Introduction to Algebraic Quantum Field Theory
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Abstract

We give an overview of mathematical approaches to quantum field theory. We start with some preliminaries on von Neumann algebras, the Tomita-Takesaki modular theory and the Minkowski space. We present the Araki-Haag-Kaster axioms and prove some consequences. As a first example, we construct the two-dimensional massive bosonic and fermionic free field nets. We discuss the Bisognano-Wichmann property, the modular nuclearity condition and the split property, and show that the two-dimensional massive free fermionic net satisfies modular nuclearity. As a consequence, we construct a AHK new net, which turns out to be the massive Ising model. We also briefly discuss a construction of a class of interacting models on the two-dimensional Minkowski space and some advanced topics.

1 Lecture 1 (5/15 14:45). Overview

1.1 Classical/quantum physics

Quantum Field Theory (QFT) is a physical theory to describe elementary particles (electrons, quarks (that constitute protons and neutrons), photons...): the standard model of elementary particles is considered as the most precise theory of all the interactions except gravity. QFT is also very useful in the study of critical phenomena in statistical mechanics and it captures universal properties of condensed matter physics. On the other hand, the most realistic models (the standard model of particle physics or its components, such as the Yang-Mills theory) remain to be constructed mathematically.

At the classical level, the motion of particles is described by classical system (ordinary differential equations). To study particles at atomic or subatomic scales, we need quantum mechanics (Hilbert spaces, self-adjoint operators). More in general, we also need to consider the change of the (background) fields. A field is a function of the configuration space,

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and it changes in time (partial differential equation). At atmic and subatomic scales, we would also need to "quantize" the fields. Although we know what quantized fields should be (Wightman/Osterwalder-Schrader/Araki-Haag-Kastler axioms), we do not have the most important examples (the standard model, Quantum Electrodynamics, the Yang-Mills theories in the 4 spacetime dimensions).

To be more specific, the Newton equation in Classical Mechanics, in its simplest form where there is only one particle in a potential, is given by

$$m\frac{d^2}{dt^2}x(t) = -V'(x(t)),$$

where m is the mass of the particle, $x(t)$ is the position of the particle at time t and $V(x)$ is the potential energy at the position x. This is a specific form of the "$F = ma$" formula. For a given initial condition, one can solve this ordinary differential equation and predict the motion of the particle. It can be equivalently written in the Hamiltonian formalism, where the equation of motion is given by $\frac{dq}{dt} = \frac{\partial H}{\partial p} = \{H, p\}, \frac{dp}{dt} = -\frac{\partial H}{\partial q} = \{H, q\}$, where $H(p, q) = \frac{p^2}{2m} + V(q)$ is the Hamiltonian of the system.

The simplest system in Quantum Mechanics is formulated on a Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Correspondingly to the previous classical system, its "quantization" is given by the Schrödinger equation (with $\hbar = 1$)

$$i\frac{\partial}{\partial t}\Psi(t, x) = \left[\frac{1}{2m}P^2 + V(Q)\right]\Psi(t, x),$$

where $P = i\frac{\partial}{\partial x}$, Q is the multiplication operator by x. Therefore, the system is given by this partial differential equation. In the "Heisenberg picture", where operators change in time (instead of the states Ψ), the evolution of the system is equivalently given by the Heisenberg equation $\frac{dQ}{dt}(t) = i[H, Q(t)], \frac{dP}{dt}(t) = i[H, P(t)]$, making the correspondence from the classical mechanics more straightforward. The basic observables Q, P satisfy the canonical commutation relation $[Q, P] = i$.

There are important classical theories involving fields on the continuum, such as the Electrodynamics, fluid mechanics, or General relativity. These theories are based a set of partial differential equations (Maxwell's equations, the Navier-Stokes equation or the Einstein equation, respectively). As a simplest example, one may consider the non-linear Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi(t, x) = -V'(\phi(t, x))$$

This is a partial differential equation about a function $\phi(t, x)$. Informally, for each x, there is a function $\phi(t, x)$ of t satisfying the equation of motion. In this sense, a field theory can be seen as a classical mechanics with infinite degrees of freedom.

If we try to quantize the Klein-Gordon equation, formally $\phi(t, x)$ should be an operator for each (t, x). Yet, if we impose the canonical commutation relations in analogy with quantum mechanics (considering the Poisson brackets), it should read $[\phi(0, x), \pi(0, y)] = \delta(x - y)$. Therefore, $\phi(t, x)$ should be an operator-valued distribution, rather than an operator-valued function.

Yet, if $\phi(t,x)$ is a distribution, then the expression for the Hamiltonian is not straightforward:

$$H \overset{?}{=} \int_{\mathbb{R}^{d-1}} \frac{1}{2}(\pi(0,x)^2 + (\nabla\phi(0,x))^2 + m^2\phi(0,x)^2 + V(\phi(0,x)))dx$$

As $\phi(0,x)$ is an operator-valued distribution, the meaning of $\phi(0,x)^2$ is not clear. It turns out that one can make sense of it by the so-called Wick product, but the potential term $V(\phi(0,x))$ is even more problematic. This is the ultraviolet problem in QFT, and it is severer in higher dimensions and with stronger interactions. The intergral over $\mathbb{R}^{d-1}$ poses another, infrared problem.

Classical mechanics	"quantization" →	Quantum mechanics
Cannonballs, planets, stars, galaxies...		Atoms, molecules, metals...
Newton's equation of motion		The Schrödinger equation
$m\ddot{x} = F$		$\frac{d}{dt}\psi = iH\psi$

Classical fields	quantization? ⇢	Quantum field theory
Electromagnetism, gravitation...		Elementary particles
Field equations		Quantized field equations?
(e.g. ϕ^4 theory, $V(\phi(x)) = \phi(x)^4$)		?
$(\Box + m^2)\phi(t,x) = -\lambda\phi(t,x)^3$		

1.2 The status of mathematical quantum field theory

As we have seen, a quantum field should be an operator-valued distribution, a mathematical object not easy to handle. Mathematicians started with the minimal requirements that a quantum field satisfy. One such set of requirements are called the **Wightman axioms** (the literature says that Wightman formulated them in 1950s but they are only published in 1964, the first edition of [SW00]): they consist of Poincaré covariance, locality (Einstein causality), positivity of energy and existence of vacuum (see below for the precise statements). There is also an operator-algebraic formalism, called Haag-Kastler axioms (1964) [HK64], that are representation-independend, or when we require that the operators are represented in the vacuum Hilbert space, they are called Araki-Haag-Kastler axioms [Ara99]. We will be mainly interested in the latter.

It is not difficult to show that these axioms are nonempty: in any spacetime dimension d, there are so-called free fields. However, they represent particles that do not interact and hence of no physical interest. The first interacting examples have been constructed in $d = 1 + 1$ by Glimm-Jaffe in 1972 by the Hamiltonian method [GJ72].

In 1975, Osterwalder and Schrader found a connection between Wightman fields and statistical mechanics in the Euclidean space (the Osterwalder-Schrader axioms) [OS75]. As statistical mechanics is concerned with (commutative) probability theory, therefore, it is more tractable. Many interacting examples have been constructed in the Osterwalder-Schrader axioms (this research programme is referred to as Constructive QFT). There are corresponding Wightman fields. Just to name a few, there have been constructed the Gross-Neveu model and the sine-Gordon model in $d = 1 + 1$, the ϕ_3^4-model ($d = 2 + 1$), the Abelian Higgs model ($d = 1 + 1, 2 + 1$), see [Sum12] and the references therein.

Apart from the constructive QFT, there are also interesting QFT, called conformal field theories. These models have a large symmetry group in $d = 1+1$ and been studied in an algebraic framework such as vertex operator algebras, and recently their Wightman counterparts have been found [CKLW18, RTT22, AGT].

However, in the realistic dimension $d = 1 + 3$, no interacting QFT has been constructed. The main reason why interacting QFTs are difficult in higher dimensions is that they suffer from the UV divergence. Most recently, the triviality of the ϕ_4^4-model has been proven [ADC21]. Roughly speaking, one cannot construct interacting scalar ϕ_4^4-model from lattice approximation. Other models that are considered plausible for nontrivial construction are the Yang-Mills theories. Constructing the Yang-Mills theories and proving the mass gap is a Millemium problem. The most important theory, the standard model of particle physics, contains some Yang-Mills theories and Quantum Electrodynamics and the Higgs particles. Constructing the standard model is widely open. One can say that the mathematical definitions of QFT are there, but examples are missing.

In these lectures, we present the AHK axioms. There are good reasons to study the algebraic approach:

- one can consider various states and representations
- there are interesting relationships between the theory of operator algebras (in particular the Tomita-Takesaki modular theory) and physics such as the Lorentz group and (relative) entropy
- theories can be defined on curved spacetimes
- some new examples have been constructed in the AHK axioms, using operator-algebraic techniques

To exhibit some of these aspects, we will study the axioms, the simplest example (the free field in $d = 1 + 1$), some algebraic and analytic properties of the free field and a construction of interacting examples.

2 Lecture 2 (5/15). Preliminaries

We briefly review some basic mathematical ingredients needed for Algebraic Quantum Field Theory. In quantum physics, observables are represented by self-adjoint operators on a Hilbert space $\mathcal{H}$. Here we consider "quantum fields" $\Phi(x)$, which should be an operator-valued distribution: for any test function f on the spacetime, we have an (unbounded) operator $\phi(f)$ on $\mathcal{H}$. In the so-called Wightman axioms, we consider such operator-valued distributions. In Algebraic QFT, we are instead interested in operator algebras (either C^*- or von Neumann) and the family of operator algebras should obey the so-called Araki-Haag-Kastler axioms.

2.1 Operator algebras

Most material should be found in the books by Bratteli-Robinson [BR87, BR97] or those by Takesaki [Tak02, Tak03] or by Strătilă-Zsidó [SZ79].

Let $\mathcal{H}$ be a (separable) Hilbert space. The set $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ can be equipped with the operator norm, defined by

$$\|x\| = \sup_{\xi\in\mathcal{H},\xi\neq 0} \frac{\|x\xi\|}{\|\xi\|}.$$

Then it holds that $\|xy\| \le \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$. The space $\mathcal{B}(\mathcal{H})$ is a **C^*-algebra** with this norm.

Actually, we use other topologies on $\mathcal{B}(\mathcal{H})$ as well.

- The **strong operator topology** (SOT) is generated by open base of the form $U_{x,\xi,\epsilon} = \{y \in \mathcal{B}(\mathcal{H}) : \|(y-x)\xi\| < \epsilon\}$, for $x \in \mathcal{B}(\mathcal{H}), \xi \in \mathcal{H}, \epsilon > 0$.
- The **weak operator topology** (WOT) is generated by open base of the form $U_{x,\xi,\eta,\epsilon} = \{y \in \mathcal{B}(\mathcal{H}) : |\langle\eta,(y-x)\xi\rangle| < \epsilon\}$, for $x \in \mathcal{B}(\mathcal{H}), \xi,\eta \in \mathcal{H}, \epsilon > 0$.

We denote the identity operator (on some Hilbert space) by $\mathbb{1}$. A **von Neumann algebra** $\mathcal{M}$ on $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathbb{1} \in \mathcal{M}$ and closed in WOT.

