Assignment 2: solutions

Q01

Determine the domains of the functions.

- $f(x) = \sqrt{1-x}$
- $f(x) = \frac{1}{x^2-4}$
- $f(x) = \log(1-x^2)$

Solution

- As $\sqrt{y}$ makes sense for $y \geq 0$, we should have $1-x \geq 0$, that is, $x \leq 1$.
- As $\frac{1}{y}$ makes sense for $y \neq 0$, we should have $x^2 - 4 \neq 0$, or $x^2 \neq 4$, that is, $x \neq \pm 2$.
- As $\log y$ makes sense for $y > 0$, we should have $1 - x^2 > 0$, that is, $1 > x^2$ or $-1 < x < 1$.

Q02

Calculate the limits.

- $\lim_{x\to 2} \frac{x^2+\sqrt{x+\sqrt{2x}}}{5x}$.
- $\lim_{x\to\infty} \frac{2x^3+2x+4}{3x^3-5x^2+x}$

Solution

- Both the numerator and the denominators are continuous at $x = 2$, thus we can just substitute x by 2 and $\lim_{x\to 2} \frac{x^2+\sqrt{x+\sqrt{2x}}}{5x} = \frac{2^2+\sqrt{2+\sqrt{2\cdot 2}}}{5\cdot 2} = \frac{3}{5}$.
- By extracting the largest terms in the denominator and the numerator, respectively, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^3+2x+4}{3x^3-5x^2+x} &= \lim_{x\to\infty} \frac{x^3}{x^3} \frac{2+\frac{2}{x^2}+\frac{4}{x^3}}{3-\frac{5}{x}+\frac{1}{x^2}} \\ &= \frac{2}{3}\end{aligned}$$

Q03

Compute $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x}$.

Solution We calculate

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} &= \lim_{x\to 0} \frac{(\sqrt{1-x}-\sqrt{1+x})(\sqrt{1-x}+\sqrt{1+x})}{x(\sqrt{1-x}+\sqrt{1+x})} \\ &= \lim_{x\to 0} \frac{-2x}{x(\sqrt{1-x}+\sqrt{1+x})} \\ &= \frac{-2}{2} = -1.\end{aligned}$$

Q04

Compute $\lim_{x\to 1} \frac{x^2-1}{x^3-1}$. Define $f(1)$ in such a way that $f(x) = \frac{x^2-1}{x^3-1}, x \neq 1$, is continuous.

Solution We calculate

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x^3-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}\\ &= \lim_{x\to 1}\frac{x+1}{x^2+x+1}\\ &= \frac{2}{3}.\end{aligned}$$

To have the continuity, we should have $\lim_{x\to 1} f(x) = f(1)$, so we must choose $f(1) = \frac{2}{3}$.

Q05

Calculate

- $(\frac{3}{2})^4$
- $9^{-\frac{5}{2}}$
- $\log_2 \sqrt{128}$
- $\log_{\frac{1}{2}} \frac{1}{8}$

Solution They are

- $(\frac{3}{2})^4 = \frac{81}{16}$
- $9^{-\frac{5}{2}} = (3^2)^{-\frac{5}{2}} = 3^{-5} = \frac{1}{243}$
- $\log_2 \sqrt{128} = \log_2(2^{\frac{7}{2}}) = \frac{7}{2}$
- $\log_{\frac{1}{2}} \frac{1}{8} = \log_{\frac{1}{2}} (\frac{1}{2})^3 = 3$

Q06

Calculate

- $\sin \frac{13\pi}{3}$
- $\cos(-\frac{3\pi}{4})$
- $\sin \frac{\pi}{8}$

Solution They are

- Using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin(x+2\pi) = \sin(x)$, we have $\sin\frac{13\pi}{3} = \sin(\frac{\pi}{3}+4\pi) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.
- Using $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos(x-\pi) = -\cos(x)$, $\cos(-\frac{3\pi}{4}) = \cos(\frac{\pi}{4}-\pi) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.
- Using $\cos(2x) = 1-2\sin^2(x)$, thus $\sin^2 x = \frac{1-\cos(2x)}{2}$, and the fact that $\sin\frac{\pi}{8} > 0$, we have $\sin\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Q07

