$" group, thus it is related somehow with translation.

5.7.4 Construction of wedge-local nets through inner symmetry in chiral CFT

Inner symmetry

Let $\mathcal{A}_0$ be a conformal (Möbius) net on S^1. Recall that an **automorphism** of $\mathcal{A}_0$ is a family of automorphisms $\{\alpha_{0,I}\}$ of local algebras $\{\mathcal{A}_0(I)\}$ with the consistency condition

$\alpha_{0,J}|_{\mathcal{A}_0(I)} = \alpha_{0,I}$ for $I \subset J$. If each $\alpha_{0,I}$ preserves the vacuum state ω, then α_0 is said to be an **inner symmetry**. An inner symmetry α_0 is implemented by a unitary U_{α_0} defined by $U_{\alpha_0}x\Omega = \alpha_{0,I}(x)\Omega$, where $x \in \mathcal{A}_0(I)$. This definition does not depend on the choice of I by the consistency condition. If $\alpha_{0,t}$ is a one-parameter family of weakly continuous automorphisms, then the implementing unitaries satisfy $V_{\alpha_0}(t)V_{\alpha_0}(s) = V_{\alpha_0}(t+s)$ and $V_{\alpha_0}(0) = \mathbb{1}$, hence there is a self-adjoint operator Q_0 such that $V_{\alpha_0}(t) = e^{itQ_0}$ and $Q_0\Omega = 0$. Furthermore, e^{itQ_0} commutes with modular objects [82]: $J_0 e^{itQ_0} J_0 = e^{itQ_0}$, or $J_0 Q_0 J_0 = -Q_0$ (note that J_0 is an anti-unitary involution). If α_t is periodic with period 2π, namely $a_{0,t} = a_{0,t+2\pi}$ then it holds that $V_{\alpha_0}(t) = V_{\alpha_0}(t+2\pi)$ and the generator Q_0 has a discrete spectrum $\mathrm{sp}\, Q_0 \subset \mathbb{Z}$. For the technical simplicity, we restrict ourselves to the study of periodic inner symmetries. We may assume that the period is 2π by a rescaling of the parameter.

Example 5.7.10. We consider the loop group net $\mathcal{A}_{G,k}$ of a (simple, simply connected) compact Lie group G at level k [41, 91], the net generated by vacuum representations of loop groups LG [75]. On this net, the original group G acts as a group of inner symmetries. We fix a maximal torus in G and choose a one-parameter group in the maximal torus with a rational direction, then it is periodic. Any one-parameter group is contained in a maximal torus, so there are a good proportion of periodic one-parameter groups in G (although generic one-parameter groups have irrational direction, hence not periodic). In particular, in the $SU(2)$-loop group net $\mathcal{A}_{SU(2),k}$, any one-parameter group in $SU(2)$ is periodic since $SU(2)$ has rank 1.

An inner automorphism α_0 commutes with Möbius symmetry because of Bisognano-Wichmann property. Hence it holds that $U_0(g)Q_0U_0(g)^* = Q_0$. Furthermore, if the net $\mathcal{A}_0$ is conformal, then α_0 commutes also with the diffeomorphism symmetry [26]. Let G be a group of inner symmetries and $\mathcal{A}_0^G$ be an assignment: $I \mapsto \mathcal{A}_0(I)^G|_{\mathcal{H}_0^G}$, where $\mathcal{A}_0(I)^G$ denotes the fixed point algebra of $\mathcal{A}_0(I)$ with respect to G and $\mathcal{H}_0^G := \overline{\{x\Omega_0 : x \in \mathcal{A}_0^G(I), I \subset S^1\}}$. Then it is easy to see that $\mathcal{A}_0^G$ is a Möbius covariant net and it is referred to as the **fixed point subnet** of $\mathcal{A}_0$ with respect to G.

We can describe the action α_0 of a periodic one-parameter group of inner symmetries in a very explicit way, which can be considered as the "spectral decomposition" of α_0. Although it is well-known, we summarize it here with a proof for the later use. This will be the basis of the subsequent analysis.

Proposition 5.7.11. *Any element $x \in \mathcal{A}_0(I)$ can be written as $x = \sum_n x_n$, where $x_n \in \mathcal{A}_0(I)$ and $\alpha_{0,t}(x_n) = e^{int}x_n$. We denote $\mathcal{A}_0(I)_n = \{x \in \mathcal{A}_0(I) : \alpha_{0,t}(x) = e^{int}x\}$. It holds that $\mathcal{A}_0(I)_m\mathcal{A}_0(I)_n \subset \mathcal{A}_0(I)_{m+n}$ and $\mathcal{A}_0(I)_m E_0(n)\mathcal{H}_0 \subset E_0(m+n)\mathcal{H}_0$, where $E_0(n)$ denotes the spectral projection of Q_0 corresponding to the eigenvalue $n \in \mathbb{Z}$.*

Proof. Let us fix an element $x \in \mathcal{A}_0(I)$. The Fourier transform

$$x_n := \int_0^{2\pi} \alpha_s(x)e^{-ins}\, ds$$

(here we consider the weak integral using the local normality of $\alpha_{0,t}$) is again an element of $\mathcal{A}_0(I)$, since $\mathcal{A}_0(I)$ is invariant under $\alpha_{0,t}$. Furthermore it is easy to see that

$$\begin{aligned}
\alpha_{0,t}(x_n) &= \alpha_{0,t}\left(\int_0^{2\pi} \alpha_{0,s}(x)e^{-ins}\,ds\right) \\
&= \int_0^{2\pi} \alpha_{0,s+t}(x)e^{-ins}\,ds \\
&= e^{int}\int_0^{2\pi} \alpha_{0,s}(x)e^{-ins}\,ds = e^{int}x_n,
\end{aligned}$$

hence we have $x_n \in \mathcal{A}_0(I)_n$.

By assumption, $\alpha_{0,t}(x) = \mathrm{Ad}e^{itQ_0}(x)$ and $\mathrm{sp}\,Q_0 \subset \mathbb{Z}$. If we define $x_{l,m} = E_0(l)xE_0(m)$, it holds that $\mathrm{Ad}e^{itQ_0}x_{l,m} = e^{i(l-m)t}x_{l,m}$. The integral and this decomposition into matrix elements are compatible, hence for $x \in \mathcal{A}_0(I)$ we have

$$x_n = \sum_{l-m=n} x_{l,m}.$$

Now it is clear that $x = \sum_n x_n$ where each summand is a different matrix element, hence the sum is strongly convergent. Furthermore from this decomposition we see that $\mathcal{A}_0(I)_m\mathcal{A}_0(I)_n \subset \mathcal{A}_0(I)_{m+n}$ and $\mathcal{A}_0(I)_m E_0(n)\mathcal{H}_0 \subset E_0(m+n)\mathcal{H}_0$. □

At the end of this Section, we exhibit a simple formula for the adjoint action $\mathrm{Ad}e^{i\kappa Q_0\otimes Q_0}$ on the tensor product Hilbert space $\mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_0$.

