Assignment 1: solutions

Q01

Let $A = \{0, 1, 2, 3, 4, 6, 7\}, B = \{n \in \mathbb{Z} : \text{there is } m \in \mathbb{Z} \text{ such that} n = 2m\}$. Determine the elements of $A \cup B$ and $A \cap B$ in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Solution B is the set of even numbers, that is, $B = \{\cdots, -4, -2, 0, 2, 4, 6, 8, 10, \cdots\}$.

$A \cup B$ is the union, the set that contains the elements of A and B and nothing else. Thus, in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cup B$ contains $0, 1, 2, 3, 4, 6, 7, 8$.

$A \cap B$ is the intersection, the set that contains the elements that belong both to A and B and nothing else. Thus, in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cup B$ contains $0, 2, 4, 6$.

Q02

Let $A = \{x \in \mathbb{R} : x^2 - 6x + 8 > 0\}$. Write A as a union of intervals.

Solution The condition for A can be written as follows:

$$\begin{aligned}
&x^2 - 6x + 8 > 0\\
&\Leftrightarrow (x-2)(x-4) > 0\\
&\Leftrightarrow (x - 2 > 0 \text{ and } x - 4 > 0) \text{ or } (x - 2 < 0 \text{ and } x - 4 < 0)\\
&\Leftrightarrow (x > 2 \text{ and } x > 4) \text{ or } (x < 2 \text{ and } x < 4)\\
&\Leftrightarrow (x > 4) \text{ or } (x < 2)\\
&\Leftrightarrow x \in (-\infty, 2) \cup (4, \infty)
\end{aligned}$$

Q03

Let $A = \{x \in \mathbb{R} : x^3 - x^2 - 2x > 0\}$. Write A as a union of intervals:

Solution The condition for A can be written as follows:

$$\begin{aligned}
&x^3 - x^2 - 2x > 0\\
&\Leftrightarrow (x+1)x(x-2) > 0
\end{aligned}$$

One can proceed as before, but it is more convenient to consider the following cases:

- $x < -1$. Then $(x+1)x(x-2) < 0$, so x does not belong to A.
- $x = -1$. Then $(x+1)x(x-2) = 0$, so x does not belong to A.
- $-1 < x < 0$. Then $(x+1)x(x-2) > 0$, so x belongs to A.
- $x = 0$. Then $(x+1)x(x-2) = 0$, so x does not belong to A.
- $0 < x < 2$. Then $(x+1)x(x-2) < 0$, so x does not belong to A.
- $x = 2$. Then $(x+1)x(x-2) = 0$, so x does not belong to A.
- $2 < x$. Then $(x+1)x(x-2) > 0$, so x belongs to A.

Altogether, we have $A = (-1, 0) \cup (2, \infty)$.

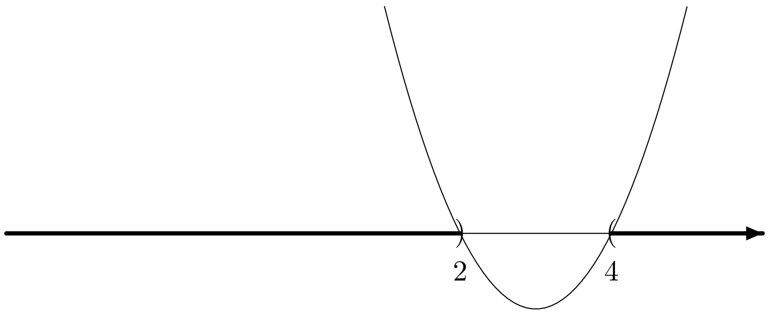

Figure 1: Solution to Q2. $x \in A$ if and only if the graph of $x^3 - 6x - 8$ is above the x-axis.

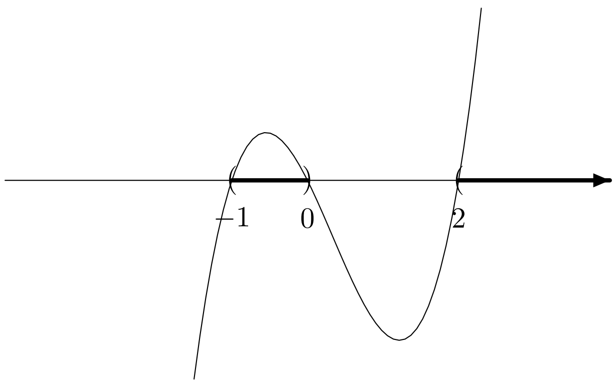

Figure 2: Solution to Q3. $x \in A$ if and only if the graph of $x^3 - x^2 - 2x$ is above the x-axis.

Q04

Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 1\}$. Choose all $(x, y) \in A$ among the following. $(-2, 0), (-2, 3), (-1, 2), (-1, 4), (0, -2), (0, -1), (1, 2), (1, 3), (2, -2), (2, 5), (3, 0)$.

Solution For each of (x, y), one should chech whether $y = x^1 + 1$ holds or not. The answer is $(-1, 2), (1, 2), (2, 5)$.

Q05

Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x^3 - 1\}$. Choose all $(x, y) \in A$ among the following. $(-2, 0), (-2, 3), (-1, 2), (-1, 4), (0, -2), (0, -1), (1, 2), (1, 3), (2, -2), (2, 5), (3, 0)$.

