Assignment 4: solutions

Q01

Let us calculate the integral $\int_0^1 8^{-x}dx$ based on the definition. Let us take $I = [0,1]$, $P_n = \{[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots [\frac{n-1}{n}, 1]\}$ and $f(x) = 8^{-x}$.

With $n = 3$, calculate $\underline{S}_I(f, P_n)$ and $\overline{S}_I(f, P_n)$.

Solution Note that $f(x) = 8^{-x}$ is monotonically decreasing, so when we consider sup and inf, we can simply take the values of f at the boundary.

$$\underline{S}_I(f, P_3) = \sum_{k=1}^{3} 8^{-\frac{k}{3}} \cdot \frac{1}{3} = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{7}{24},$$
$$\overline{S}_I(f, P_3) = \sum_{k=1}^{3} 8^{-\frac{k-1}{3}} \cdot \frac{1}{3} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{7}{12}.$$

Q02

Calculate the following definite integral. $\int_0^2 (x^3 + 4x^2 - 4)dx$.

Solution We have

$$\int_0^2 (x^3 + 4x^2 - 4)dx = \left[\frac{x^4}{4} + \frac{4x^3}{3} - 4x\right]_0^2 = \frac{20}{3}.$$

Q03

Calculate the following definite integral. $\int_0^{\frac{\pi}{4}} \sin(2(x - \pi))dx$.

Solution We have

$$\int_0^{\frac{\pi}{4}} \sin(2(x - \pi))dx = \left[-\frac{1}{2}\cos(2(x - \pi))\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$

Q04

Calculate the following definite integral. $\int_0^2 xe^{x^2}dx$.

Solution By substitution,

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}\int_0^2 2x \cdot e^{x^2}dx = \frac{1}{2}\left[e^{x^2}\right]_0^2 = \frac{1}{2}(e^4 - 1).$$

Q05

Calculate the following definite integral. $\int_0^2 xe^{-x}dx$.

Solution By integrating by parts,

$$\begin{aligned}\int_0^2 xe^{-x}dx &= \left[x \cdot (-e^{-x})\right]_0^2 - \int_0^2 (-e^{-x})dx \\ &= -2e^{-2} - \left[e^{-x}\right]_0^2 = -3e^{-1} + 1.\end{aligned}$$

Q06

Calculate the following definite integral. $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$.

Solution By the change of the variables $x = \sin t = \varphi(t), \varphi'(t) = \cos t$ and for the limits of the integral $\varphi^{-1}(0) = 0, \varphi^{-1}(\frac{1}{2}) = \frac{\pi}{6}$, we have

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx &= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{1-\sin^2 t}} \cos t dt \\ &= \int_0^{\frac{\pi}{6}} 1 dt = \frac{\pi}{6}. \end{aligned}$$

Q07

Calculate the following definite integral. $\int_2^3 \frac{1}{x^2-x} dx$.

Solution We have

$$\frac{1}{x^2-x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x-A}{x(x-1)},$$

and solving $A + B = 0, -A = 1$, we have $A = -1, B = 1$, so

$$\begin{aligned} \int_2^3 \frac{1}{x^2-x} dx &= \int_2^3 \left(\frac{-1}{x} + \frac{1}{x-1} \right) dx \\ &= [-\log|x| + \log|x-1|]_2^3 \\ &= (-\log 3 + \log 2) - (-\log 2 + \log 1) = 2\log 2 - \log 3 = \log \frac{4}{3}. \end{aligned}$$

Q08

Calculate the following improper integral. $\int_1^\infty xe^{-x^2} dx$.

Solution This integral is over an infinite interval of a bounded function. So this is by definition,

$$\begin{aligned} \int_1^\infty xe^{-x^2} dx &= \lim_{\beta \to \infty} \int_1^\beta xe^{-x^2} dx \\ &= \lim_{\beta \to \infty} -\frac{1}{2} \int_1^\beta -2xe^{-x^2} dx \\ &= \lim_{\beta \to \infty} -\frac{1}{2} \left[e^{-x^2} \right]_1^\beta \\ &= \lim_{\beta \to \infty} -\frac{1}{2} (e^{-\beta^2} - (-e^{-1})) = \frac{1}{2} e^{-1}. \end{aligned}$$

Q09

Choose all convergent improper integrals.

- $\int_0^\infty e^{-x} dx$
- $\int_0^\infty e^{x} dx$
- $\int_0^\infty x^{100} e^{-x} dx$

- $\int_0^\infty e^{-x^2} dx$
- $\int_0^\infty e^{x^2} dx$
- $\int_1^\infty x^{-\frac{3}{2}} dx$
- $\int_0^1 x^{-\frac{3}{2}} dx$
- $\int_{-\infty}^\infty \frac{1}{x^2+1} dx$

Solution

- $\int_0^\infty e^{-x} dx$ can be directly calculated and is convergent.
- $\int_0^\infty e^{x} dx$. e^x is monotonically increasing, so it cannot converge.
- $\int_0^\infty x^{100} e^{-x} dx$. It holds that $x^{100}e^{-\frac{1}{2}} \to 0$ as $x \to \infty$, therefore, $x^{100}e^{-\frac{1}{2}} < C$ for x sufficiently large. As $\int_a^\infty Ce^{-\frac{1}{2}} dx$ is convergent, and $x^{100}e^{-x}$ is bounded on any bounded interval, the whole integral is convergent as well.
- $\int_0^\infty e^{-x^2} dx$. We have $e^{-x^2} \le e^{-x}$ for $x \ge 1$, so we have $\int_0^\infty e^{-x^2} dx \le \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-1} dx$ and the latter is convergent, thus the former is convergent as well.
- $\int_0^\infty e^{x^2} dx$. e^{x^2} is monotonically increasing, so it cannot converge.
- $\int_1^\infty x^{-\frac{3}{2}} dx$ can be directly calculated and is convergent.
- $\int_0^1 x^{-\frac{3}{2}} dx = \lim_{\alpha \to 0^+} \int_\alpha^1 x^{-\frac{3}{2}} dx$ can be directly calculated and is divergent.
- $\int_{-\infty}^\infty \frac{1}{x^2+1} dx$. By noting that $\frac{1}{x^2+1} < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ and $\int_{-\infty}^{1} \frac{1}{x^2} dx$ are convergent, and $\frac{1}{x^2+1}$ is bounded on $[-1, 1]$, we conclude that the former intergral is convergent as well.

Q10

Using the Taylor formula of $\cos(2x)$ as $x \to 0$, determine whether the following improper integral $\int_0^1 \frac{x^{\frac{1}{2}}}{\cos(2x)+x^2-1}$ convergens or diverges.

Solution We have

$$\cos y = 1 - \frac{y^2}{2} + o(y^2)$$
$$\cos(2x) = 1 - \frac{4x^2}{2} + o(x^2) = 1 - 2x^2 + o(x^2)$$

Therefore, $\cos(2x) + x^2 - 1 = -x^2 + o(x^2)$ as $x \to 0$ and

$$\lim_{x \to 0^+} \frac{\frac{x^{\frac{1}{2}}}{\cos(2x)+x^2-1}}{x^{-\frac{3}{2}}} \lim_{x \to 0^+} \frac{\frac{x^{\frac{1}{2}}}{-x^2+o(x^2)}}{x^{-\frac{3}{2}}} = -1$$

As $\int_0^1 x^{-\frac{3}{2}} dx$ diverges, $\int_0^1 \frac{x^{\frac{1}{2}}}{\cos(2x)+x^2-1}$ diverges as well.