Lemma 5.7.12. *For $x_m \in \mathcal{A}_0(I)_m, y_n \in \mathcal{A}_0(I)_n$, it holds that $\mathrm{Ad}e^{i\kappa Q_0\otimes Q_0}(x_m \otimes \mathbb{1}) = x_m \otimes e^{im\kappa Q_0}$ and $\mathrm{Ad}e^{i\kappa Q_0\otimes Q_0}(\mathbb{1} \otimes y_n) = e^{in\kappa Q_0} \otimes y_n$.*

Proof. Recall that $\mathrm{sp}\,Q_0 \in \mathbb{Z}$. Let $Q_0 = \sum_l l \cdot E_0(l)$ be the spectral decomposition of Q_0. As in the proof of Lemma 5.7.1, we decompose only the second component of $Q_0 \otimes Q_0$ to see that

$$\begin{aligned}
Q_0 \otimes Q_0 &= Q_0 \otimes \left(\sum_l l \cdot E_0(l)\right) = \sum_l lQ_0 \otimes E_0(l) \\
e^{i\kappa Q_0\otimes Q_0} &= \sum_l e^{il\kappa Q_0} \otimes E_0(l) \\
\mathrm{Ad}e^{i\kappa Q_0\otimes Q_0}(x_m \otimes \mathbb{1}) &= \sum_l \mathrm{Ad}e^{il\kappa Q_0}(x_m) \otimes E_0(l) \\
&= \sum_l e^{iml\kappa}x_m \otimes E_0(l) \\
&= x_m \otimes e^{im\kappa Q_0}.
\end{aligned}$$

□

Proposition 5.7.13. *For each $l \in \mathbb{Z}$ there is a cyclic and separating vector $v \in E_0(l)\mathcal{H}_0$ for a local algebra $\mathcal{A}_0(I)$.*

Proof. It is enough to note that the decomposition $\mathbb{1} = \sum_l E_0(l)$ is compatible with the decomposition of the whole space with respect to rotations, since inner symmetries commute with any Möbius transformation. Hence each space $E_0(l)\mathcal{H}_0$ is a direct sum of eigenspace of rotation. It is a standard fact that a eigenvector of rotation which has positive spectrum is cyclic and separating for each local algebra (see the standard proof of Reeh-Schlieder property, e.g. [3]). □

We put $E(l,l') := E_0(l) \otimes E_0(l')$.

Corollary 5.7.14. *Each space $E(l,l')\mathcal{H}$ contains a cyclic and separating vector v for $\mathcal{A}_0(I) \otimes \mathcal{A}_0(J)$ for any pair of intervals I, J.*

Construction of wedge-local nets and their intersection property

Let $\mathcal{A}_0$ be a Möbius covariant net and $\alpha_{0,t}$ be a periodic one-parameter group of inner symmetries. The automorphisms can be implemented as $\alpha_{0,t} = \mathrm{Ad}e^{itQ_0}$ as explained in Section 5.7.4. The self-adjoint operator Q_0 is referred to as the generator of the inner symmetry.

We construct a Borchers triple as in Section 5.7.2. Let $\kappa \in \mathbb{R}$ be a real parameter (this time κ is not necessarily positive) and we put

$$\begin{aligned}
\mathcal{M}_{Q_0,\kappa} &:= \{x \otimes \mathbb{1}, \mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(\mathbb{1} \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}(\mathbb{R}_+)\}'' \\
T(t_0,t_1) &:= T_0\left(\frac{t_0 - t_1}{\sqrt{2}}\right) \otimes T_0\left(\frac{t_0 + t_1}{\sqrt{2}}\right) \\
\Omega &:= \Omega_0 \otimes \Omega_0
\end{aligned}$$

Theorem 5.7.15. *The triple $(\mathcal{M}_{Q_0,\kappa}, T, \Omega)$ above is a Borchers triple with a nontrivial scattering operator $S_{Q_0,\kappa} = e^{i\kappa Q_0 \otimes Q_0}$.*

Proof. As remarked in Section 5.7.4, Q_0 commutes with Möbius symmetry U_0, hence $Q_0 \otimes Q_0$ and the translation $T = T_0 \otimes T_0$ commute. Since $(\mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}(\mathbb{R}_+), T, \Omega)$ is a wedge-local net (see Section 5.5), it holds that $\mathrm{Ad}T(t_0,t_1)\mathcal{M} \subset \mathcal{M}$ for $(t_0,t_1) \in W_{\mathrm{R}}$ and $T(t_0,t_1)\Omega = \Omega$ and T has the joint spectrum contained in V_+.

Since $\alpha_{0,t}$ is a one-parameter group of inner symmetries, it holds that $\alpha_{0,s}(\mathcal{A}_0(\mathbb{R}_-)) = \mathcal{A}_0(\mathbb{R}_-)$ and $\alpha_{0,t}(\mathcal{A}_0(\mathbb{R}_+)) = \mathcal{A}_0(\mathbb{R}_+)$ for $s,t \in \mathbb{R}$. By Lemma 5.7.1, for $x \in \mathcal{A}_0(\mathbb{R}_-)$ and $x' \in \mathcal{A}_0(\mathbb{R}_+)$ it holds that

$$[x \otimes \mathbb{1}, \mathrm{Ad}e^{i\kappa Q_0 \otimes Q_0}(x' \otimes \mathbb{1})] = 0.$$

Then one can show that $(\mathcal{M}_{Q_0,\kappa}, T, \Omega)$ is a wedge-local net as in the proof of Theorem 5.7.2. □

We now proceed to completely determine the intersection property of $\mathcal{M}_{Q_0,\kappa}$. As a preliminary, we describe the elements in $\mathcal{M}_{Q_0,\kappa}$ in terms of the original algebra $\mathcal{M}$ componentwise. On $\mathcal{M} = \mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+)$, there acts the group $S^1 \otimes S^1$ by the tensor product action: $(s,t) \mapsto \alpha_{s,t} := \alpha_{0,s} \otimes \alpha_{0,t} = \mathrm{Ad}(e^{isQ_0} \otimes e^{itQ_0})$. According to this action, we have a decomposition of an element $z \in \mathcal{M}$ into Fourier components as in Section 5.7.4:

$$z_{m,n} := \int_{S^1 \times S^1} \alpha_{s,t}(z) e^{-i(ms+nt)}\, ds\, dt,$$

which is still an element of $\mathcal{M}$, and with $E(l,l') := E_0(l) \otimes E_0(l')$, these components can be obtained by

$$z_{m,n} = \sum_{\substack{l-k=m \\ l'-k'=n}} E(l,l') z E(k,k').$$

One sees that $\mathrm{Ad}(e^{isQ_0} \otimes e^{itQ_0})$ acts also on $\mathcal{M}_{Q_0,\kappa}$ since it commutes with $\mathrm{Ad} e^{i\kappa Q_0 \otimes Q_0}$. We still write this action by α. We can take their Fourier components by the same formula and the formula with spectral projections still holds.