For a subset $M \subset \mathcal{B}(\mathcal{H})$, define the **commutant** of M by

$$M' = \{x \in \mathcal{B}(\mathcal{H}) : [x,y] = 0 \text{ for all } y \in M\}.$$

If $M = M^*$ (that is, $x \in M$ if and only if $x^* \in M$), then it is easy to see that M' is a von Neumann algebra (that is, it is a $*$-algebra, contains $\mathbb{1}$ and closed in WOT).

A fundamental result is the following.

Theorem 2.1 (von Neumann's bicommutant theorem)**.** *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra, $\mathbb{1} \in \mathcal{M}$. The following are equivalent.*

1. *$\mathcal{M}$ is closed in WOT.*
2. *$\mathcal{M}$ is closed in SOT.*
3. *$\mathcal{M} = \mathcal{M}''$.*

The first two conditions are topological, while the last one is algebraic (the commutant is defined only algebraically). This theorem is fundamental because it shows the interplay between analysis and algebra.

A **state** φ on $\mathcal{M}$ is a linear functional $\mathcal{M} \to \mathbb{C}$, $\varphi(x^*x) \ge 0$ (positive) and $\varphi(\mathbb{1}) = 1$ (normalized). Any state is automatically continuous in norm. If a state is in the norm-closure of the set of WOT-continuous states, it is called a **normal state**.

A von Neumann algebra $\mathcal{M}$ is called a **factor** if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathbb{1}$. A factor is classified into three types.

- Type I. $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. There is a "trace" on $\mathcal{M}$ which takes discrete values.
- Type II. There is a "trace" on $\mathcal{M}$ which takes continuous values.
- Type III. There is no trace. Any orthogonal projection $p \in \mathcal{M}$ is equivalent to $\mathbb{1}$ in $\mathcal{M}$, that is, there is $u \in \mathcal{M}$ such that $u^*u = \mathbb{1}, uu^* = p$.

2.2 Review of von Neumann algebras and the Tomita-Takesaki modular theory

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be **cyclic** for $\mathcal{M}$ if $\overline{\mathcal{M}\Omega} = \mathcal{H}$, and **separating** for $\mathcal{M}$ if there is no $x \neq 0, x \in \mathcal{M}$ such that $x\Omega = 0$.

We have the following.

Lemma 2.2. *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$. A vector Ω is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}'$.*

Let Ω be cyclic and separating for $\mathcal{M}$ on $\mathcal{H}$. We define the map

$$S : \mathcal{M}\Omega \ni x\Omega \longmapsto x^*\Omega \in \mathcal{M}\Omega \subset \mathcal{H}.$$

This is a well-defined (because Ω is separating) densely defined (because Ω is cyclic) operator on $\mathcal{H}$. It is actually closable, and we denote its closure with the same symbol S. It has the polar decomposition $J\Delta^{\frac{1}{2}}$. The main result of the Tomita-Takesaki modular theory is the following.

Theorem 2.3 (Tomita)**.** *It holds that* $\operatorname{Ad}\Delta^{it}(\mathcal{M}) = \mathcal{M}, J\mathcal{M}J = \mathcal{M}$.

We call Δ the **modular operator**, Δ^{it} the **modular group**, J the **modular conjugation** and $\sigma_{\mathcal{M},\Omega}(x) = \operatorname{Ad}\Delta^{it}(x)$ on $\mathcal{M}$ the **modular automorphism group**.

We will need the following later [BBS01].

Lemma 2.4 (Borchers-Buchholz-Schroer)**.** *If $\Psi \in \operatorname{Dom}(\Delta^{\frac{1}{2}})$, then there is a closed operator F affliate with $\mathcal{M}$ such that $\Psi = F\Omega, S\Psi = F^*\Omega$.*

Proof. Let Ψ be such a vector. As $\mathcal{M}\Omega$ is a core for S, there is a sequence $x_n \in \mathcal{M}$ such that $\Psi = \lim_n x_n\Omega, S\Psi = \lim_n Sx_n\Omega = \lim_n x_n^*\Omega$. We define the operator F defined on $\mathcal{M}'\Omega$ by $Fy\Omega = y\Psi = \lim_n yx_n\Omega$. It has an adjoint F^* defined on a dense domain $\mathcal{M}'\Omega$ by $F^*y\Omega = yS\Psi = \lim_n yx_n^*\Omega$. Thus F is closable.

It is clear that $yF \subset Fy$ for any $y \in \mathcal{M}'$. If $\Phi \in \operatorname{Dom}(F^*)$, then $\langle y\Phi, Fz\Omega\rangle = \langle \Phi, y^*Fz\Omega\rangle = \langle y^*S\Psi, z\Omega\rangle$, which shows that $y\Phi \in \operatorname{Dom}(F^*)$ and $F^*y\Phi = y^*S\Psi$. Altogether, F is affiliated with $\mathcal{M}$. □

2.3 The Minkowski space and the Poincaré group

Next we introduce the geometry where our quantum field theories are defined. Let $d \in \mathbb{N}, d \geq 1$. The $(1+d)$**-dimensional Minkowski space** is the space $\mathbb{R}^{1+d}$ equipped with the indefinite metric $(x, y) = x_0 y_0 - \sum_{j=1}^d x_j y_j$, where $x, y \in \mathbb{R}^{1+d}$. The group of (affine) transformations γ of $\mathbb{R}^{1+d}$ that preserve the metric (that is, $(\gamma x, \gamma y) = (x, y)$ for all $x, y \in \mathbb{R}^{1+d}$) is called the **Poincaré group**. We are mostly interested in the elements in the Poincaré group that preserve also the time-orientation (orthochronous) and parity (proper). They form a subgroup, denoted by $\mathcal{P}_+^\uparrow$.

We say that two points $x, y \in \mathbb{R}^{1+d}$ are

- **spacelike separated** if $(x - y, x - y) < 0$.
- **lightlike separated** if $(x - y, x - y) = 0$.
- **timelike separated** if $(x - y, x - y) > 0$.

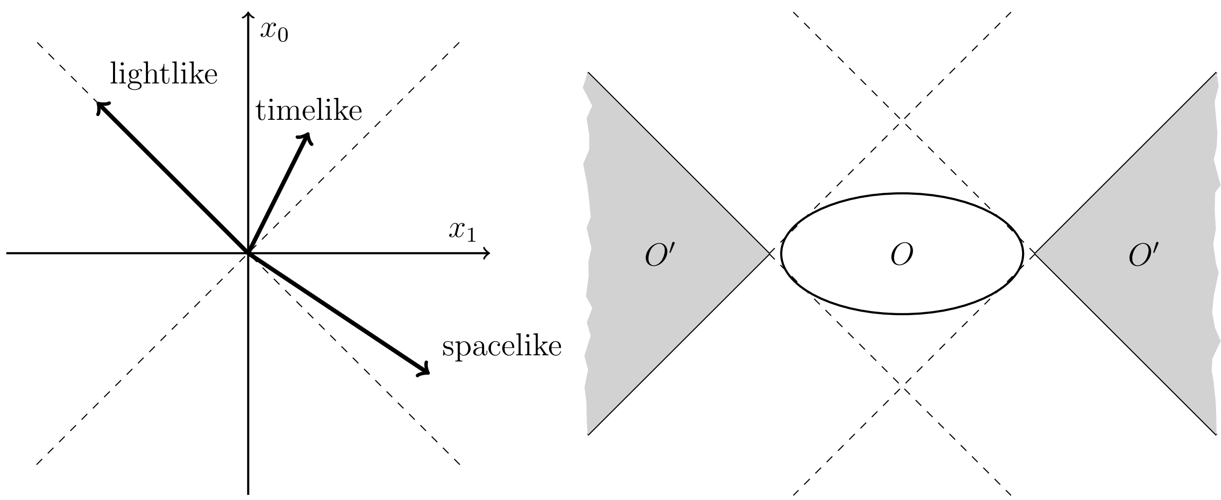

Figure 1: Left:The Minkowski space and vectors x, separation from 0. Right: A region O and its causal complement O'.

Let $O \subset \mathbb{R}^{1+d}$. We denote by O' the **causal complement**, that is, the interior of the set of all the points spacelike to O.

We have the decomposition $\mathcal{P}_+^\uparrow = \mathbb{R}^{1+d} \rtimes \mathcal{L}_+^\uparrow$, where $\mathbb{R}^{1+d}$ acts on $\mathbb{R}^{1+d}$ by translations and $\mathcal{L}_+^\uparrow$ is the group of linear transformations of $\mathbb{R}^{1+d}$ that preserve the metric. The group $\mathcal{L}_+^\uparrow$ is called the (proper orthochronous) Lorentz group, and is generated by the spacelike rotations (the elements preserving the time component x_0 and acting as the usual rotations on $(x_1, \cdots x_d)$) and the Lorentz boosts, which mix the time and space components.

In the case $d = 1$, the Lorentz boosts have the following form

$$\Lambda(\lambda) = \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

It is easy to check that this is a group homomorphism from $\mathbb{R}$ (as the additive group) into the set of 2×2-matrices $\mathrm{M}(\mathbb{R}, 2)$, therefore, we can identify the $1+1$-dimensional (proper orthochronous) Lorentz group with $\mathbb{R}$, and denote its element by $\lambda \in \mathbb{R}$. A generic element in $\mathcal{P}_+^\uparrow$ can be written as $\gamma = (a, \lambda)$, and it acts on $x \in M$ by $\Lambda(\lambda)x + a$, where $\Lambda(\lambda)x$ is the matrix multiplication.

Let $\mathcal{H}$ be a Hilbset space. A **unitary representation** of a group G is a map $U : G \to \mathcal{U}(\mathcal{H})$ (the group of unitary operators on $\mathcal{H}$) such that $U(\gamma_1\gamma_2) = U(\gamma_1)U(\gamma_2)$. We always assume that such U are continuous from a natural topology on G into SOT on $\mathcal{U}(\mathcal{H})$.

If $G = \mathbb{R}$ and U is a unitary representation of $\mathbb{R}$, then there is a "generator" H such that $U(t) = e^{itH}$ (the Stone theorem). Using the spectral decomposition of $H = \int_{\mathbb{R}} p dE(p)$, we have $U(t) = e^{itH} = \int e^{itp} dE(p)$.

If $G = \mathbb{R}^{1+d}$ and U is a unitary representation of $\mathbb{R}^{1+d}$, then there is a projection-valued measure E on $\mathbb{R}^{1+d}$ (the dual of G, which in this case coicides with G itself) such that

$$U(x) = \int_{\mathbb{R}^{1+d}} e^{i(x,p)} dE(p),$$

where (x, p) is the metric, but is considered as the coupling between the abelian group $\mathbb{R}^{1+d}$ and its continuous dual group $\mathbb{R}^{1+d}$. This is a generalization of the Stone theorem, known as the SNAG theorem [BdlH20, Theorem 2.C.2]. The support of dE is referred to as the **spectrum** of U.

A representation U is said to be **irreducible** if $\{U(\gamma) : \gamma \in G\}'' = \mathcal{B}(\mathcal{H})$. Let $m > 0$. An irreducible representation of $\mathcal{P}_+^\uparrow$ is given on $\mathcal{H}_m = L^2(\mathbb{R})$, and for $\Psi \in \mathcal{H}_m$, the action is given by

$$(U_m(a, \lambda)\Psi)(\theta) := e^{i(a,p(\theta))}\Psi(\theta - \lambda),$$

where $p(\theta) = (m \cosh \theta, m \sinh \theta) \in \mathbb{R}^{1+1}$.