Calculate

- $\lim_{n\to\infty} \frac{2^n+1}{3^n}$
- $\lim_{n\to\infty} \frac{\sqrt{4^n+3}}{2^n}$
- $\lim_{n\to\infty}(1+2^n)^{\frac{1}{n}}$
- $\lim_{n\to\infty}(1+\frac{1}{2^n})^n$

Solution

- $\lim_{n\to\infty} \frac{2^n+1}{3^n} = \lim_{n\to\infty} \frac{2^n}{3^n}\frac{1+\frac{1}{2^n}}{1} = 0.$
- $\lim_{n\to\infty} \frac{\sqrt{4^n+3}}{2^n} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{4^n}}}{1} = 1.$
- $\lim_{n\to\infty}(1+2^n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1+2^n}{2^n}\cdot 2^n\right)^{\frac{1}{n}} = 2$ because $1 < (\frac{1+2^n}{2^n})^{\frac{1}{n}} < (1+\frac{1}{2^n})^{\frac{1}{n}} < (1+\frac{1}{n^2})^{\frac{1}{n}} \to 1$ as $n\to\infty$.
- As $n^2 < 2^n$, we have $1 < \lim_{n\to\infty}(1+\frac{1}{2^n})^n < \lim_{n\to\infty}(1+\frac{1}{n^2})^n \to 1$ therefore, by squeezing, $\lim_{n\to\infty}(1+\frac{1}{2^n})^n = 1$.

Q08

Calculate

- $\lim_{x\to 0} \frac{x}{\sin(2x)}$
- $\lim_{x\to 0} \frac{\log(x+1)}{\sin(x)}$
- $\lim_{x\to\infty} x(e^{\frac{1}{x}}-1)$

Solution

- $\lim_{x\to 0} \frac{x}{\sin(2x)} = \lim_{x\to 0} \frac{1}{2}\frac{2x}{\sin(2x)} = \frac{1}{2}.$
- $\lim_{x\to 0} \frac{\log(x+1)}{\sin(x)} = \lim_{x\to 0} \frac{\log(x+1)}{x}\frac{x}{\sin(x)} = 1.$
- $\lim_{x\to\infty} x(e^{\frac{1}{x}}-1) = \lim_{x\to 0^+} \frac{(e^x-1)}{x} = 1.$

Q09

Let $f(x) = x^2 - 4x + 5$ defined on $[0,5]$. Find its minimum and maximum.

Solution We have $f(x) = x^2 - 4x + 5 = (x-2)^2 + 1$. It takes the minumum at $x = 2$, which is in the domain and $f(2) = 1$.. As for maximum, the value is larger when x is farther from 2, thus it takes the maximum at $x = 5$ and $f(5) = (5-2)^2 + 1 = 10$.

Q10

Let $f(x) = \sin(x^2), g(x) = e^{-x^2}, h(x) = \log(x^2+1)$ on $(-\infty,\infty)$. Choose the one which takes both maximum and minumum, and calculate these values.

Choose all that are not bounded.

Solution As $|\sin(y)| \leq 1, |f(x)| = |\sin(x^2)| \leq 1$ and these values are taken at $x = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{3}}$, respectively, they are 1 and -1.

As $0 < e^{-y} \geq 1$ for $y \geq 0$, e^{-x^2} is bounded. It takes the maximum, $g(0) = e^{-0^2} = 1$, but takes no minimum because as x becomes large, e^{-x^2} can be arbitrarily small but positive.

As $0 \leq \log(y)$ for $y \geq 1$, we have $h(x) = \log(x^2 + 1) \geq 0$. It takes the minumum $h(0) = \log(0^2+1) = \log 1 = 0$, but no minimum because as x becomes large, $\log(x^2+1)$ can be arbitrarily large. This is not bounded.