Lemma 5.7.16. *An element $z_\kappa \in \mathcal{M}_{Q_0,\kappa}$ has the components of the form*

$$(z_\kappa)_{m,n} = z_{m,n}(e^{in\kappa Q_0} \otimes \mathbb{1}),$$

where $z = (z_{m,n})$ is some element in $\mathcal{M}$. Similarly, an element $z'_\kappa \in \mathcal{M}'_{Q,\kappa}$ has the components of the form

$$(z'_\kappa)_{m,n} = z_{m,n}(\mathbb{1} \otimes e^{im\kappa Q_0}),$$

where $z' = (z_{m,n})$ is some element in $\mathcal{M}'$.

Proof. We will show only the former statement since the latter is analogous. First we consider an element of a simple form $(x_m \otimes \mathbb{1})S(\mathbb{1} \otimes y_n)S^*$, where $x_m \in \mathcal{A}(\mathbb{R}_-)_m$ and $y_n \in \mathcal{A}(\mathbb{R}_+)_n$. We saw in Proposition 5.7.12 that this is equal to $(x_m \otimes y_n)(e^{i\kappa n Q_0} \otimes \mathbb{1})$, thus this has the asserted form. Note that the linear space spanned by these elements for different m,n is closed even under product. For a finite product and sum, the thesis is linear with respect to x and y, hence we obtain the desired decomposition. The von Neumann algebra $\mathcal{M}_{Q,\kappa}$ is linearly generated by these elements. Recalling that $z_{m,n}$ is a matrix element with respect to the decomposition $\mathbb{1} = \sum_{l,l'} E(l,l')$, we obtain the Lemma. □

Now we are going to determine the intersection of wedge algebras. At this point, we need to use unexpectedly strong additivity and conformal covariance (see Section 1.1.1). The fixed point subnet $\mathcal{A}_0^{\alpha_0}$ of a strongly additive net $\mathcal{A}_0$ on S^1 with respect to the action α_0 of a compact group G of inner symmetry is again strongly additive [98].

Example 5.7.17. The loop group nets $\mathcal{A}_{SU(N),k}$ are completely rational [41, 97], hence in particular they are strongly additive. Moreover, they are conformal [75].

If $\mathcal{A}$ is diffeomorphism covariant, the strong additivity follows from the split property and the finiteness of μ-index [65]. We have plenty of examples of nets which satisfy strong additivity and conformal covariance since it is known that complete rationality passes to finite index extensions and finite index subnets [62].

Theorem 5.7.18. *Let $\mathcal{A}_0$ be strongly additive and conformal. We write, with a little abuse of notation, $T(t_+, t_-) := T_0(t_+) \otimes T_0(t_-)$. For $t_+ < 0$ and $t_- > 0$ we have*

$$\mathcal{M}_{Q_0,\kappa} \cap \left(\mathrm{Ad}T(t_+, t_-)(\mathcal{M}'_{Q,\kappa})\right) = \mathcal{A}_0^G(t_+, 0) \otimes \mathcal{A}_0^G(0, t_-),$$

where G is the group of automorphisms generated by $\mathrm{Ad}e^{i\kappa Q_0}$.

Proof. Let us consider an element from the intersection. From Lemma 5.7.16, we have two descriptions of such an element, namely,

$$\begin{aligned} (z_\kappa)_{m,n} &= z_{m,n}(e^{in\kappa Q_0} \otimes \mathbb{1}), \;\; z \in \mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+), \\ (z'_\kappa)_{m,n} &= z'_{m,n}(\mathbb{1} \otimes e^{im\kappa Q_0}), \;\; z' \in \mathcal{A}_0(\mathbb{R}_+ + t_0) \otimes \mathcal{A}(\mathbb{R}_- + t_1). \end{aligned}$$

If these elements have to coincide, each (m, n) component has to coincide. Or equivalently, it should happen that $z_{m,n}(e^{in\kappa Q_0} \otimes e^{-im\kappa Q_0}) = z'_{m,n}$.

Recall that an inner symmetry commutes with diffeomorphisms [26]. This implies that the fixed point subalgebra contains the representatives of diffeomorphisms. Furthermore, the fixed point subalgebra by a compact group is again strongly additive [98]. This means that

$$\mathcal{A}_0^{\alpha_0}(-\infty, t_+) \vee \mathcal{A}_0^{\alpha_0}(0, \infty) = \mathcal{A}_0^{\alpha_0}((t_+, 0)'),$$
$$\mathcal{A}_0^{\alpha_0}(-\infty, 0) \vee \mathcal{A}_0^{\alpha_0}(t_-, \infty) = \mathcal{A}_0^{\alpha_0}((0, t_-)'),$$

where I' means the complementary interval in S^1.

We claim that if for $z \in \mathcal{A}_0(\mathbb{R}_-) \otimes \mathcal{A}_0(\mathbb{R}_+)$ and $z' \in \mathcal{A}_0(\mathbb{R}_+ + t_+) \otimes \mathcal{A}(\mathbb{R}_- + t_-)$ there holds $z \cdot (e^{im\kappa Q} \otimes e^{-in\kappa Q}) = z'$, then $z = z' \in (\mathcal{A}(\mathbb{R}_-) \cap \mathcal{A}(\mathbb{R}_+ + t_0)) \otimes (\mathcal{A}(\mathbb{R}_+) \cap \mathcal{A}(\mathbb{R}_- + t_-))$. Indeed, since $z \in \mathcal{A}(\mathbb{R}_-) \otimes \mathcal{A}(\mathbb{R}_+)$, it commutes with $U(g_+ \times g_-)$ with $\mathrm{supp}(g_+) \subset \mathbb{R}_+$ and $\mathrm{supp}(g_-) \subset \mathbb{R}_-$. Similarly, $z' \in \mathcal{A}(\mathbb{R}_+ + t_0) \otimes \mathcal{A}(\mathbb{R}_- + t_1)$ commutes with $U(g_+ \times g_-)$ with $\mathrm{supp}(g_+) \subset \mathbb{R}_- + t_+$ and $\mathrm{supp}(g_-) \subset \mathbb{R}_+ + t_-$. Furthermore, the unitary $e^{im\kappa Q} \otimes e^{-in\kappa Q}$ which implements an inner symmetry commutes with any action of diffeomorphism [26]. Recall that the fixed point subalgebra is strongly additive, hence by the assumed equality $z \cdot (e^{im\kappa Q} \otimes e^{-in\kappa Q}) = z'$, this element commutes with $\mathcal{A}_0^{\alpha_0}((t_+, 0)') \otimes \mathcal{A}_0^{\alpha_0}((0, t_-)')$. In particular, it commutes with any diffeomorphism of $S^1 \times S^1$ supported in $(t_+, 0)' \times (0, t_-)'$. There is a sequence of diffeomorphisms g_i which take $\mathbb{R}_- \times \mathbb{R}_+$ to $(t_+ - \varepsilon_i, 0) \times (0, t_- + \varepsilon_i)$ with support disjoint from $(t_+, 0) \times (0, t_-)$ for arbitrary small $\varepsilon_i > 0$. This fact and the diffeomorphism covariance imply that z is indeed contained in $\mathcal{A}_0(t_+, 0) \otimes \mathcal{A}_0(t_-, 0)$. By a similar reasoning, one sees that $z' \in \mathcal{A}_0(t_+, 0) \otimes \mathcal{A}_0(0, t_-)$ as well. Now by Reeh-Schlieder property for $\mathcal{A}_0(t_+, 0) \otimes \mathcal{A}_0(0, t_1)$ we have $z = z'$ since $z\Omega = z \cdot (e^{im\kappa Q} \otimes e^{-in\kappa Q})\Omega = z'\Omega$.