Exercise: Show that U_m is indeed a representation of $\mathcal{P}_+^\uparrow$.

Exercise: Determine the spectrum of U_m defined above.

3 Lecture 3 (5/16). The axioms

3.1 The Wightman axioms

Let us start with the most conservative set of axioms. A quantum theory should act on a Hilbert space $\mathcal{H}$ and the symmetry group should be represented as unitary operators. By the observation in Lecture 1, a quantum field ϕ should be an operator-valued tempered distribution. This means that, for any Ψ_1, Ψ_2 in the "domain" of ϕ, the map $\mathscr{S}(\mathbb{R}^{1+d}) \ni f \mapsto \langle \Psi_1, \phi(f)\Psi_2 \rangle$ is a tempered distribution. Moreover, there should be the distinguished "vacuum" state in $\mathcal{H}$ that represents the physical state without particles. The quantum fields should be characterized by the correlation functions, that are the values of their product in the vacuum. In order to define correlation functions, we need to assume that the vacuum is in the domain $\mathscr{D}$ which is invariant under $\phi(f)$ for any $f \in \mathscr{S}(\mathbb{R}^{1+d})$ (that is, $\phi(f)\mathscr{D} \subset \mathscr{D}$). For simplicity, we consider the scalar case, that is, there is only one field ϕ and we have $\phi(f) \subset \phi(\bar{f})^*$. In particular, $\phi(f)$ is a symmetric operator if f is real.

A (scalar) **Wightman field theory** is $(\phi, U, \Omega, \mathscr{D} \subset \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, $\mathscr{D}$ is a dense subspace in $\mathcal{H}$, ϕ is an operator-valued distribution, U is a unitary representation of $\mathcal{P}_+^\uparrow$ such that $U(\gamma)\mathscr{D} = \mathscr{D}$ for all $\gamma \in \mathcal{P}_+^\uparrow$ and $\Omega \in \mathscr{D}$ such that

(1) **Locality**. If $f, g \in \mathscr{S}(\mathbb{R}^{1+d})$ such that $\operatorname{supp} f$ and $\operatorname{supp} g$ are spacelike separated, then $[\phi(f), \phi(g)] = 0$ (on $\mathscr{D}$).

(2) **Covariance**. For $\gamma \in \mathcal{P}_+^\uparrow$, it holds that $\operatorname{Ad} U(\gamma)(\phi(f)) = \phi(f_\gamma)$, where $f_\gamma(x) = f(\gamma^{-1}x)$.

(3) **Positivity of energy**. The restriction $U|_{\mathbb{R}^{1+d}}$ has the spectrum in $\overline{V_+}$, where $V_+ := \{p \in \mathbb{R}^{1+d} : p_0 > \|p_\perp\|\}$ is the future light cone and $p_\perp = (p_1, \cdots p_d) \in \mathbb{R}^d$.

(4) **Vacuum**. For all $\gamma \in \mathcal{P}_+^\uparrow$, $U(\gamma)\Omega = \Omega$ and $\operatorname{span}\{\phi(f_1)\cdots\phi(f_n)\Omega : n \in \mathbb{N}, f_j \in \mathscr{S}(\mathbb{R}^{1+d})\}$ is dense in $\mathcal{H}$.

If there is no confusion, we just call ϕ a Wightman field, without specifying $U, \Omega, \mathscr{D}$.

Remark 3.1. The Millenium problem asks to construct the quantum Yang-Mills theory in the Wightman axioms, or a similarly strong framework (such as the Osterwalder-Schrader axioms).

We are mainly interested in the operator-algebraic approach, the (Araki-)Haag-Kaster axioms. Note, however, that many examples are first constructed in the Wightman axioms, then corresponding operator algebras are constructed. We will see how this can be done.

By the closed graph theorem, one can show that $f \mapsto \phi(f)\Psi$ is continuous in norm for any fixed $\Psi \in \mathscr{D}$: we assume that ϕ is a tempered distribution defined on $\mathscr{S}(\mathbb{R}^{1+d})$ which is an F-space and the range of the map $\phi(f)\Psi$ is a Hilbert space. If $f_n \to 0$, $\phi(f_n)\Psi \to \Phi$, then $\langle \Psi_1, \phi(f_n)\Psi\rangle \to \langle \Psi_1, \Phi\rangle$, but the former is a tempered distribution, thus must tend to 0. This implies that $\langle \Psi_1, \Phi\rangle = 0$ for any $\Psi_1 \in \mathscr{D}$, and as $\mathscr{D}$ is dense, $\Phi = 0$. Then we can apply the closed graph theorem [Rud91, Theorem 2.15].

3.2 The Araki-Haag-Kastler axioms

An **Araki-Haag-Kastler net** is $(\mathcal{A}, U, \Omega, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a map from bounded open regions $\mathcal{O} = \{O \subset \mathbb{R}^{1+d} : O \text{ is open and bounded}\}$ into the set of von Neumann algebras on $\mathcal{H}$, U is a unitary representation of $\mathcal{P}_+^\uparrow$ and $\Omega \in \mathcal{H}$ such that

(1) **Isotony**. If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

(2) **Locality**. If O_1, O_2 are spacelike separated, then $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.

(3) **Covariance**. For $\gamma \in \mathcal{P}_+^\uparrow$ and $O \in \mathcal{O}$, $\operatorname{Ad} U(\gamma)(\mathcal{A}(O)) = \mathcal{A}(gO)$.

(4) **Positivity of energy**. The restriction $U|_{\mathbb{R}^{1+d}}$ has the spectrum in $\overline{V_+}$.

(5) **Vacuum**. For all $\gamma \in \mathcal{P}_+^\uparrow$, $U(\gamma)\Omega = \Omega$ and $\bigcup_{O\in\mathcal{O}} \mathcal{A}(O)\Omega$ is dense in $\mathcal{H}$.

(6) **Weak additivity**. If $O \subset \bigcup_j O_j$, then $\mathcal{A}(O) \subset \bigvee_j \mathcal{A}(O_j)$, where $\bigvee$ denotes the von Neumann algebra generated by the following set.

The term "net" is indeed a net from the set $\mathcal{O}$ into the set of von Neumann algebras on $\mathcal{H}$, preserving the order relations defined by inclusion. Isotony means that a larger region corresponds to a larger algebra, and local algebras $\mathcal{A}(O)$ should be interpreted as the algebras generated by observables that can be measured in the corresponding regions O.

As in the case of Wightman fields, we may call $\mathcal{A}$ an Araki-Haag-Kastler net if no confusion arises.

Remark 3.2. When one wants to study the properties of the net of algebras that do not depend on a particular (vacuum) representation, one can take local algebras to be C^*-algebras (rather than von Neumann algebras) and do not require that the vacuum axiom and positivity of energy (and replace the adjoint action by $U(\gamma)$ by isomorphisms between local C^*-algebras). Such a set of axioms is sometimes referred to as the Haag-Kastler axioms.

3.3 From Wightman to AHK

Let ϕ be a Wightman field. We would like to define

$$\mathcal{A}(O) = \{e^{i\phi(f)} : \operatorname{supp} f \subset O\}''.$$

However, it is not automatic that $\phi(f)$ is (essentially) self-adjoint, and the meaning of $e^{i\phi(f)}$ is unclear. One might try to define using the polar decomposition $\mathcal{A}(O) = \{U, e^{itA} : UA = \overline{\phi(f)}, \operatorname{supp} f \subset O, t \in \mathbb{R}\}''$. Then it is easy to show isotony, covariance, positivity of energy and cyclicity of vacuum, but locality and weak additivity do not follow easily.

We say that two symmetric operators commute on the domain $\mathscr{D}$ if $\mathscr{D}$ is invariant for A, B and $[A, B]\Psi = 0$ for all $\Psi \in \mathscr{D}$. We say that two self-adjoint operators A, B **commute strongly** if e^{isA} and e^{itB} commute for all $t, s \in \mathbb{R}$. This is equivalent to the condition that all spectral projections of A and B commute.

The main problem is that commutation on a domain does not imply strong commutation. Indeed, there are self-adjoint operators A, B, defined on a common invariant domain such that $[A, B]\Psi = 0$ for any vector Ψ in the common invariant domain, yet A, B do not strongly commute (Nelson's counterexample [Nel59, Section 10][RS80, Section VIII.5]). This means that, even if ϕ satisfies locality in the Wightman sense, we cannot infer that its exponentials $e^{i\phi(f)}, e^{i\phi(g)}$ commute even if $\operatorname{supp} f$ and $\operatorname{supp} g$ are spacelike.

The following is due to [GJ87, Theorem 19.4.4], with a slight variation on the assumptions [AGT, Theorem A.2].

Theorem 3.3 (Glimm-Jaffe)**.** *The following hold.*

- *Let H be a positive self-adjoint operator, $\mathscr{D}$ a core of H and A a symmetric operator on $\mathscr{D}$ such that $A\mathscr{D} \subset \mathscr{D}$ and suppose that $\|A\Psi\| \le C\|(H + \mathbb{1})\Psi\|, \|[H, A]\Psi\| \le C\|(H + \mathbb{1})\Psi\|, \|[H, [H, A]]\Psi\| \le C\|(H + \mathbb{1})\Psi\|$ for $\Psi \in \mathrm{Dom}(H)$. Then A is essentially self-adjoint on any core of H.*

- *Let H, A as above and assume that B satisfies parallel assumptions as A, and suppose that the operators*

– $A, \delta(A) = i[H, A], \delta^2(A) = -[H, [H, A]], \delta^3(A) = -i[H, [H, [H, A]]]$

– $B, \delta(B) = i[H, B], \delta^2(B) = -[H, [H, B]], \delta^3(B) = -i[H, [H, [H, B]]]$

are defined on $\mathscr{D}$ *and*

– $\|A\Psi\| \le C\|(H + \mathbb{1})\Psi\|, \|\delta^k(A)\Psi\| \le C\|(H + \mathbb{1})\Psi\|, k = 1, 2, 3$

– $\|B\Psi\| \le C\|(H + \mathbb{1})\Psi\|, \|\delta^k(B)\Psi\| \le C\|(H + \mathbb{1})\Psi\|, k = 1, 2, 3$

Suppose furthermore that $AB = BA$ *on* $\mathscr{D}$*. Then* A, B *are essentially self-adjoint on any core of* H *and their closures commute strongly.*

Let (ϕ, U, Ω) be a Wightman field theory and H be the generator of the one-parameter group $U((t, 0, \cdots 0), 0)$ (called the **Hamiltonian**). We say that ϕ satisfies a **linear energy bound** if for any $f \in \mathscr{S}(\mathbb{R}^{1+d})$ there is $C_f > 0$ such that $\|\phi(f)\Psi\| \le C_f\|(H + \mathbb{1})\Psi\|$ for all $\Psi \in \mathscr{D}$.

We need the following [RS80, Theorem VIII.7].

