Assignment 3: solutions

Q01

Let $f(x) = \tanh \frac{1}{x}$ for $x > 0$ and $e^x - x$ for $x < 0$. Choose the value of $f(0)$ such that f is differentiable at $x = 0$.

With that value, calculate $f'(0)$.

Solution In order to make f differentiable, we must make it continuous. As $\lim_{x\to 0^+} \tanh \frac{1}{x} = \lim_{x\to\infty} \tanh x = 1$, we need to choose $f(0) = 1$.

With this choice,

$$
\begin{aligned}
D_+f(0) &= \lim_{h\to 0^+} \frac{\tanh 1h + 0 - 1}{h} = \lim_{h\to\infty} \frac{\frac{e^h+e^{-h}}{e^h-e^{-h}} - 1}{h} \\
&= \lim_{h\to\infty} \frac{\frac{e^h+e^{-h}}{e^h-e^{-h}} - 1}{h} = \lim_{h\to\infty} \frac{\frac{2e^{-h}}{e^h-e^{-h}}}{h} = 0, \\
D_-f(0) &= \lim_{h\to 0^-} \frac{e^{h+0} - (h+0) - 1}{h} = \lim_{h\to 0^-} \frac{e^h - h - 1}{h} = 0
\end{aligned}
$$

and they coincide, so $f'(0) = 0$.

Q02

Calculate the following derivatives.

- $f(x) = x^3 + 4x^2 - 4, f'(2)$
- $g(x) = \sin x, g'(\frac{\pi}{6})$
- $h(x) = \sqrt{x}, h'(9)$.

Solution

- $f'(x) = 3x^2 + 8x, f'(2) = 28$
- $g'(x) = \cos x, g'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
- $h'(x) = \frac{1}{2}x^{-\frac{1}{2}}, h'(9) = \frac{1}{6}$

Q03

Calculate the following derivatives.

- $f(x) = x \sin x, f'(\frac{\pi}{4})$
- $g(x) = \log(x^2 + 1), g'(2)$
- $h(x) = \frac{x}{x^2-1}, h'(2)$

Solution

- $f'(x) = \sin x + x \cos x, f'(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$
- $g'(x) = \frac{2x}{x^2+1}, g'(2) = \frac{4}{5}$
- $h'(x) = \frac{(x^2-1)+x\cdot 2x}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}, h'(2) = \frac{-5}{9}$

Q04

Determine the stationary points.

- $f(x) = x^3 - 3x^2 - 2$
- $g(x) = xe^{x^2-3x}$

Solution

- $f'(x) = 3x^2 - 6x$. At a stationary point x_0, it holds that $f'(x_0) = 3x_0^2 - 6x_0 = 3x_0(x_0 - 2) = 0$, so $x_0 = 0, 2$.
- $g'(x) = e^{x^2-3x} + x \cdot (2x-3)e^{x^2-3x} = (2x^2 - 3x + 1)e^{x^2-3x}$. At a stationary point x_0, it holds that $g'(x_0) = (2x_0^2 - 3x_0 + 1)e^{x_0^2-3x_0} = 0$. As e^y never takes 0, it must hold $0 = 2x_0^2 - 3x_0 + 1 = (2x_0 - 1)(x_0 - 1)$, so $x_0 = 1, \frac{1}{2}$.

Q05

Determine the minumum and the maximum for the function $f(x) = \frac{x-1}{x^2-2x+2}$.

Solution The denominator $x^2 - 2x + 2 = (x-1)^2 + 1$ never takes 0, so f is defined for all $x \in \mathbb{R}$. We have

$$f'(x) = \frac{(x^2 - 2x + 2) - (x-1)(2x-2)}{(x^2 - 2x + 2)^2} = \frac{-x^2 + 2x}{(x^2 - 2x + 2)^2} = \frac{-x(x-2)}{(x^2 - 2x + 2)^2}.$$

It has two stationary points $x_0 = 0, 2$, and $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $0 < x < 2$ and $f'(x) < 0$ for $x > 2$, therefore, $x_0 = 0$ is a local minumum with $f(0) = -\frac{1}{2}$ and $x_0 = 2$ is a local maximum with $f(2) = \frac{1}{2}$. Furthermore, $\lim_{x\to\pm\infty} f(x) = 0$, therefore, they are the global minimum and global maximum, respectively.

Q06

Determine the three intervals where $f(x) = \cosh(x^3 - x)$ is monotonically increasing.