Thus, if $z_{m,n}(e^{im\kappa Q} \otimes e^{-in\kappa Q}) = z'_{m,n}$, then $z_{m,n} = z'_{m,n} \in \mathcal{A}_0(t_+, 0) \otimes \mathcal{A}_0(0, t_-)$. Furthermore, by Corollary 5.7.14, there is a separating vector $v \in E(l, l')\mathcal{H}$. Now it holds that $e^{inl\kappa - iml'\kappa} z_{m,n} v = z'_{m,n} v$, hence from the separating property of v it follows that $e^{inl\kappa - iml'\kappa} z_{m,n} = z'_{m,n}$ for each pair $(l, l') \in \mathbb{Z} \times \mathbb{Z}$. This is possible only if both $n\kappa$ and $m\kappa$ are 2π multiple of an integer or $z_{m,n} = z'_{m,n} = 0$. This is equivalent to that $\mathrm{Ad}e^{i\kappa m Q_0} \otimes e^{i\kappa n Q_0}(z) = z$, namely, z is an element of the fixed point algebra $\mathcal{A}_0^G(I) \otimes \mathcal{A}_0^G(J)$ by the action $\mathrm{Ad}e^{i\kappa m Q_0} \otimes e^{i\kappa n Q_0}$ of $G \times G$. $\square$

Note that the size of the intersection is very sensitive to the parameter κ: If κ is 2π-multiple of a rational number, then the inclusion $[\mathcal{A}_0, \mathcal{A}_0^G]$ has finite index. Otherwise, it has infinite index.

Finally, we comment on the net generated by the intersection. The intersection takes a form of chiral net $\mathcal{A}_0^G \otimes \mathcal{A}_0^G$ where G is generated by $\mathrm{Ad}e^{i\kappa Q_0}$, hence the S-matrix is trivial [34]. This result is expected also from [86], where Möbius covariant net has always trivial S-matrix. Our deformation is based on inner symmetries which commute with Möbius symmetry, hence the net of strictly local elements is necessarily Möbius covariant, then it should have trivial S-matrix. But from this simple argument one cannot infer that the intersection should be asymptotically complete, or equivalently chiral. This exact form of the intersection can be found only by the present argument.

Construction through cyclic group actions

Here we briefly comment on the actions by the cyclic group $\mathbb{Z}_k$. In previous Sections, we have constructed wedge-local nets for the action of S^1. It is not difficult to replace S^1 by a finite group $\mathbb{Z}_k$. Indeed, the main ingredient was the existence of the Fourier components. For $\mathbb{Z}_k$ actions, the discrete Fourier transform is available and all the arguments work parallelly (or even more simply). For the later use, we state only the result without repeating the obvious modification of definitions and proofs.

Theorem 5.7.19. *Let $\mathcal{A}_0$ be a strongly additive conformal net on S^1 and $\alpha_{0,n} = \mathrm{Ad}e^{i\frac{2\pi n}{k}Q_0}$ be an action of $\mathbb{Z}_k$ as inner symmetries. Then, for $n \in \mathbb{Z}_k$, the triple*

$$\begin{aligned}
\mathcal{M}_{Q_0,n} &:= \{x \otimes \mathbb{1}, \mathrm{Ad}e^{i\frac{2\pi n}{k}Q_0\otimes Q_0}(\mathbb{1} \otimes y) : x \in \mathcal{A}_0(\mathbb{R}_-), y \in \mathcal{A}(\mathbb{R}_+)\}'' \\
T(t_0, t_1) &:= T_0\left(\frac{t_0 - t_1}{\sqrt{2}}\right) \otimes T_0\left(\frac{t_0 + t_1}{\sqrt{2}}\right) \\
\Omega &:= \Omega_0 \otimes \Omega_0
\end{aligned}$$

is an asymptotically complete wedge-local net with S-matrix $e^{i\frac{2\pi n}{k}Q_0\otimes Q_0}$. As for strictly local elements, we have

$$\mathcal{M}_{Q_0,\kappa} \cap \left(\mathrm{Ad}T(t_+, t_-)(\mathcal{M}'_{Q,\kappa})\right) = \mathcal{A}_0^G(t_+, 0) \otimes \mathcal{A}_0^G(0, t_-),$$

where G is the group of automorphisms of $\mathcal{A}_0$ generated by $\mathrm{Ad}e^{i\frac{2\pi n}{k}Q_0}$.

5.8 Construction through a family of endomorphisms on the $U(1)$-current net

In this Section, we construct a wedge-local net based on the $U(1)$-current model for a fixed φ, the boundary value of an inner symmetric function (see Section 1.5.1). Many operators are naturally defined on the unsymmetrized Fock space, hence we always keep in mind the inclusion $\mathcal{H}_s^\Sigma \subset \mathcal{H}^\Sigma$. The full Hilbert space for the two-dimensional wedge-local nets will be $\mathcal{H}_s^\Sigma \otimes \mathcal{H}_s^\Sigma$.

On $\mathcal{H}^m$, there act m commuting operators

$$\{\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1} : 1 \le i \le m\}.$$

We construct a unitary operator by the functional calculus on the corresponding spectral measure. We set

- $P_{i,j}^{m,n} := (\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1}) \otimes (\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1})$, which acts on $\mathcal{H}^m \otimes \mathcal{H}^n$, $1 \le i \le m$ and $1 \le j \le n$.

- $S_\varphi^{m,n} := \prod_{i,j} \varphi(P_{i,j}^{m,n})$, where $\varphi(P_{i,j}^{m,n})$ is the functional calculus on $\mathcal{H}^m \otimes \mathcal{H}^n$.

- $S_\varphi := \bigoplus_{m,n} S^{m,n} = \bigoplus_{m,n} \prod_{i,j} \varphi(P_{i,j}^{m,n})$

By construction, the operator S_φ acts on $\mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma$. Furthermore, it is easy to see that S_φ commutes with both $P_s \otimes \mathbb{1}$ and $\mathbb{1} \otimes P_s$: In other words, S_φ naturally restricts to partially symmetrized subspaces $\mathcal{H}_s^\Sigma \otimes \mathcal{H}^\Sigma$ and $\mathcal{H}^\Sigma \otimes \mathcal{H}_s^\Sigma$ and to the totally symmetrized space $\mathcal{H}_s^\Sigma \otimes \mathcal{H}_s^\Sigma$. Note that $S_\varphi^{m,n}$ is a unitary operator on the Hilbert spaces $\mathcal{H}^m \otimes \mathcal{H}^n$ and S_φ is the direct sum of them.