Lemma 3.4. *Let* H *be a self-adjoint operator on* $\mathrm{Dom}(H)$*,* $\Psi \in \mathcal{H}$*.* $\Psi \in \mathrm{Dom}(H)$ *if and only if* $\frac{e^{itH}\Psi - \Psi}{t}$ *converges (to* $iH\Psi$*, a posteriori) as* $t \to 0$*.*

Lemma 3.5. *Assume that* ϕ *satisfies linear energy bounds. Then, for any* $f \in \mathscr{S}(\mathbb{R}^{1+d})$*,* $\phi(f)$ *preserves* $C^\infty(H)$*, where* $C^\infty(H) = \bigcap_{n\in\mathbb{N}} \mathrm{Dom}(H^n)$*, and* $[H, \phi(f)] = i\phi(f')$*.*

Proof. Let $\Psi \in \mathrm{Dom}(H^2)$ and $f \in \mathscr{S}(\mathbb{R}^{1+d})$. We denote $f_t(x) = f(x - (t, 0, \cdots, 0)), V(t) = U((t, 0, \cdots, 0), 0)$. By covariance, we have $\phi(f_t)\Psi = V(t)\phi(f)V(t)^*\Psi$, or

$$V(t)^*\phi(f_t)\Psi = \phi(f)V(t)^*\Psi. \tag{1}$$

As for the right-hand side, note that

$$\begin{aligned}\left\|\frac{\phi(f)V(t)^*\Psi - \phi(f)\Psi}{t} - i\phi(f)H\Psi\right\| &\le C_f\left\|(H + \mathbb{1})\frac{(V(t)^* - \mathbb{1})}{t}\Psi - iH\Psi\right\| \\ &= C_f\left\|\left(\frac{(V(t)^* - \mathbb{1})}{t} - iH\right)(H + \mathbb{1})\Psi\right\|\end{aligned}$$

therefore, as $\Psi \in C^\infty(H)$, hence $(H + \mathbb{1})\Psi \in \mathrm{Dom}(H)$ and we conclude that the derivative of the right-hand side of (1) is $i\phi(f)H\Psi$. As for the left-hand side,

$$\lim_{t\to 0}\frac{V(t)^*\phi(f_t)\Psi - \phi(f)\Psi}{t} = \lim_{t\to 0}\left(\frac{V(t)^*\phi(f_t)\Psi - V(t)^*\phi(f)\Psi}{t} + \frac{V(t)^*\phi(f)\Psi - \phi(f)\Psi}{t}\right).$$

As the first term and the whole expression (which is equal to the right-hand side of (1)) converge, the second term must converge. This implies that $\phi(f)\Psi \in \mathrm{Dom}(H)$ and $H\phi(f)\Psi = \phi(f)H\Psi + i\phi(f')\Psi$. The last expression shows that, if $\Psi \in \mathrm{Dom}(H^n)$, then $\phi(f)\Psi \in \mathrm{Dom}(H^{n-1})$, by induction. Indeed, assume that $\phi(f)\Psi \in \mathrm{Dom}(H^{n-1})$ follows from $\Psi \in \mathrm{Dom}(H^n)$. Then if $\Psi \in \mathrm{Dom}(H^{n+1})$, then $H\Psi \in \mathrm{Dom}(H^n)$ and by induction $H\phi(f)\Psi = \phi(f)H\Psi + i\phi(f')\Psi \in \mathrm{Dom}(H^{n-1})$, hence $\phi(f)\Psi \in \mathrm{Dom}(H^n)$. □

The following is [RS80, Theorem VIII.11], whose proof is essentially contained in the proof of Stone's theorem [RS80, Theorem VIII.8].

Lemma 3.6. *Let $V(t)$ be a unitary representation of $\mathbb{R}$ (continuous in SOT) and $\mathscr{D}$ be a dense domain, $V(t)\mathscr{D} = \mathscr{D}$, and assume that vectors in $\mathscr{D}$ can be differentiated, that is, $\lim_{t\to 0} \frac{V(t)-\mathbb{1}}{t}\Psi$ converges for any $\Psi \in \mathscr{D}$. Then the generator H of $V(t) = e^{itH}$ is essentially self-adjoint on $\mathscr{D}$.*

By this Lemma, $\mathrm{span}\{\phi(f_1)\cdots\phi(f_n)\Omega : n \in \mathbb{N}, f_j \in \mathscr{S}(\mathbb{R}^{1+d})\}$ is a core of H. Thus so is the domain $\mathscr{D}$ of the Wightman field theory.

The following is known as the Trotter product formula [RS80, Theorem VIII.31], see also the reference therein.

Lemma 3.7. *Let A, B two self-adjoint operators, $A+B$ essentially self-adjoint on* $\mathrm{Dom}(A)\cap \mathrm{Dom}(B)$. *Then $e^{i(A+B)} = \lim_{n\to\infty}(e^{iA/n}e^{iB/n})^n$.*

Theorem 3.8. *Assume that a Wightman field theory (ϕ, U, Ω) satisfies linear energy bounds. Then $(\mathcal{A}, U, \Omega)$ constructed above satisfies the AHK axioms.*

Proof. Locality is immediate by Theorem 3.3, by extending the locality of ϕ to $C^\infty(H)$ by linear energy bounds (the conclusion that $\phi(f)$ is essentially self-adjoint on any core of H follows without the commutation of $\phi(f), \phi(g)$). As for weak additivity, we may assume that $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp}\, f \subset O, \mathrm{supp}\, f \text{ compact}\}''$. If $O \subset \bigcup_j O_j, \mathrm{supp}\, g \subset O$ compact, we can find finitely many g_j such that $\mathrm{supp}\, g_j \subset O_j$ and $g = \sum g_j$. Therefore, it is enought to show that $O \subset O_1 \cup O_2$ implies $\mathcal{A}(O) \subset \mathcal{A}(O_1) \vee \mathcal{A}(O_2)$.

Let $g = g_1 + g_2, \mathrm{supp}\, g_j \subset O_j$. By linear energy bounds, $\phi(g_1), \phi(g_2), \phi(g)$ are essentially self-adjoint on $\mathrm{Dom}(H)$. By the Trotter formula, $e^{i\phi(g)} = \lim_{n\to\infty}(e^{i\phi(g_1)/n}e^{i\phi(g_2)/n})^n \in \mathcal{A}(O_1) \vee \mathcal{A}(O_2)$. □

This is a typical case where one can associate a AHK net to a Wightman field. The $\mathscr{P}(\phi)_2$-models, the ϕ^4_3-model, the Yukawa model in $d = 1+1$ have been shown to satisfy the AHK axioms [Sum12]

3.4 The Reeh-Schlieder property

Let $(\mathcal{A}, U, \Omega)$ be an AHK net. An important consequence of the axioms is the **Reeh-Schlieder property**, which allows one to apply the Tomita-Takesaki modular theory to local algebras and the vacuum vector.

Theorem 3.9 (Reeh, Schlieder)**.** *For any $O \subset \mathbb{R}^{1+d}$ open, Ω is cyclic for $\mathcal{A}(O)$. If O' is nonempty, then Ω is cyclic for $\mathcal{A}(O)$.*

Proof. Let us show the cyclicity of Ω for $\mathcal{A}(O)$, O open. Let $\Psi \in (\mathcal{A}(O)\Omega)^\perp$ and we prove that $\Psi = 0$.

Let $\underline{O}$ an open whose closure is included in O. Take $x \in \mathcal{A}(\underline{O})$. Then for $a \in \mathbb{R}^{1+d}$ sufficiently small, $\mathrm{Ad}\, U(a,0)(x) \subset \mathcal{A}(O)$ and hence

$$\langle \Psi, U(a,0)\Omega\rangle = \langle \Psi, U(a,0)xU(a,0)^*\Omega\rangle = 0.$$

Let a be a lightlike vector with $a_0 > 0$. For $t \in \mathbb{R}$, we introduce

$$f(t) = \langle \Psi, U(ta, 0)x\Omega \rangle,$$

and it vanishes for t small.

On the other hand, recall that

$$U(a, 0) = \int e^{i(a,p)} dE(p)$$

and the support of dE is contained in $\overline{V_+}$. As a is lightlike and $a_0 > 0$, the map $t \mapsto i(ta, p) \in \mathbb{C}$ can be analytically extended to $\mathbb{C}$, and if $\Im t > 0$, then $\Re(i(ta, p)) < 0$. This implies that

$$t \longmapsto \langle \Psi, U(ta, 0)x\Omega \rangle = \int e^{i(ta,p)} \langle \Psi, dE(p)x\Omega \rangle$$

has a bounded analytic continuation to $\Im t > 0$. But $f(t) = 0$ for $t \in \mathbb{R}$ small, hence it must vanish for all $\Im t > 0$, and by continuity, for all $t \in \mathbb{R}$.

This holds for any lightlike vector a, hence by repeating this argument, we obtain that $\langle \Psi, U(a, 0)x\Omega \rangle = 0$ for any $a \in \mathbb{R}^{1+d}$.

The same argument can be applied to $\langle \Psi, U(a_n, 0)x_n U(a_{n-1}, 0)x_{n-1} \cdots U(a_1, 0)x_1\Omega \rangle$ separately for a_j, and we conclude that such a scalar product vanishes for any $a_j \in \mathbb{R}^{1+d}$. By weak additivity, for any $D \subset \bigcup_j O + a_j$, we obtain that $\langle \Psi, x\Omega \rangle = 0$ for $x \in \mathcal{A}(D)$. By the cyclicity of Ω for $\bigcup_{D \subset \mathbb{R}^{1+d}} \mathcal{A}(D)$, $\Psi = 0$.

The second part follows from the first part immediately. □

It is a difficult question under which reasonable conditions one can construct Wightman fields starting with an AHK net, see [Bos05, FJ96].

4 Lecture 4 (5/17). The massive free fields

In this Section, we introduce the simplest examples satisfying the Wightman axioms and the AHK axioms: the massive free fields and their associated nets. For simplicity, we restrict to the case $d = 1$, but the results on the bosonic free fields can be generalized to arbitrary dimensions. We also discuss the massive free fermions. We follow [Lec03].

4.1 One-particle space

Let $m > 0$, called the mass of the scalar particle. We take $\mathcal{H}_m = L^2(\mathbb{R}, d\theta)$ as before. Let U_m be the representation of the Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1+1} \rtimes \mathbb{R}$, as we constructed in Section 2, given by

$$(U_m(a, \lambda)\Psi)(\theta) = e^{i(a,p(\theta))}\Psi(\theta - \lambda),$$

where $p(\theta) = (m \cosh \theta, m \sinh \theta) \in \mathbb{R}^{1+1}$. This has positive energy, as we have seen (as an exercise). The spectrum is $\Omega_m = \{(m \cosh \theta, m \sinh \theta) : \theta \in \mathbb{R}\} \subset V_+ \subset \mathbb{R}^{1+1}$.

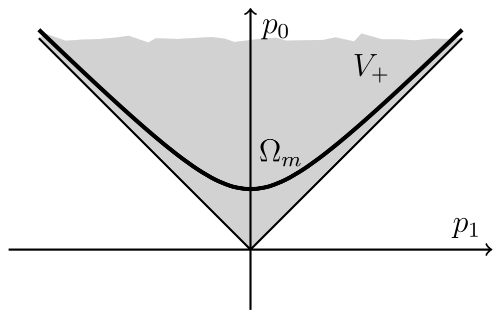

Figure 2: The spectrum of U_m, called the mass shell, included in V_+.