Solution We have

$$f'(x) = (3x^2 - 1)\sinh(x^3 - x),$$

and

- $3x^2 - 1 > 0$ for $x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < 3$, while $3x^2 - 1 < 0$ for $-\frac{1}{\sqrt{3}} < x\frac{1}{\sqrt{3}}$.
- $\sinh y > 0$ if and only if $y > 0$, and $x^3 - x = x(x-1)(x+1) > 0$ if and only if $-1 < x < 0$ or $x > 0$. Similarly, $\sinh(x^3 - x) < 0$ if and only if $x < 0$ or $0 < x < 1$.
- Altogether, $f'(x) > 0$ if and only if $-1 < x < -\frac{1}{\sqrt{3}}, 0 < x < \frac{1}{\sqrt{3}}, 1 < x$.

Q07

Determine asymptotes of $f(x) = \sqrt{\frac{x^4}{x^2-1}}$.

Solution

- Vertical. $f(x)$ is defined on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. We should check $x = -1, 1$, and $f(x)$ diverges there. So $x = -1, 1$ are vertical asymptotes.
- Horizontal. $\lim_{x\to\pm\infty} \sqrt{\frac{x^4}{x^2-1}} = \infty$, so there is no horizontal asymptote.
- Oblique. $\lim_{x\to\infty} \frac{\sqrt{\frac{x^4}{x^2-1}}}{x} = 1$ and

$$\begin{aligned}\lim_{x\to\infty} \sqrt{\frac{x^4}{x^2-1}} - x &= \lim_{x\to\infty} \frac{x(x-\sqrt{x^2-1})}{\sqrt{x^2-1}} \\ &= \lim_{x\to\infty} \frac{x(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(x+\sqrt{x^2-1})} \\ &= \lim_{x\to\infty} \frac{x(x^2-(x^2-1))}{\sqrt{x^2-1}(x+\sqrt{x^2-1})} \\ &= 0\end{aligned}$$

  So $y = x$ is an oblique asymptote. Similarly, $\lim_{x\to-\infty} \frac{\sqrt{\frac{x^4}{x^2-1}}}{x} = \lim_{x\to-\infty} -\sqrt{\frac{x^4}{x^2(x^2-1)}} = -1$ and

$$\begin{aligned}\lim_{x\to\infty} \sqrt{\frac{x^4}{x^2-1}} - (-x) &= \lim_{x\to\infty} \frac{x(x+\sqrt{x^2-1})}{\sqrt{x^2-1}} \\ &= \lim_{x\to\infty} \frac{x(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}{\sqrt{x^2-1}(x-\sqrt{x^2-1})} \\ &= \lim_{x\to\infty} \frac{x(x^2-(x^2-1))}{\sqrt{x^2-1}(x-\sqrt{x^2-1})} \\ &= 0\end{aligned}$$

  So $y = -x$ is an oblique asymptote.

Q08

Compute the limit. $\lim_{x\to\infty} \frac{\log(e^x+1)}{\sqrt{4x^2+1}}$.

Solution We have

$$\lim_{x\to\infty} \frac{\frac{e^x}{e^x+1}}{8x \cdot \frac{1}{2}\frac{1}{\sqrt{4x^2+1}}} = \frac{1}{2},$$

so by Bernoulli-de l'Hôpital formula, $\lim_{x\to\infty} \frac{\log(e^x+1)}{\sqrt{4x^2+1}} = \frac{1}{2}$.

Q09

Write the Taylor formula as $x \to 1$ to the third order for $f(x) = (x-1)\log(x)$.

Solution Put $g(x) = \log x$, then

- $g(1) = 0$.
- $g'(x) = \frac{1}{x}$, $f'(1) = 1$,
- $g''(x) = -\frac{1}{x^2}$, $f''(1) = -1$.

So we have $g(x) = (x-1) - \frac{(x-1)^2}{2} + o((x-1)^2)$ as $x \to 1$. Then $f(x) = (x-1)g(x) = (x-1)^2 - \frac{(x-1)^3}{2} + o((x-1)^3)$.

Q10

Compute the limit. $\lim_{x\to 0} \frac{x(e^x-1)}{\sin(3x^2)}$.

Solution We have, as $x \to 0$,

- $e^x - 1 = x + o(x)$
- $x(e^x - 1) = x^2 + o(x)$
- $\sin(y) = y + o(y)$, thus $\sin(3x^2) = 3x^2 + o(x^2)$

Altogether,

$$\lim_{x\to 0} \frac{x(e^x-1)}{\sin(3x^2)} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{3x^2 + o(x^2)} = \lim_{x\to 0} \frac{1 + \frac{o(x^2)}{x^2}}{3 + \frac{o(x^2)}{x^2}} = \frac{1}{3}.$$