Let $E_1 \otimes E_1 \otimes \cdots \otimes E_1$ be the joint spectral measure of operators $\{\mathbb{1} \otimes \cdots \otimes \underset{j\text{-th}}{P_1} \otimes \cdots \otimes \mathbb{1} : 1 \le j \le n\}$. The operators $\{\varphi_{i,j}^{m,n} : 1 \le i \le m, 1 \le j \le n\}$ and $S^{m,n}$ are compatible with the spectral measure $\left(\overbrace{E_1 \otimes E_1 \otimes \cdots \otimes E_1}^{m\text{-times}}\right) \otimes \left(\overbrace{E_1 \otimes E_1 \otimes \cdots \otimes E_1}^{n\text{-times}}\right)$ and one has

$$\varphi(P_{i,j}^{m,n}) = \int \left(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\varphi(p_j P_1)} \otimes \cdots \mathbb{1}\right) \otimes \left(\mathbb{1} \otimes \cdots \underset{j\text{-th}}{dE_1(p_j)} \otimes \cdots \mathbb{1}\right).$$

Note that for $m = 0$ or $n = 0$ we have $\varphi_{i,j}^{m,n} = \mathbb{1}$ since we have only $p_j = 0$ or $p_i = 0$, respectively.

According to this spectral decomposition, we decompose S_φ with respect only to the right component as in the commutativity Lemma 5.7.1:

$$
\begin{aligned}
S_\varphi &= \bigoplus_{m,n} \prod_{i,j} \varphi(P_{i,j}^{m,n}) \\
&= \bigoplus_{m,n} \prod_{i,j} \int \Big(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\varphi(p_j P_1)} \otimes \cdots \mathbb{1} \Big) \otimes dE_0(p_1) \otimes \cdots \otimes dE(p_n) \\
&= \bigoplus_{m,n} \int \prod_{i,j} \Big(\mathbb{1} \otimes \cdots \otimes \underset{i\text{-th}}{\varphi(p_j P_1)} \otimes \cdots \mathbb{1} \Big) \otimes dE_0(p_1) \otimes \cdots \otimes dE(p_n) \\
&= \bigoplus_{n} \int \bigoplus_{m} \prod_{j} (\varphi(p_j P_1))^{\otimes m} \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n) \\
&= \bigoplus_{n} \int \prod_{j} \bigoplus_{m} (\varphi(p_j P_1))^{\otimes m} \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n) \\
&= \bigoplus_{n} \int \prod_{j} \Gamma(\varphi(p_j P_1)) \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n),
\end{aligned}
$$

where the integral and the product commute in the third equality since the spectral measure is disjoint for different values of p's, and the sum and the product commute in the fifth equality since the operators in the integrand act on mutually disjoint spaces, namely on $\mathcal{H}^m \otimes \mathcal{H}^\Sigma$ for different m. Since all operators appearing in the integrand in the last expression are the second quantization operators, this formula naturally restricts to the partially symmetrized space $\mathcal{H}_s^\Sigma \otimes \mathcal{H}^\Sigma$.

Lemma 5.8.1. *It holds for $x \in \mathcal{A}_{U(1)}(\mathbb{R}_-)$ and $x' \in \mathcal{A}_{U(1)}(\mathbb{R}_+)$ that*

$$[x \otimes \mathbb{1}, \mathrm{Ad}S_\varphi(x' \otimes \mathbb{1})] = 0,$$

on the Hilbert space $\mathcal{H}_s^\Sigma \otimes \mathcal{H}_s^\Sigma$.

Proof. The operator S_φ is disintegrated into second quantization operators as we saw above. If φ is an inner symmetric function, then so is $\varphi(p_j \cdot)$, $p_j \geq 0$, thus each $\Gamma(\varphi(p_j P_1))$ implements a Longo-Witten endomorphism.

Note that S_φ restricts naturally to $\mathcal{H}_s^\Sigma \otimes \mathcal{H}^\Sigma$ by construction and $x \otimes \mathbb{1}$ and $x' \otimes \mathbb{1}$ extend naturally to $\mathcal{H}_s^\Sigma \otimes \mathcal{H}^\Sigma$ since the right-components of them are just the identity operator $\mathbb{1}$. Then we calculate the commutation relation on $\mathcal{H}_s^\Sigma \otimes \mathcal{H}^\Sigma$. This is done in the same way as Lemma 5.7.1: Namely, we have

$$\mathrm{Ad}S_\varphi(x' \otimes \mathbb{1}) = \bigoplus_n \int \prod_j \mathrm{Ad}\Gamma(\varphi(p_j P_1))(x') \otimes dE_1(p_1) \otimes \cdots \otimes dE_1(p_n).$$

And this commutes with $x \otimes \mathbb{1}$. Indeed, since $x \in \mathcal{A}_{U(1)}(\mathbb{R}_-)$ and $x' \in \mathcal{A}_{U(1)}(\mathbb{R}_+)$, hence $\mathrm{Ad}\Gamma(\varphi(p_j))(x') \in \mathcal{A}_{U(1)}(\mathbb{R}_+)$ for any $p_j \geq 0$ by Theorem 1.5.1 of Longo-Witten, and by the fact that the spectral support of E_1 is positive. Precisely, we have $[x\otimes\mathbb{1}, \mathrm{Ad}S_\varphi(x'\otimes\mathbb{1})] = 0$ on $\mathcal{H}_s^\Sigma\otimes\mathcal{H}^\Sigma$.

Now all operators S_φ, $x\otimes\mathbb{1}$ and $x'\otimes\mathbb{1}$ commute with $\mathbb{1}\otimes P_s$, we obtain the thesis just by restriction. □

Finally we construct a wedge-local net by following the prescription at the end of Section 5.6.1.

Theorem 5.8.2. *The triple*

- $\mathcal{M}_\varphi := \{x\otimes\mathbb{1}, \mathrm{Ad}S_\varphi(\mathbb{1}\otimes y) : x\in\mathcal{A}_{U(1)}(\mathbb{R}_-), y\in\mathcal{A}_{U(1)}(\mathbb{R}_+)\}''$
- T *of* $\mathcal{A}_{U(1)}\otimes\mathcal{A}_{U(1)}$
- Ω *of* $\mathcal{A}_{U(1)}\otimes\mathcal{A}_{U(1)}$

is an asymptotically complete wedge-local net with S-matrix S_φ.