4.2 Second quantization (general)

Let $\mathcal{K}$ be a Hilbert space and consider its n-th tensor product $\mathcal{K}^{\otimes n}$. There is a natural acton π_n^+ of the symmetric group $\mathfrak{S}_n$ by permutation of the components:

$$\pi_n^+(\sigma)\Psi_1 \otimes \cdots \otimes \Psi_n = \Psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \Psi_{\sigma^{-1}(n)}.$$

Put $P_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \pi_n^+(\sigma)$, the orthogonal projection onto the symmetric n-particle subspace. We introduce $\mathcal{F}^+(\mathcal{K}) = \bigoplus_{n\in\mathbb{N}} P_n\mathcal{K}^{\otimes n}$, where we make the convention that $0 \in \mathbb{N}$ and $\mathcal{K}^{\otimes 0} = \mathbb{C}\Omega$ and Ω is just a symbolic basis of a one-dimensional complex vector space, called the **(Fock) vacuum**. This $\mathcal{F}^+(\mathcal{K})$ is called the **Bosonic Fock space** (based on $\mathcal{K}$),

Let $\xi \in \mathcal{K}$. We define the creation and annihilation operators on $\mathcal{F}^+(\mathcal{K})$ as follows: for $\Phi_n \in P_n\mathcal{K}^{\otimes n}$,

$$\begin{aligned} a^\dagger(\xi)\Phi_n &= \sqrt{n+1}P_{n+1}(\xi \otimes \Phi_n), \\ a(\xi) &= a^\dagger(\xi)^*. \end{aligned}$$

Note that $a^\dagger$ is linear in ξ, while $a(\xi)$ is antilinear in ξ.

Exercise: Show that $a^\dagger, a$ satisfy

$$[a^\dagger(\xi), a^\dagger(\eta)] = 0, [a(\xi), a(\eta)] = 0, [a(\xi), a^\dagger(\eta)] = \langle \xi, \eta\rangle,$$

where our convention of the inner product is that it is linear in the second argument.

For a unitary operator U on $\mathcal{K}$, we define

$$\Gamma(U) := \bigoplus_{n\in\mathbb{N}} P_n U^{\otimes n}.$$

It is straightforward to see that $\Gamma(U)$ is a unitary operator $\mathcal{F}^+(\mathcal{K})$. Moreover, if U is a unitary representation of a group G on $\mathcal{K}$, then $\Gamma(U)(\gamma) := \Gamma(U(\gamma))$ is a unitary representation of G on $\mathcal{F}^+(\mathcal{K})$. They are called the **(multiplicative) second quantization** of U. For a self-adjoint operator A, we define the **(additive) second quantization** by

$$d\Gamma(A) = \bigoplus_{n\in\mathbb{N}} \sum_{k=1}^n \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underset{k\text{-th}}{A} \otimes \mathbb{1} \cdots \otimes \mathbb{1},$$

where the $n = 0$ case the sum gives 0.

Exercise: Show that $\mathrm{Ad}\,\Gamma(U)(a^\dagger(\xi)) = a^\dagger(U\xi)$.

Exercise: Show that, if U is a representation of $\mathbb{R}^{1+1}$, then $\mathrm{sp}\,\Gamma(U)$ is contained in the closure of the closure of the set additively generated by $\mathrm{sp}\,U$. If A is a self-adjoint operator, it holds that $e^{id\Gamma(A)} = \Gamma(e^{iA})$.

4.3 Free massive scalar field

Now we define the free quantum field satisfying the Wightman axioms. Here we fix $m > 0$ and take $\mathcal{K} = \mathcal{H}_m$.

For $f \in \mathscr{S}(\mathbb{R}^{1+1})$, we define

$$f^+(\theta) = \int e^{i(p(\theta),x)} f(x) d^2x.$$

It is easy to show that $f^+(\theta) \in \mathcal{H}_m = L^2(\mathbb{R}, d\theta)$, and furthermore, the map $f \mapsto f^+$ is an L^2-valued distribution. Considering $p(\theta) = (m\cosh\theta, m\sinh\theta)$ on the mass shell Ω_m, this is simply the restriction of the Fourier transform of f to Ω. Then we define

$$\phi(f) = a^\dagger(f^+) + a(f^+).$$

This is real linear in f, defined on $\mathscr{D} = \bigoplus_{n\in\mathbb{N},\mathrm{alg}} P_n \mathcal{H}_m^{\otimes n}$.

Furthermore, for $\gamma \in \mathcal{P}_+^\uparrow$, let $U(\gamma) = \Gamma(U_m(\gamma))$. As the vacuum vector, we choose $\Omega \in \mathcal{F}^+(\mathcal{H}_m)$.

We need a technical lemma, see [Tan16, Proposition A.1]: for $\mathbb{S}_{0,\pi} = \mathbb{R} + i(0,\pi)$, we consider

$$H^2(\mathbb{S}_{0,\pi}) = \{\Psi \in L^2(\mathbb{R}, d\theta) : \Psi \text{ continues analytically to } \mathbb{S}_{0,\pi}, \|\Psi(\cdot + i\lambda)\|_{L^2} \le C\}.$$

Lemma 4.1. *If $\xi, \eta \in H^2(\mathbb{S}_{0,\pi})$, then the Cauchy theorem holds: $\int \xi(\theta)\eta(\theta)d\theta = \int \xi(\theta + i\pi)\eta(\theta + i\pi)d\theta$.*

Theorem 4.2. *ϕ is a Wightman field on $\mathbb{R}^{1+1}$.*

Proof. It is clear that ϕ is an operator-valued distribution with an invariant domain $\mathscr{D}$ and $U(\gamma)\mathscr{D} = \mathscr{D}$.

Let us first show covariance. We have

$$(U_m(a,\lambda)f^+)(\theta) = e^{i(p(\theta),a)} \int e^{i(p(\theta-\lambda),x)} f(x) d^2x.$$

Using

$$\begin{aligned} p(\theta-\lambda) = \begin{pmatrix} m\cosh(\theta-\lambda) \\ m\sinh(\theta-\lambda) \end{pmatrix} &= m \begin{pmatrix} \cosh\theta\cosh\lambda - \sinh\theta\sinh\lambda \\ \sinh\theta\cosh\lambda - \cosh\theta\sinh\lambda \end{pmatrix} \\ &= \begin{pmatrix} \cosh\lambda & \sinh(-\lambda) \\ \sinh(-\lambda) & \cosh\lambda \end{pmatrix} \begin{pmatrix} m\cosh\theta \\ m\sinh\theta \end{pmatrix} = \Lambda(-\lambda)p(\theta), \end{aligned}$$

we have

$$\begin{aligned}(U_m(a,\lambda)f^+)(\theta) &= e^{i(p(\theta),a)}\int e^{i(\Lambda(-\lambda)p(\theta),x)}f(x)d^2x \\ &= e^{i(p(\theta),a)}\int e^{i(p(\theta),\Lambda(\lambda)x)}f(x)d^2x \\ &= e^{i(p(\theta),a)}\int e^{i(p(\theta),x)}f(\Lambda(-\lambda)x)d^2x \\ &= \int e^{i(p(\theta),x+a)}f(\Lambda(-\lambda)x)d^2x \\ &= \int e^{i(p(\theta),x)}f(\Lambda(-\lambda)(x-a))d^2x,\end{aligned}$$

where we used the translation-invariance of the integral, the invariance of the Lorentz metric and the fact that $\det\Lambda(-\lambda) = 1$. This implies $U_m(a,\lambda)f^+ = (f_{(a,\lambda)})^+$.

Next we prove locality. By covariance, we may assume that $\mathrm{supp}\, f \subset W_\mathrm{L}, \mathrm{supp}\, g \subset W_\mathrm{R}$, where

$$W_\mathrm{L} := \{(a_0,a_1) \in \mathbb{R}^{1+1} : |a_0| \le -a_1\} W_\mathrm{R} := \{(a_0,a_1) \in \mathbb{R}^{1+1} : |a_0| \le a_1\} = W'_\mathrm{L}.$$

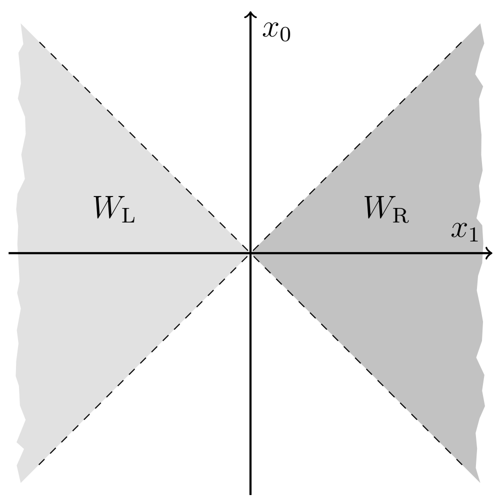

Figure 3: Letf and right wedges $W_\mathrm{L}, W_\mathrm{R}$.

As we know that the commutators between $a^\dagger$ and those between a vanish, to know the commutator $[\phi(f),\phi(g)]$, we only have fo calculate $[a^\dagger(f^+), a(g^+)]$ and $[a(f^+), a^\dagger(g^+)]$. Note that

$$\begin{aligned}p(\theta+i\zeta) &= (m(\cosh\theta\cosh(i\zeta)+\sinh\theta\sinh(i\zeta)), m(\sinh\theta\cosh(i\zeta)+\cosh\theta\sinh(i\zeta))) \\ &= (m(\cosh\theta\cos(\zeta)+i\sinh\theta\sin(\zeta)), m(\sinh\theta\cos(\zeta)+i\cosh\theta\sin(\zeta))),\end{aligned}$$

therefore, its imaginary part $m(\sinh\theta\sin(\zeta), \cosh\theta\sin(\zeta))$ is in W_R if $0<\zeta<\pi$ and $p(\theta + i\pi) = -p(\theta) = -p(\theta - i\pi)$.

We have $[a^\dagger(f^+), a(g^+)] = -\int f^+(\theta)\overline{g^+(\theta)}d\theta$. As $(x,y) > 0$ if $x \in W_{\mathrm{L}}, y \in W_{\mathrm{R}}$, $i(p(\theta + i\zeta), x)$ has negative real part, and therefore,

$$f^+(\theta + i\zeta) = \int e^{i(p(\theta+i\zeta,x)} f(x) d^2x$$

is continuous on $\overline{\mathbb{S}_{0,\pi}}$ (as for the boundary, one can just use the Lebesgue dominated convergence theorem), has an analytic continuation to $0 < \Im\zeta < \pi$, and is rapidly decreasing for each such π, and it holds that $f^+(\theta + i\pi) = f^+(-\theta) = \overline{f^+(\theta)}$. Then by Lemma A.1, $f^+ \in H^2(\mathbb{S}_{0,\pi})$. Similarly, $\overline{g^+(\theta)}$ has an analytic continuation to $0 < \Im\zeta < \pi$ ($\zeta \mapsto \overline{g^+(\bar\zeta)}$ and $\overline{g^+}(\theta + i\pi) = g^+(-\theta) = \overline{g^+(\theta)}$. Therefore, we can apply the Cauchy theorem, and

$$\begin{aligned}
[a^\dagger(f^+), a(g^+)] &= -\int f^+(\theta)\overline{g^+(\theta)}d\theta \\
&= -\int f^+(\theta + i\pi)\overline{g^+(\theta - i\pi)}d\theta \\
&= -\int \overline{f^+(\theta)}g^+(\theta)d\theta = -[a(f^+), a^\dagger(g^+)].
\end{aligned}$$

Altogether, we have $[\phi(f), \phi(g)] = 0$ in this situation.