Solution For each of (x, y), one should chech whether $y < x^3 - 1$ holds or not. The answer is $(0, -2), (2, -2), (2, 5), (3, 0)$.

Q06

Let $A = \{x \in \mathbb{R} : x^3 - x^2 - 2x > 0\}$. Determine whether A is bounded below or above. Determine $\inf A$ (if it exists).

Solution This is the same set as in Q03, and we know that $A = (-1, 0) \cup (2, \infty)$. This is bounded below, for example, by -1. But it is not bounded above. -1 is $\inf A$, because for any number $x > -1$, there is a number a such that $-1 < a < x$ and $a \in (-1, 0) \subset A$.

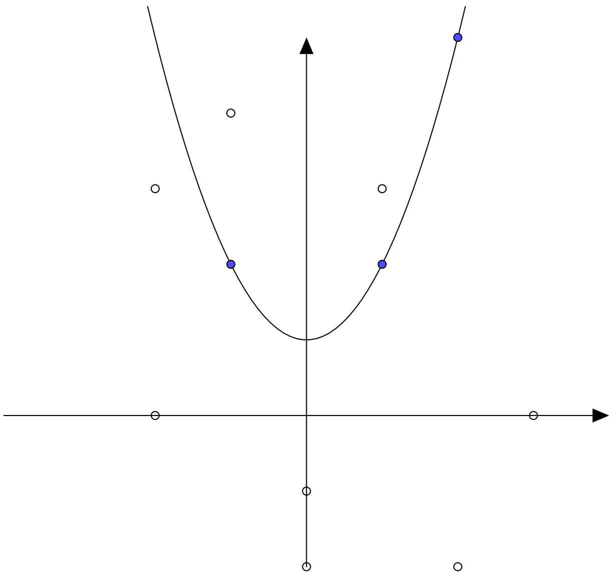

Figure 3: Solution to Q4. One should pick the points on the graph of $y = x^2 + 1$.

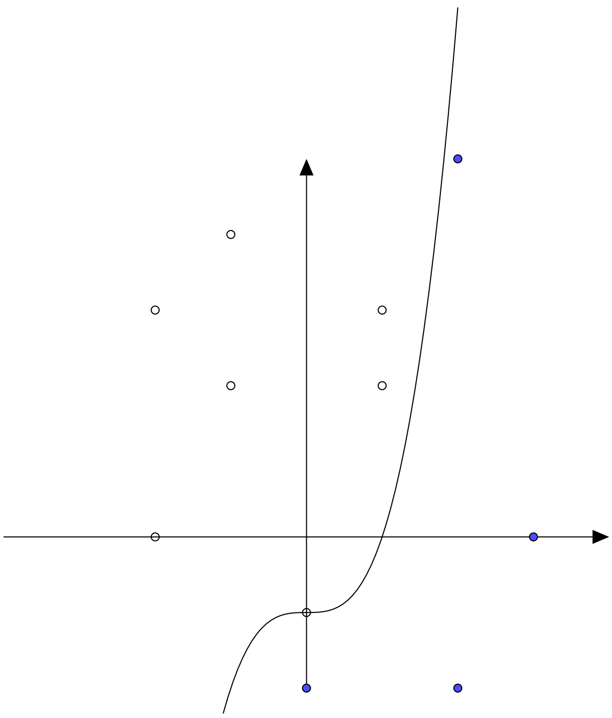

Figure 4: Solution to Q5. One should pick the points below the graph of $y = x^3 - 1$.

Q07

Let $A = \{\sum_{k=1}^{n}(\frac{1}{2})^k \in \mathbb{R} : n \in \mathbb{N}\}$. Determine $\sup A = \boxed{\text{a}}$ and $\inf A = \frac{\boxed{\text{b}}}{\boxed{\text{c}}}$.

Solution We saw in the lecture that $\sum_{k=1}^{n}(\frac{1}{2})^k = \frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = 1 - (\frac{1}{2})^n$. Therefore, $A = \{1 - (\frac{1}{2})^n : n \in \mathbb{N}\} = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots\}$. The sequence $1 - (\frac{1}{2})^n$ is monotonically increasing and converges to 1. Therefore, $\inf A = \frac{1}{2}, \sup A = 1$.

Q08

Compute the sum. $\sum_{n=1}^{4}(n^2 + 1)$.

Solution This is

$$\sum_{n=1}^{4}(n^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34.$$

Q09

Compute the product. $\prod_{n=1}^{3}(1 - \frac{1}{2^n})$.

Solution This is

$$\textstyle\prod_{n=1}^{3}(1 - \frac{1}{2^n}) = (1 - \frac{1}{2^1})(1 - \frac{1}{2^2})(1 - \frac{1}{2^3}) = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} = \frac{21}{64}.$$

Q10

Expand $(2x + y)^4$

Solution Using $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ with $a = 2x, b = y, n = 4$, we get

$$\begin{aligned}
(2x + y)^4 &= \sum_{k=0}^{4} \binom{4}{k} (2x)^k y^{n-k} \\
&= y^4 + 4 \cdot 2xy^3 + 6 \cdot (2x)^2 y^2 + 4(2x)^3 y + (2x)^4 \\
&= 16x^4 + 32x^3 y + 24x^2 y^2 + 8xy^3 + y^4.
\end{aligned}$$