Proof. This is almost a repetition of the proof of Theorem 5.7.2. Namely, the conditions on T and Ω are readily satisfied since they are same as the chiral net. The operators S and T commute since both are the functional calculus of the same spectral measure, hence $T(t_0,t_1)$ sends $\mathcal{M}_\varphi$ into itself for $(t_0,t_1)\in W_\mathrm{R}$. The vector Ω is cyclic for $\mathcal{M}_\varphi$ since $\mathcal{M}_\varphi\Omega\supset\{x\otimes\mathbb{1}\cdot S_\varphi\cdot\mathbb{1}\otimes y\cdot\Omega\} = \{x\otimes\mathbb{1}\cdot\mathbb{1}\otimes y\cdot\Omega\}$ and the latter is dense by the Reeh-Schlieder property of the chiral net. The separating property of Ω is shown through Lemma 5.8.1. □

Remark 5.8.3. In this approach, the function φ itself appears in two-particle scattering, not the square as in [58]. Thus, although the formulae look similar, the present construction contains much more examples.

Intersection property for constant functions φ

For the simplest cases $\varphi(p) = 1$ or $\varphi(p) = -1$, we can easily determine the strictly local elements. Indeed, for $\varphi(p) = 1$, $S_\varphi = \mathbb{1}$ and the wedge-local net coincides with the original chiral net. For $\varphi(p) = -1$, $S_\varphi^{m,n} = (-1)^{mn}\cdot\mathbb{1}$ and it is not difficult to see that if one defines an operator $Q_0 := \mathbb{1}-P_e$, where P_e is a projection onto the "even" subspace $\bigoplus_n\mathcal{H}_s^{2n}$ of $\mathcal{H}_s^\Sigma$, then $e^{i\pi Q_0}$ implements a $\mathbb{Z}_2$-action of inner symmetries on $\mathcal{A}_{U(1)}$ and $S_\varphi = e^{i\pi Q_0\otimes Q_0}$. Then Theorem 5.7.19 applies to find that the strictly local elements are of the form $\mathcal{A}_{U(1)}^{\mathbb{Z}_2}\otimes\mathcal{A}_{U(1)}^{\mathbb{Z}_2}$ where the action of $\mathbb{Z}_2$ is realized by $\mathrm{Ad}e^{i\pi nQ_0}$.

Free fermionic case

As explained in [64], one can construct a family of endomorphisms on the Virasoro net Vir_c with the central charge $c=\frac{1}{2}$ by considering the free fermionic field. With a similar construction using the one-particle space on which the Möbius group acts irreducibly and projectively with the lowest weight $\frac{1}{2}$, one considers the free fermionic (nonlocal) net on S^1, which contains $\mathrm{Vir}_{\frac{1}{2}}$ with index 2.

The endomorphisms are implemented again by the second quantization operators. By “knitting up” such operators as is done for bosonic $U(1)$-current case, then by restricting to the observable part $\mathrm{Vir}_{\frac{1}{2}}$, we obtain a family of wedge-local nets with the asymptotic algebra $\mathrm{Vir}_{\frac{1}{2}} \otimes \mathrm{Vir}_{\frac{1}{2}}$ with nontrivial S-matrix.

5.9 Open problems

Non-chiral CFT

We identified the space of collision states of waves with the space generated by the maximal chiral subalgebra from the vacuum. The orthogonal complement of the space of collision states, which may be quite large as we explained in Section 5.2.2, is a natural subject of future research. Fortunately, we have tools to investigate this orthogonal complement: They include the theory of particle weights [17, 74], developed to study infraparticles. With the help of this theory it has been confirmed that infraparticles are present in all states in product representations of the chiral subnet, hence in the orthogonal complement of the space of collision states of waves in any completely rational net [33, 54]. The question of interaction and asymptotic completeness of these excitations remains open to date (for a general account on asymptotic completeness, see [13]). However, the fact that the incoming and outgoing asymptotic fields coincide in Möbius covariant theories on the entire Hilbert space suggests the absence of interaction.

Intersection property

One important lesson from Section 5.7.4 is that construction of wedge-local nets should be considered as an intermediate step to construct strictly local nets: Indeed, any Möbius covariant net has trivial S-matrix as seen in Section 5.1, hence the triviality of S-matrix in the construction through inner symmetries is interpreted as a natural consequence. Although the S-matrix as a wedge-local net is nontrivial, this should be treated as a false-positive. The true nontriviality should be inferred by examining the strictly local part. On the other hand, we believe that the techniques developed in this thesis will be of importance in the further explorations in strictly local nets.

More wedge-local nets

Apart from the problem of strict locality, a more systematic study of the necessary or sufficient conditions for S-matrix is desired. Such a consideration could lead to a classification result of certain classes of massless asymptotically complete models. For the moment, a more realistic problem would be to construct S-matrix with the asymptotic algebra $\mathcal{A}_N \otimes \mathcal{A}_N$, where $\mathcal{A}_N$ is a local extension of the $U(1)$-current net [16, 64]. A family of Longo-Witten endomorphisms has been constructed also for $\mathcal{A}_N$, hence a corresponding family of wedge-local net is expected and recently a similar kind of endomorphisms has been found for a more general family of nets on S^1 [4]. Or a general scheme of deforming a given Wightman-field theoretic net has been established [58].

Bibliography

[1] Huzihiro Araki, Daniel Kastler, Masamichi Takesaki, and Rudolf Haag. Extension of KMS states and chemical potential. *Comm. Math. Phys.*, 53(2):97–134, 1977.

[2] Huzihiro Araki and László Zsidó. Extension of the structure theorem of Borchers and its application to half-sided modular inclusions. *Rev. Math. Phys.*, 17(5):491–543, 2005.

[3] Hellmut Baumgärtel. *Operator algebraic methods in quantum field theory.* Akademie Verlag, Berlin, 1995.

[4] Marcel Bischoff. Boundary quantum field theory associated to lattice models. in preparation.

[5] Jens Böckenhauer. Localized endomorphisms of the chiral Ising model. *Comm. Math. Phys.*, 177(2):265–304, 1996.

[6] H.-J. Borchers. The CPT-theorem in two-dimensional theories of local observables. *Comm. Math. Phys.*, 143(2):315–332, 1992.

[7] H. J. Borchers. On revolutionizing quantum field theory with Tomita's modular theory. *J. Math. Phys.*, 41(6):3604–3673, 2000.

[8] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 1. C^*- and W^*-algebras, symmetry groups, decomposition of states.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987.

[9] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2. Equilibrium states. Models in quantum statistical mechanics.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997.

[10] R. Brunetti, D. Guido, and R. Longo. Modular structure and duality in conformal quantum field theory. *Comm. Math. Phys.*, 156(1):201–219, 1993.

[11] D. Buchholz. Collision theory for waves in two dimensions and a characterization of models with trivial S-matrix. *Comm. Math. Phys.*, 45(1):1–8, 1975.

[12] D. Buchholz, G. Lechner, and S.J. Summers. Warped convolutions, Rieffel deformations and the construction of quantum field theories. *Commun. Math. Phys.*, to appear.