Positivity of energy follows from the additivity of the spectrum for second quantization.

The properties of the vacuum Ω is easy: it is straightforward that $\{\phi(f)\Omega : f \in \mathscr{S}(\mathbb{R}^{1+1})\}$ spans the one-particle space $\mathcal{H}_m$. The rest can be done by induction and the structure $\phi(f) = a^\dagger(f^+) + a(f^+)$. □

To conclude the discussion, let us show that ϕ satisfies linear energy bounds, therefore, it generates a AHK net.

Theorem 4.3. *The field ϕ satisfies linear energy bounds.*

Proof. By the construction, it is clear that the generator H_m of the time-translation is given by the multiplication operator by the function $\cosh\theta \geq 1$. Then its second quantization is bounded below by $d\Gamma(\mathbb{1})$, which gives the particle number N.

On the other hand, it is clear that the norm of $a^\dagger(\xi)$ is bounded by $\sqrt{n+1}\|\xi\|$ on $P_n\mathcal{H}_m^{\otimes n}$, hence it is bounded by $\|\xi\|(N + \mathbb{1})$. Therefore, $a(\xi)$ is bounded by $\|\xi\|(N + \mathbb{1})$ as well. Altogether, $\phi(f)$ is bounded by $2\|f^+\|(N + \mathbb{1})$. □

Note that, we can repeat the same argument for the representation π^- of $\mathfrak{S}_n$, except locality. The operators $a^\dagger(\xi), a(\xi)$ satisfy the anticommutations relations, instad of the commutation relations.

5 Lecture 5 (5/18). The Tomita-Takesaki theory in AHK nets

Here we study some properties of AHK nets related with the Tomita-Takesaki modular theory and prove them for the massive free fields.

5.1 The Borchers(-Florig) theorem

We will show that, in the massive free field nets, the modular group of a wedge algebra with respect to the vacuum state coincides with the Lorentz boosts. More precisely, if $\Delta^{it}_{\mathcal{A}(W_{\mathrm{L}}),\Omega} = U(0, 2\pi t)$, we say that $(\mathcal{A}, U, \Omega)$ satisfies the **Bisognano-Wichmann property** [BW75, BW76]. It is known that the Bisognano-Wichmann property holds automatically if a net is generated by a Wightman field in dimension $d = 1 + 3$. In $d = 1 + 1$, it is easy to construct a counterexample. Therefore, we need to study the free fields explicitly. The following is due to [Bor92], whose proof has been drastically simplified in [Flo98]

Lemma 5.1 (Borchers-Florig). *Let $\mathcal{M}$ be a von Neumann algebra, Ω be a cyclic and separating vector for $\mathcal{M}$ on $\mathcal{H}$, and Δ, J be the modular objects of $\mathcal{M}$ with respect to Ω. Assume that there is a one-parameter group of unitaries $V(t) = e^{itH}$ with positive generator H satisfying $\mathrm{Ad}\, V(t)(\mathcal{M}) \subset \mathcal{M}$ for $t \geq 0$. Then $\Delta^{it}V(s)\Delta^{-it} = V(e^{-2\pi t}s)$ and $JV(s)J = V(-s)$.*

Proof. Let $x \in \mathcal{M}, y \in \mathcal{M}'$ and define, for a fixed $s \in \mathbb{R}$,

$$f(t) = \langle \Delta^{-it}y\Omega, V(e^{2\pi t}s)\Delta^{-it}x\Omega\rangle.$$

Recall that $x\Omega \in \mathrm{Dom}(\Delta^{\frac{1}{2}}), y\Omega \in \mathrm{Dom}(\Delta^{-\frac{1}{2}})$. Therefore, the map $t \mapsto \Delta^{it}x\Omega$ has an $\mathcal{H}$-valued bounded analytic continuation to $\mathbb{R} + i(0, \frac{1}{2})$. Similarly, the map $t \mapsto \Delta^{-it}y\Omega$ has an antianalytic continuation to $\mathbb{R} + i(-\frac{1}{2}, 0)$. In addition, $V(e^{2\pi t}s) = \int_{\mathbb{R}_+} e^{i\lambda s e^{2\pi t}} dE(\lambda)$ also has a bounded analytic continuation to $\mathbb{R} + i(0, \frac{1}{2})$ because $e^{2\pi t}$ has positive imaginary part. Therefore, f has an analytic continuation to $\mathbb{R} + i(0, \frac{1}{2})$, and on the boundary it holds that

$$\begin{aligned} f(t + \tfrac{i}{2}) &= \langle \Delta^{-i(t-\frac{i}{2})}y\Omega, V(e^{2\pi(t+\frac{i}{2})}s)\Delta^{-i(t+\frac{i}{2})}x\Omega\rangle \\ &= \langle \Delta^{-it}\Delta^{-\frac{1}{2}}y\Omega, V(-e^{2\pi t}s)\Delta^{-it}\Delta^{\frac{1}{2}}x\Omega\rangle \\ &= \langle \Delta^{-it}Jy^*\Omega, V(-e^{2\pi t}s)\Delta^{-it}Jx^*\Omega\rangle \\ &= \langle JV(-e^{2\pi t}s)J\Delta^{-it}x^*\Omega, \Delta^{-it}y^*\Omega\rangle \end{aligned}$$

Observe that $V_1(s) = JV(-s)J$ has the property that $\mathrm{Ad}\, V_1(s)(\mathcal{M}) \subset \mathcal{M}$ for $s \geq 0$. Therefore,

$$\begin{aligned} f(t + \tfrac{i}{2}) &= \langle \Delta^{-it}y\Delta^{it}\Omega, V_1(e^{2\pi t}s)\Delta^{-it}x\Delta^{it}\Omega\rangle \\ &= \langle \Delta^{-it}y\Omega, V_1(e^{2\pi t}s)\Delta^{-it}x\Omega\rangle =: f_1(t). \end{aligned}$$

Note that, by replacing V by V_1, we can repeat the same argument, that is, $f_1(t)$ has an analytic continuation in $\mathbb{R} + i(0, \frac{1}{2})$ and

$$\begin{aligned} f_1(t + \tfrac{i}{2}) &= \langle \Delta^{-it}y\Delta^{it}\Omega, V_1(e^{2\pi t}s)\Delta^{-it}x\Delta^{it}\Omega\rangle \\ &= \langle \Delta^{-it}y\Omega, JV_1(-e^{2\pi t}s)J\Delta^{-it}x\Omega\rangle \\ &= f(t). \end{aligned}$$

This implies that f is constant, and we obtain that $\Delta^{it}V(e^{-2\pi t}s)\Delta^{-it} = V(s), JV(-s)J = V(s)$. □

5.2 The Bisognano-Wichmann property

Using this Lemma, we prove the Bisognano-Wichmann property for the massive free fields in $d = 1+1$. We can apply this to $\mathcal{M} = \mathcal{A}(W_\mathrm{L})$ and $V(s) = U((s,-s),0)$. The conclusion is that $\Delta^{it}V(s)\Delta^{-it} = V(e^{-2\pi t}s)$. On the other hand, by the Poincaré group relations, we also have that $U(0,2\pi t)V(s)U(0,2\pi t)^* = V(e^{-2\pi t}s)$. This implies that $\Delta^{it}U(0,-2\pi t)$ commutes with $V(s) = U((s,-s),0)$. We can apply a parallel argument to the negative lightlike translations $U((-s,-s)),0)$, which preserves $\mathcal{M}$ with negative generator, and conclude that Δ^{it} and $U(0,2\pi t)$ satisfy the same commutation relations with $U((-s,-s),0)$. Furthermore, $U(0,2\pi t)$ implements a vacuum-preserving automorphism of $\mathcal{A}(W_\mathrm{L})$, therefore, it commutes with Δ^{it}. Altogether, $I(t)$ commutes with $U(\gamma), \gamma \in \mathcal{P}_+^\uparrow$.

Theorem 5.2 (Bisognano-Wichmann property for the free field)**.** *For the massive free field net $(\mathcal{A}, U, \Omega)$, the Bisognano-Wichmann property holds: $U(0,2\pi t) = \Delta_{\mathcal{A}(W_\mathrm{L}),\Omega}$.*

Proof. We put $\mathcal{M} = \mathcal{A}(W_\mathrm{L}), I(t) = \Delta^{it}U(0,-2\pi t)$. Our goal is to prove that $I(t) = \mathbb{1}$ for $t \in \mathbb{R}$. As $I(t)$ commutes both with translations and Lorents boosts, it commutes with the whole $U(\gamma), \gamma \in \mathcal{P}_+^\uparrow$.

It is clear that $I(t)\Omega = \Omega$. Next, the one-particle space $\mathcal{H}_m$ is an irreducible subspace of $U(\gamma)$ with the mass m with no multiplicity, therefore, as $I(t)$ commutes with $U(\gamma)$, $I(t)$ must act as a scalar on $\mathcal{H}_m$, say, $I(t)|_{\mathcal{H}_m} = e^{it\lambda}$ for some $\lambda \in \mathbb{R}$.

Note that $\mathrm{Ad}\, I(t)(\mathcal{M}) = \mathcal{M}$. We define $\phi_t(f) = \mathrm{Ad}\, I(t)(\phi(f))$ for $f \in \mathscr{S}(\mathbb{R}^{1+1})$ such that $\mathrm{supp}\, f \subset W_\mathrm{L}$, then $\phi_t(f)$ is still affiliated to $\mathcal{M}$.

For $g \in \mathscr{S}(\mathbb{R}^{1+1})$ such that $\mathrm{supp}\, f \subset W_\mathrm{R}$, we have

$$
\begin{aligned}
I(t)\phi(f)\phi(g)\Omega &= \phi_t(f)I(t)\phi(g)\Omega \\
&= e^{it\lambda}\phi_t(f)\phi(g)\Omega \\
&= e^{it\lambda}\phi(g)\phi_t(f)\Omega \\
&= e^{it\lambda}\phi(g)I(t)\phi(f)\Omega \\
&= e^{i2t\lambda}\phi(g)\phi(f)\Omega \\
&= e^{i2t\lambda}\phi(f)\phi(g)\Omega
\end{aligned}
$$

and this holds for all such f, g, we must have $\lambda = 0$, because the vector $\phi(f)\phi(g)\Omega$ has Ω component in general.

