Assignment 5: solutions

Q01

Calculate the series $\sum_{k=1}^{\infty} \frac{5}{3^k}$.

Solution We know that $\sum_{k=1}^{n} a^k = \frac{a-a^{n+1}}{1-a}$, so for $|a| < 1$ we have $\sum_{k=1}^{\infty} a^k = \frac{1}{1-a}$. Here with $a = \frac{1}{3}$

$$\sum_{k=1}^{\infty} \frac{5}{3^k} = 5 \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = 5 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{5}{2}.$$

Q02

Determine for which x the series $\sum_{k=1}^{\infty} \left(\frac{x}{3}\right)^k$ converges.

Solution By the root test, $\left(\left(\frac{|x|}{3}\right)^n\right)^{\frac{1}{n}} = \frac{|x|}{3}$.

If $\frac{|x|}{3} < 1$, then the original series $\sum_{k=1}^{\infty} \left(\frac{x}{3}\right)^k$ converges absolutely. This happenes if and only if $-3 < x < 3$.

If $\frac{|x|}{3} > 1$, it diverges. If $\frac{|x|}{3} = 1$, so $x = -3, 3$, the series is either $\sum_k 1^k$ or $\sum_k (-1)^k$ and they are not convergent.

Q03

Determine for which x the series $\sum_{k=1}^{\infty} \frac{1}{k^x+1}$ converges.

Solution Let us consider the case $x > 0$. Then $\frac{k^x+1}{k^x} \to 1$ as $k \to \infty$, and we know that $\sum_{k=1}^{\infty} \frac{1}{k^x}$ converges if and only if $x > 1$.

For $x \leq 0$, $\frac{1}{k^x+1} > \frac{1}{2}$, so it diverges.

Q04

Calculate $(1 + i\sqrt{3})^5$.

Solution Note that $(1+i\sqrt{3}) = 2(\frac{1}{2}+i\frac{\sqrt{3}}{2}) = 2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$. Thus $(1+i\sqrt{3})^5 = 2^5(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = 32(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 16 - i16\sqrt{3}$.

Q05

Calculate $(-\frac{\sqrt{3}}{2} + \frac{1}{2})^{\frac{1}{5}}$, where both the real and the imaginary parts are positive.

Solution Note that $(-\frac{\sqrt{3}}{2} + \frac{1}{2}) = (\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$. Thus $(-\frac{\sqrt{3}}{2} + \frac{1}{2})^{\frac{1}{5}} = (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

Q06

Find a general solution of the differential equation $y''(x) + 2y'(x) + 5y(x) = 0$.

Solution Consider the algebraic equation $z^2+2z+5=0$, and this has the solutions $z=-1\pm 2i$. So in this case a general solution is

$$y(x)=C_1e^{-x}\cos(2x)+C_2e^{-1}\sin(2x)$$

Q07

Find a general solution of the differential equation $y'(x)=xy(x)^2$ and $y(0)=1$.

Solution This is a separable differential equation. We can write it as

$$\frac{y'(x)}{y(x)^2}=x,$$

and by integrating the both sides,

$$-\frac{1}{y(x)}=\frac{x^2}{2}+C,$$

or

$$y(x)=-\frac{1}{\frac{x^2}{2}+C}.$$

With $y(0)=1$, we should have $-\frac{1}{\frac{0^2}{2}+C}=1$, thus $C=-1$ and

$$y(x)=-\frac{1}{\frac{x^2}{2}-1}=\frac{-2}{x^2-2}.$$