[13] Detlev Buchholz. On particles, infraparticles, and the problem of asymptotic completeness. In *VIIIth international congress on mathematical physics (Marseille, 1986)*, pages 381–389. World Sci. Publishing, Singapore, 1987.

[14] Detlev Buchholz, Claudio D'Antoni, and Roberto Longo. Nuclear maps and modular structures. I. General properties. *J. Funct. Anal.*, 88(2):233–250, 1990.

[15] Detlev Buchholz and Klaus Fredenhagen. Dilations and interaction. *J. Math. Phys.*, 18(5):1107–1111, 1977.

[16] Detlev Buchholz, Gerhard Mack, and Ivan Todorov. The current algebra on the circle as a germ of local field theories. *Nuclear Phys. B Proc. Suppl.*, 5B:20–56, 1988. Conformal field theories and related topics (Annecy-le-Vieux, 1988).

[17] Detlev Buchholz, Martin Porrmann, and Ulrich Stein. Dirac versus Wigner. Towards a universal particle concept in local quantum field theory. *Phys. Lett. B*, 267(3):377–381, 1991.

[18] Detlev Buchholz and Hanns Schulz-Mirbach. Haag duality in conformal quantum field theory. *Rev. Math. Phys.*, 2(1):105–125, 1990.

[19] Detlev Buchholz and Stephen J. Summers. Warped convolutions: a novel tool in the construction of quantum field theories. In *Quantum field theory and beyond*, pages 107–121. World Sci. Publ., Hackensack, NJ, 2008.

[20] Paolo Camassa, Roberto Longo, Yoh Tanimoto, and Mihály Weiner. *Commun. Math. Phys.*

[21] Paolo Camassa, Roberto Longo, Yoh Tanimoto, and Mihály Weiner. submitted.

[22] Sebastiano Carpi. Classification of subsystems for the Haag-Kastler nets generated by $c = 1$ chiral current algebras. *Lett. Math. Phys.*, 47(4):353–364, 1999.

[23] Sebastiano Carpi. The Virasoro algebra and sectors with infinite statistical dimension. *Ann. Henri Poincaré*, 4(3):601–611, 2003.

[24] Sebastiano Carpi. On the representation theory of Virasoro nets. *Comm. Math. Phys.*, 244(2):261–284, 2004.

[25] Sebastiano Carpi, Yasuyuki Kawahigashi, and Roberto Longo. Structure and classification of superconformal nets. *Ann. Henri Poincaré*, 9(6):1069–1121, 2008.

[26] Sebastiano Carpi and Mihály Weiner. On the uniqueness of diffeomorphism symmetry in conformal field theory. *Comm. Math. Phys.*, 258(1):203–221, 2005.

[27] Claudio D'Antoni, Roberto Longo, and Florin Radulescu. Conformal nets, maximal temperature and models from free probability. *J. Operator Theory*, 45(1):195–208, 2001.

[28] G. F. Dell'Antonio. On dilatation invariance and the Wilson expansion. *Nuovo Cimento A (11)*, 12:756–762, 1972.

[29] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[30] Jacques Dixmier. *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1981.

[31] S. Doplicher and R. Longo. Standard and split inclusions of von Neumann algebras. *Invent. Math.*, 75(3):493–536, 1984.

[32] Wojciech Dybalski and Yoh Tanimoto. Asymptotic completeness with respect to infraparticles in two-dimensional CFT. in preparation.

[33] Wojciech Dybalski and Yoh Tanimoto. Infraparticles with superselected direction of motion in two-dimensional conformal field theory. in preparation.

[34] Wojciech Dybalski and Yoh Tanimoto. Asymptotic completeness in a class of massless relativistic quantum field theories. *Comm. Math. Phys.*, to appear.

[35] D. B. A. Epstein. Commutators of Csp$\,\infty$-diffeomorphisms. Appendix to: "A curious remark concerning the geometric transfer map" by John N. Mather [Comment. Math. Helv. **59** (1984), no. 1, 86–110; MR0743944 (86c:58017)]. *Comment. Math. Helv.*, 59(1):111–122, 1984.

[36] B. L. Feigin and D. B. Fuks. Skew-symmetric invariant differential operators on the line and Verma modules over the Virasoro algebra. *Funct. Anal. Appl.*, 16(2):114–126, 1982.

[37] Christopher J. Fewster and Stefan Hollands. Quantum energy inequalities in two-dimensional conformal field theory. *Rev. Math. Phys.*, 17(5):577–612, 2005.

[38] Martin Florig. On Borchers' theorem. *Lett. Math. Phys.*, 46(4):289–293, 1998.

[39] Jürgen Fuchs. Lectures on conformal field theory and Kac-Moody algebras. In *Conformal field theories and integrable models (Budapest, 1996)*, volume 498 of *Lecture Notes in Phys.*, pages 1–54. Springer, Berlin, 1997.

[40] P. Furlan, G. M. Sotkov, and I. T. Todorov. Two-dimensional conformal quantum field theory. *Riv. Nuovo Cimento (3)*, 12(6):1–202, 1989.

[41] Fabrizio Gabbiani and Jürg Fröhlich. Operator algebras and conformal field theory. *Comm. Math. Phys.*, 155(3):569–640, 1993.

[42] Howard Garland. The arithmetic theory of loop algebras. *J. Algebra*, 53(2):480–551, 1978.

[43] Roe Goodman and Nolan R. Wallach. Projective unitary positive-energy representations of Diff(Ssp 1). *J. Funct. Anal.*, 63(3):299–321, 1985.

[44] Harald Grosse and Gandalf Lechner. Wedge-local quantum fields and noncommutative Minkowski space. *J. High Energy Phys.*, (11):012, 26, 2007.

[45] Harald Grosse and Gandalf Lechner. Noncommutative deformations of Wightman quantum field theories. *J. High Energy Phys.*, (9):131, 29, 2008.

[46] Rudolf Haag. *Local quantum physics*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996. Fields, particles, algebras.

[47] G. H. Hardy. *Ramanujan: twelve lectures on subjects suggested by his life and work.* Chelsea Publishing Company, New York, 1959.

[48] Masaki Izumi. Subalgebras of infinite C^*-algebras with finite Watatani indices. I. Cuntz algebras. *Comm. Math. Phys.*, 155(1):157–182, 1993.

[49] V. G. Kac and A. K. Raina. *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, volume 2 of *Advanced Series in Mathematical Physics*. World Scientific Publishing Co. Inc., Teaneck, NJ, 1987.

[50] Daniel Kastler. Equilibrium states of matter and operator algebras. In *Symposia Mathematica, Vol. XX (Convegno sulle Algebre Csp$*$ e loro Applicazioni in Fisica Teorica, Convegno Sulla Teoria degli Operatori Indice e Teoria K, INDAM, Rome, 1975)*, pages 49–107. Academic Press, London, 1976.

[51] Yasuyuki Kawahigashi. Classification of operator algebraic conformal field theories in dimensions one and two. In *XIVth International Congress on Mathematical Physics*, pages 476–485. World Sci. Publ., Hackensack, NJ, 2005.