Assume that $I(t)$ acts as $\mathbb{1}$ on the k-particle spaces, $k = 0, \cdots n$. This implies that the restriction of Δ^{it} is equal to $U(0,2\pi t)$. On the n-particle space, there are dense set of vectors that are in the domain of $U(0,i\pi)$. For such a vector Ψ, there is a closed operator F affliated with $\mathcal{M}$ such that $\Psi = F\Omega, J\Delta^{\frac{1}{2}}\Psi\Omega$. Then $\mathrm{Ad}\, I(t)(F)$ is affiliated with $\mathcal{M}$ and we have

$$
\begin{aligned}
I(t)\phi(g)F\Omega &= \mathrm{Ad}\, I(t)(F)\phi(g)\Omega \\
&= \mathrm{Ad}\, I(t)FI(t)^*\phi(g)\Omega \\
&= \phi(g)\, \mathrm{Ad}\, I(t)FI(t)^*\Omega \\
&= \phi(g)F\Omega
\end{aligned}
$$

which shows that $I(t)$ is trivial on the $n+1$-particle space. This completes the induction. □

We used the following in the proof.

Exercise. Let A, B be (possibly unbounded) self-adjoint operators. Suppose that $\mathcal{M}$ is a von Neumann algebra, A is affiliated with $\mathcal{M}$ and B is affiliated with $\mathcal{M}'$. Prove that, if $\Psi \in \mathrm{Dom}(B) \cap \mathrm{Dom}(BA)$, then $\Psi \in \mathrm{Dom}(AB)$ and $AB\Psi = BA\Psi$.

With the uniqueness of Ω, we have the following [Lon79], or in the context of AHK net, [Dri75].

Theorem 5.3. *Let $\mathcal{M}$ be a von Neumann algebra, Ω be a cyclic and separating vector for $\mathcal{M}$ on $\mathcal{H}$, and Δ, J be the modular objects of $\mathcal{M}$ with respect to Ω. Assume that there is a one-parameter group of unitaries $V(t) = e^{itH}$ with positive generator H satisfying $\mathrm{Ad}\, V(t)(\mathcal{M}) \subset \mathcal{M}$ for $t \geq 0$ and Ω is the only vector (up to scalar) that $V(t)\Omega = \Omega$. Then, if $\mathcal{M} \neq \mathbb{1}\mathbb{C}$, then it is a type III factor.*

Proof. We introduce $\mathcal{M}^\Omega = \{x \in \mathcal{M} : \langle \Omega, xy\Omega \rangle = \langle \Omega, yx\Omega \rangle \text{ for } y \in \mathcal{M}\}$, the centralizer of $\mathcal{M}$ with respect to Ω. We claim that $\mathcal{M}^\Omega = \mathbb{C}\mathbb{1}$.

Let $p \in \mathcal{M}^\Omega$ a projection. Define $f(t) = \langle \Omega, pV(t)p\Omega \rangle$. Then, for $t \geq 0$, $V(t)pV(t)^* \in \mathcal{M}$ for $t \geq 0$ and $f(t) = \langle \Omega, pV(t)pV(t)^*\Omega \rangle = \langle \Omega, V(t)pV(t)^*p\Omega \rangle = \langle \Omega, pV(-t)p\Omega \rangle = f(-t)$. But f has a bounded analytic continuation to $\mathbb{R} + i(0, \infty)$, while from the last expression it also continues to $\mathbb{R} + i(-\infty, 0)$, thus it is constant. This implies that $\langle p\Omega, V(t)p\Omega \rangle = \langle p\Omega, p\Omega \rangle$, which is possible only if $V(t)p\Omega = p\Omega$, and by assumption $p\Omega = \Omega$. As Ω is separating, $p = \mathbb{1}$.

Now we can show that $\mathcal{M}$ is of type III. Indeed, if not, the modular automorphism would be inner, say implemented by $W(t)$, and then $W(t) \in \mathcal{M}^\Omega$ because $\mathrm{Ad}\, W(t)$ preserves $\langle \Omega, \cdot\, \Omega \rangle$, thus $W(t) \in \mathbb{C}\mathbb{1}$. But the the modular group would be trivial, and $\mathcal{M}^\Omega = \mathcal{M} = \mathbb{C}\mathbb{1}$. □

Actually, from the same assumption, it follows that $\mathcal{M}$ is a type III_1 factor, see e.g. [Bau95, Corollary 1.10.8].

Corollary 5.4. $\mathcal{A}(W_\mathrm{L})$ *is a type* III_1 *factor.*

Knowing explicitly the modular group can be useful for further structural analysis. Let us start with a general result.

5.3 Modular nuclearity

Let A, B be Banach spaces. A linear map $\varphi : A \to B$ is said to be **nuclear** if there are sequences $\varphi_n \in A^*, \psi_n \in B$ such that $\varphi(x) = \sum_n \varphi(x)\psi_n$ and $\sum_n \|\varphi_n^*\|\|\psi_n\|$ is finite. If φ is normal on a von Neumann algebra A, we may assume that φ_n are normal as well, by decomposing φ_n into its normal and singular parts, then the sum of the singular parts is singular, hence the sum must be 0.

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras, and Ω be cyclic and separating for $\mathcal{M}$. Let Δ be the modular operator for $\mathcal{M}$ with respect to Ω. We say that $(\mathcal{N} \subset \mathcal{M}, \Omega)$ satisfies the **modular nuclearity** if the map

$$\mathcal{N} \ni x \longmapsto \Delta^{\frac{1}{4}} x\Omega$$

is nuclear.

We say that an inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ is **split** if there is a type I factor $\mathcal{R}$ such that $\mathcal{N} \subset \mathcal{R} \subset \mathcal{M}$.

Lemma 5.5. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of type III factors satisfying modular nuclearity with respect to Ω, and assume that Ω is cyclic for $\mathcal{N}$ as well. Then $\mathcal{N} \subset \mathcal{M}$ is split.*

Proof. Denote by Δ the modular operator for $\mathcal{M}$ with respect to Ω and define, for $x \in \mathcal{N}$, $\Xi(x) = \Delta^{\frac{1}{4}} x\Omega$. By assumption and the remark above, there are $\varphi_n \in \mathcal{N}_*, \xi_n$ such that $\Xi(x) = \sum_n \varphi_n(x)\xi_n$. We prove that there is a unitary $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $\mathrm{Ad}\, U(\mathcal{N}) = \mathcal{N} \otimes \mathbb{C}\mathbb{1}, \mathrm{Ad}\, U(\mathcal{M}') = \mathbb{C}\mathbb{1} \otimes \mathcal{M}'$.

Note that the map $\mathcal{M}' \ni y \mapsto \Delta^{\frac{1}{4}} Jy\Omega \in \mathcal{H}$ is bounded. Indeed,

$$\|\Delta^{\frac{1}{4}} Jy\Omega\|^2 = |\langle \Delta^{\frac{1}{2}} Jy\Omega, Jy\Omega\rangle| \le \langle y^*\Omega, Jy\Omega\rangle \le \|y\|^2.$$

Then it follows that its dual $\Upsilon : \mathcal{H} \ni \eta \mapsto \langle \Delta^{\frac{1}{4}} J \cdot \Omega, \eta\rangle$ is bounded as well, hence is an element of $(\mathcal{M}')^*$.

Therefore, the map $\mathcal{N} \ni x \mapsto \langle x^*\Omega, \cdot\, \Omega\rangle = \langle J\Delta^{\frac{1}{2}} x\Omega, \cdot\, \Omega\rangle = \langle \Delta^{\frac{1}{4}} J \cdot \Omega, \Delta^{\frac{1}{4}} x\Omega\rangle \in (\mathcal{M}')_*$ is nuclear, as it is $\Upsilon \circ \Xi(x) : \mathcal{N} \ni x \mapsto \langle \Delta^{\frac{1}{4}} J \cdot \Omega, \Delta^{\frac{1}{4}} x\Omega\rangle$ of a nuclear map and a bounded map. This means that there are sequences $\varphi_n \in \mathcal{N}_*, \psi_n \in (\mathcal{M}')_*$ such that for any $x \in \mathcal{N}, y \in \mathcal{M}'$ we have

$$\langle x^*\Omega, y\Omega\rangle = \langle \Omega, xy\Omega\rangle = \sum_n \varphi_n(x)\psi_n(y).$$

This shows that the state on $\mathcal{N} \otimes_{\mathrm{alg}} \mathcal{M}' \ni \sum_n x_n \otimes y_n$ (finite sum) given by $\omega = \langle \Omega, \sum_n x_n y_n \Omega\rangle$ extends to $\mathcal{N} \otimes \mathcal{M}'$. As Ω is cyclic for $\mathcal{N}\mathcal{M}'$, the GNS representation of $\mathcal{N} \otimes \mathcal{M}'$ with respect to ω is realized by $x \otimes y \longmapsto xy$. By assumption, $\mathcal{N} \otimes \mathcal{M}'$ is a type III factor, therefore, this GNS representation is actually a unitary equivalence: there is a unitary $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that $U^*(x \otimes y)U = xy$. That is, $\mathrm{Ad}\, U^*$ maps $\mathcal{N} \otimes \mathbb{C}\mathbb{1}$ to $\mathcal{N}$ and $\mathbb{C}\mathbb{1} \otimes \mathcal{M}'$ to $\mathcal{M}'$. Now we can take $\mathcal{R}$ as the image of $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}\mathbb{1}$, which satisfies

$$\begin{aligned}\mathcal{N} &= \mathrm{Ad}\, U^*(\mathcal{N} \otimes \mathbb{C}\mathbb{1}) \subset \mathrm{Ad}\, U^*(\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}\mathbb{1}) = \mathcal{R} \\ &\subset \mathrm{Ad}\, U^*(\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}) = \mathrm{Ad}\, U^*(\mathbb{C}\mathbb{1} \otimes \mathcal{M}')' = (\mathcal{M}')' = \mathcal{M},\end{aligned}$$

as desired. □

We say that a AHK net $(\mathcal{A}, U, \Omega)$ satisfies the **modular nuclearity for wedges** if the inclusion $\mathcal{A}(W_\mathrm{L} + a) \subset \mathcal{A}(W_\mathrm{L}), \Omega$ satisfy the modular nuclearity for some spacelike vector a. As we have seen, in an AHK net, the algebras for wedges $\mathcal{A}(W_\mathrm{L} + a)$ are type III_1 factors. If the inclusion $\mathcal{A}(W_\mathrm{L} + a) \subset \mathcal{A}(W_\mathrm{L})$ is split, then the relative commutant $\mathcal{A}(W_\mathrm{L}) \cap \mathcal{A}(W_\mathrm{L} + a)'$ cannot be trivial, and actually it is a type III_1 factor, because it is unitarily equivalent to $\mathcal{A}(W_\mathrm{L})' \otimes \mathcal{A}(W_\mathrm{L} + a)$. Actually, one can even show [Lec06] that Ω is cyclic for it.

In the next Section, we will show that the modular nuclearity holds if $\mathcal{A}$ is constructed throught the field ϕ^-.

If we define $\mathcal{A}(W_\mathrm{L} \cap (W_\mathrm{L} + a)) = \mathcal{A}(W_\mathrm{L}) \cap \mathcal{A}(W_\mathrm{L} + a)'$, it is easy to show that it satisfies the AHK axioms. This is an *interacting* model, and indeed, one can calculate the S-matrix, an invariant of AHK nets, and it is nontrivial.