[52] Yasuyuki Kawahigashi and Roberto Longo. Classification of local conformal nets. Case $c < 1$. *Ann. of Math. (2)*, 160(2):493–522, 2004.

[53] Yasuyuki Kawahigashi and Roberto Longo. Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories. *Comm. Math. Phys.*, 244(1):63–97, 2004.

[54] Yasuyuki Kawahigashi, Roberto Longo, and Michael Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.*, 219(3):631–669, 2001.

[55] Yasuyuki Kawahigashi, Roberto Longo, and Michael Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.*, 219(3):631–669, 2001.

[56] Hideki Kosaki. Extension of Jones' theory on index to arbitrary factors. *J. Funct. Anal.*, 66(1):123–140, 1986.

[57] Serge Lang. $\mathrm{SL}_2(\mathbf{R})$, volume 105 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985. Reprint of the 1975 edition.

[58] Gandalf Lechner. Deformations of quantum field theories and integrable models. arXiv:1104.1948.

[59] Gandalf Lechner. Construction of quantum field theories with factorizing S-matrices. *Comm. Math. Phys.*, 277(3):821–860, 2008.

[60] R. Longo and K.-H. Rehren. Nets of subfactors. *Rev. Math. Phys.*, 7(4):567–597, 1995. Workshop on Algebraic Quantum Field Theory and Jones Theory (Berlin, 1994).

[61] Roberto Longo. Notes for a quantum index theorem. *Comm. Math. Phys.*, 222(1):45–96, 2001.

[62] Roberto Longo. Conformal subnets and intermediate subfactors. *Comm. Math. Phys.*, 237(1-2):7–30, 2003. Dedicated to Rudolf Haag.

[63] Roberto Longo. Real Hilbert subspaces, modular theory, $\mathrm{SL}(2,\mathbf{R})$ and CFT. In *Von Neumann algebas in Sibiu: Conference Proceedings*, pages 33–91. Theta, Bucharest, 2008.

[64] Roberto Longo and Edward Witten. An algebraic construction of boundary quantum field theory. *Commun. Math. Phys.*, to appear.

[65] Roberto Longo and Feng Xu. Topological sectors and a dichotomy in conformal field theory. *Comm. Math. Phys.*, 251(2):321–364, 2004.

[66] M. Lüscher and G. Mack. Global conformal invariance in quantum field theory. *Comm. Math. Phys.*, 41:203–234, 1975.

[67] Gerhard Mack and Volker Schomerus. Conformal field algebras with quantum symmetry from the theory of superselection sectors. *Comm. Math. Phys.*, 134(1):139–196, 1990.

[68] J. Manuceau. étude de quelques automorphismes de la c^*-algèbre du champ de bosons libres. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, pages 117–138, 1968.

[69] John N. Mather. Commutators of diffeomorphisms. *Comment. Math. Helv.*, 49:512–528, 1974.

[70] J. Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, groups and topology, II (Les Houches, 1983)*, pages 1007–1057. North-Holland, Amsterdam, 1984.

[71] Karl-Hermann Neeb. On differentiable vectors for representations of infinite dimensional lie groups. arXiv:1002.1602.

[72] Johnny T. Ottesen. *Infinite-dimensional groups and algebras in quantum physics*, volume 27 of *Lecture Notes in Physics. New Series m: Monographs*. Springer-Verlag, Berlin, 1995.

[73] Mihai Pimsner and Sorin Popa. Entropy and index for subfactors. *Ann. Sci. École Norm. Sup. (4)*, 19(1):57–106, 1986.

[74] Martin Porrmann. Particle weights and their disintegration. I. *Comm. Math. Phys.*, 248(2):269–304, 2004.

[75] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[76] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[77] K.-H. Rehren. Chiral observables and modular invariants. *Comm. Math. Phys.*, 208(3):689–712, 2000.

[78] Karl-Henning Rehren. A new view of the Virasoro algebra. *Lett. Math. Phys.*, 30(2):125–130, 1994.

[79] Carlo Rigotti. Remarks on the modular operator and local observables. *Comm. Math. Phys.*, 61(3):267–273, 1978.

[80] John E. Roberts. Some applications of dilatation invariance to structural questions in the theory of local observables. *Comm. Math. Phys.*, 37:273–286, 1974.

[81] M. Schottenloher. *A mathematical introduction to conformal field theory*, volume 759 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, second edition, 2008.

[82] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Noncommutative Geometry, 6.

[83] Masamichi Takesaki and Marinus Winnink. Local normality in quantum statistical mechanics. *Comm. Math. Phys.*, 30:129–152, 1973.

[84] Y. Tanimoto. Deformation of chiral conformal field theory and the intersection property. submitted.

[85] Y. Tanimoto. Ground state representations of loop algebras. *Ann. Henri Poincaré*, to appear.

[86] Y. Tanimoto. Noninteraction of waves in two-dimensional conformal field theory. submitted.

[87] Yoh Tanimoto. Representation theory of the stabilizer subgroup of the point at infinity in Diff(s^1). *Internat. J. Math.*, to appear.

[88] William Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.*, 80:304–307, 1974.

[89] Valerio Toledano Laredo. Positive energy representations of the loop groups of non-simply connected Lie groups. *Comm. Math. Phys.*, 207(2):307–339, 1999.

[90] Y. Wang. Locally normal kms states of diffeomorphism covariant nets w.r.t. translation subgroups. *Ph.D. thesis, Univ. of Roma "Tor Vergata"*, 2008.

[91] Antony Wassermann. Operator algebras and conformal field theory. III. Fusion of positive energy representations of LSU(N) using bounded operators. *Invent. Math.*, 133(3):467–538, 1998.

[92] Mihály Weiner. An algebraic version of haag's theorem. *Commun. Math. Phys.*, to appear.

[93] Mihály Weiner. Conformal covariance and positivity of energy in charged sectors. *Comm. Math. Phys.*, 265(2):493–506, 2006.

[94] Mihály Weiner. Restricting positive energy representations of Diffsp $+ (S\mathrm{sp}\, 1)$ to the stabilizer of n points. *Comm. Math. Phys.*, 277(2):555–571, 2008.

[95] Hans-Werner Wiesbrock. Half-sided modular inclusions of von-Neumann-algebras. *Comm. Math. Phys.*, 157(1):83–92, 1993.

[96] Feng Xu. Algebraic coset conformal field theories. *Comm. Math. Phys.*, 211(1):1–43, 2000.

[97] Feng Xu. Jones-Wassermann subfactors for disconnected intervals. *Commun. Contemp. Math.*, 2(3):307–347, 2000.

[98] Feng Xu. Strong additivity and conformal nets. *Pacific J. Math.*, 221(1):167–199, 2005.

[99] Kai Ming Zhao. Automorphisms and homomorphisms of the Virasoro algebra (in Chinese). *J. Systems Sci. Math. Sci.*, 12(1):1–4, 1992.