6 Lecture 6 (5/19). Modular nuclearity for the free fermion

6.1 Free fermion

We follow [Lec06]. Let us recall the construction of the free field. It is based on the one-particle space $\mathcal{H}_m = L^2(\mathbb{R}, d\theta)$, where we have a natural representation U_m of $\mathcal{P}_+^\uparrow$. In order to constuct fermionic field, we take the antisymmetrization (instead of symmetrization):

$$\pi_n^-(\sigma)\Psi_1 \otimes \cdots \otimes \Psi_n = \mathrm{sgn}(\sigma)\Psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \Psi_{\sigma^{-1}(n)}$$
$$P_n^- = \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \pi_n^-(\sigma)$$

and we construct the fermionic Fock space $\mathcal{F}^-(\mathcal{H}_m) = \bigoplus_{n\in\mathbb{N}} P_n^- \mathcal{H}_m^{\otimes n}$, the creation and annihilation operators $a^\dagger, a$. In the fermionic case, we have the anticommutation relations

$$\{a^\dagger(\xi), a^\dagger(\eta)\} = 0, \{a(\xi), a^\dagger(\eta)\} = \langle \xi, \eta\rangle.$$

In particular, it holds that

$$\langle \xi, \xi\rangle = \{a(\xi), a^\dagger(\xi)\} = (a(\xi)^\dagger)^* a^\dagger(\xi) + a^\dagger(\xi)(a(\xi)^\dagger)^* \geq (a^\dagger(\xi))^* a^\dagger(\xi)$$

hence $a^\dagger(\xi)$ is bounded, and there is no question of domains.

Exercise: Prove these relations.

In this situation, we can construct the "right field" ϕ' as follows. Let $J = \bigoplus_n J_0^{\otimes n}\pi_n^+(\iota_n)$, where $\iota_n : (12\cdots n) \mapsto (n\cdots 21)$ and $\phi'(f) = J\phi(f_j)J$, where $f_j(x) = f(-x)$ for f real. It turns out that $[\phi(f), \phi'(g)] = 0$ if $\mathrm{supp}\, f \subset W_\mathrm{L}, \mathrm{supp}\, g \subset W_\mathrm{R}$. One can show that Ω is cyclic for $\mathcal{A}(W_\mathrm{L})$ and also $\mathcal{A}'(W_\mathrm{R}) := \{e^{i\phi'(f)} : \mathrm{supp}\, f \subset W_\mathrm{R}\}''$. Altogether, Ω is cyclic and separating for $\mathcal{A}(W_\mathrm{L})$. One can prove the Bisognano-Wichmann property for $\mathcal{A}(W_\mathrm{L})$ as well.

Exercise: Based on these definitions, prove $[\phi(f), \phi'(g)] = 0$ for $\mathrm{supp}\, f \subset W_\mathrm{L}, \mathrm{supp}\, g \subset W_\mathrm{R}$.

Note that, if $a \in W_\mathrm{L}$, then we have $\mathcal{A}(W_\mathrm{L} + a) \subset \mathcal{A}(W_\mathrm{L})$. Then the natural question arises whether this inclusion is split, and we can try to prove it by modular nuclearity. This will turn to be the case, and as we remarked at the end of the last section, this gives an interecting AHK net [Lec06].

6.2 General lemmas for modular nuclearity

Let $\mathcal{K}$ be a Hilbert space, $\mathcal{L}_\varphi, \mathcal{L}_\pi$ be complex subspaces of $\mathcal{K}$, $Z = Z^* = Z^{-1}$ be a unitary conjugation on $\mathcal{K}$ such that $Z\mathcal{L}_\varphi = \mathcal{L}_\varphi, Z\mathcal{L}_\pi = \mathcal{L}_\pi$. We put $\mathcal{L} := (\mathbb{1} + Z)\mathcal{L}_\varphi + (\mathbb{1} - Z)\mathcal{L}_\pi$, a real subspace of $\mathcal{K}$. Furthermore, $\mathcal{L}' = \{\xi \in \mathcal{K} : \langle \xi, \eta\rangle = \langle \eta, \xi\rangle, \eta \in \mathcal{L}\}$, the symplectic complement. Let E_φ, E_π be the orthogonal projections onto $\mathcal{L}_\varphi, \mathcal{L}_\pi$, respectively.

Furthermore, we introduce

$$\begin{aligned}\phi(\xi) &:= a^*(\xi) + a(\xi), \quad \xi \in \mathcal{L},\\ \varphi(\xi) &:= a^*(\xi) + a(Z\xi), \quad \xi \in \mathcal{K},\\ \pi(\xi) &:= i(a^*(\xi) - a(Z\xi)), \quad \xi \in \mathcal{K}.\end{aligned}$$

Note that $\varphi(\xi)$ and $\pi(\eta)$ anticommute for all ξ, η, because $\langle Z\xi, \eta\rangle = \langle Z\eta, \xi\rangle$.

Let $\mathcal{N} = \{\phi(\xi) : \xi \in \mathcal{L}\}''$. Assume that X is a densely defined, positive invertible closed operator on $\mathbb{C}\mathcal{L}$ such that $[X, Z] = 0$ and $\mathcal{N}\Omega \subset \mathrm{Dom}(\Gamma(X))$. Our goal is to show that $\Xi : \mathcal{N} \ni x \mapsto \Gamma(X)x\Omega$ is nuclear under certain assumptions on X.

Consider $\mathscr{P}(\mathcal{L}) = \mathrm{span}\{\phi(\xi_1)\cdots\phi(\xi_n) : n \in \mathbb{N}, \xi_j \in \mathcal{L}\}$. This is SOT dense in $\mathcal{N}$. Furthermore, $\mathscr{P}(\mathcal{L})^\pm$ be the even and odd operators. For any $A \in \mathscr{P}(\mathcal{L})$, we have the decomposition $A = A^+ + A^-, A^\pm \in \mathscr{P}^\pm(\mathcal{L})$. We introduce $\gamma(A) = A^+ - A^-$, then γ is implemented by the unitary $\Gamma(-1) = \bigoplus_n (-1)^n$, therefore, $\|\gamma\| = 1$ and $\|A^\pm\| = \frac{1}{2}\|A + \gamma(A)\| \le \|A\|$.

We denote $\{A, B\} = [A, B]_+, [A, B] = [A, B]_-$. For $\xi \in \mathcal{K}$, we define

$$\begin{aligned}\delta_\xi^\pm(A) = \frac{1}{2}&[\varphi((\mathbb{1} \mp Z)\xi) + i\pi((\mathbb{1} \pm Z)\xi), A^+]_-\\ &+ \frac{1}{2}[\varphi((\mathbb{1} \mp Z)\xi) + i\pi((\mathbb{1} \pm Z)\xi), A^-]_+\end{aligned}$$

Lemma 6.1. *$\delta_\xi^\pm$ are odd derivations of $\mathcal{P}(\mathcal{L})$, that is, $\delta_\xi^+(A^\pm B) = \delta_\xi^+(A^\pm)B \pm A^\pm\delta_\xi^+(B)$ and $\delta_\xi^-(A^\pm B) = \delta_\xi^-(A^\pm)B \pm A^\pm\delta_\xi^-(B)$ with $\delta_\xi^\pm(A) \in \mathcal{P}(\mathcal{L})$, real linear in ξ and satisfy the following bounds*

$$\begin{aligned}\|\delta_\xi^+(A^\pm)\| &\le (\|(\mathbb{1} - Z)E_\varphi\xi\|^2 + \|(\mathbb{1} + Z)E_\pi\xi\|^2)^{\frac{1}{2}}\|A^\pm\|,\\ \|\delta_\xi^-(A^\pm)\| &\le (\|(\mathbb{1} + Z)E_\varphi\xi\|^2 + \|(\mathbb{1} - Z)E_\pi\xi\|^2)^{\frac{1}{2}}\|A^\pm\|.\end{aligned}$$

If $\xi \in \mathcal{L}'$, then $\delta_\xi^+ = 0, \delta_{i\xi}^- = 0$.

Proof. We leave the equalities as exercises, except that

$$\begin{aligned}\delta_\xi^\pm(\phi(\eta)) &= \frac{1}{2}[\varphi((\mathbb{1} \mp Z)\xi) + i\pi((\mathbb{1} \pm Z)\xi), \phi(\eta)]_+\\ &= \frac{1}{2}(\langle\eta, (\mathbb{1} \mp Z)\xi\rangle \mp \langle(\mathbb{1} \mp Z)\xi, \eta\rangle - \langle\eta, (\mathbb{1} \pm Z)\xi\rangle \pm \langle(\mathbb{1} \pm Z)\xi, \eta\rangle)\mathbb{1}\\ &= (\langle Z\xi, \eta\rangle \mp \langle\eta, Z\xi\rangle)\mathbb{1}.\end{aligned}$$

From this, it is straightforwad to show that they are derivations, and the last equalities because $Z\mathcal{L} = \mathcal{L}$.

As for estimates, let $\xi \in \mathcal{K}$. Then

$$\xi' = \left(\frac{1}{2}(\mathbb{1} + Z)(\mathbb{1} - E_\pi) + \frac{1}{2}(\mathbb{1} - Z)(\mathbb{1} - E_\varphi)\right)\xi \in \mathcal{L}',$$

by using the assumpsions $Z\mathcal{L}_\varphi = \mathcal{L}_\varphi, Z\mathcal{L}_\pi = \mathcal{L}_\pi$ and the definition $\mathcal{L} = (1+Z)\mathcal{L}_\varphi + (1-Z)\mathcal{L}_\pi$.

Therefore, using

$$(\mathbb{1}-Z)(\xi-\xi') = (\mathbb{1}-Z)(\mathbb{1}-\tfrac{1}{2}((\mathbb{1}+Z)(\mathbb{1}-E_\pi)+(\mathbb{1}-Z)(\mathbb{1}-E_\varphi)))\xi = (\mathbb{1}-Z)E_\varphi\xi,$$
$$(\mathbb{1}+Z)(\xi-\xi') = (\mathbb{1}+Z)(\mathbb{1}-\tfrac{1}{2}((\mathbb{1}+Z)(\mathbb{1}-E_\pi)+(\mathbb{1}-Z)(\mathbb{1}-E_\varphi)))\xi = (\mathbb{1}+Z)E_\pi\xi,$$

we can estimate

$$\begin{aligned}\|\delta^+_\xi(A^\pm)\| &= \|\delta^+_{\xi-\xi'}(A^\pm)\| \\ &= \frac{1}{2}\|[\varphi((\mathbb{1}-Z)E_\varphi\xi)+i\pi((\mathbb{1}+Z)E_\pi)\xi, A^\pm]_\mp\| \\ &\le \|\varphi((\mathbb{1}-Z)E_\varphi\xi)+i\pi((\mathbb{1}+Z)E_\pi)\xi\|\|A^\pm\|.\end{aligned}$$

Furthermore, let $\chi_- = (\mathbb{1}-Z)E_\varphi\xi, \chi_+ = (\mathbb{1}+Z)E_\pi\xi$. It holds that $(\varphi(\chi_-)+i\pi(\chi_+))^* = -(\varphi_(\chi_-)+i\pi(\chi_+))$ and $\varphi(\chi_-)$ anticommutes with $\pi(\chi_+)$. Therefore,

$$\|\varphi(\chi_-)+i\pi(\chi_+)\|^2 = \|\varphi(\chi_-)^2-\pi(\chi_+)^2\| = \|\chi_-\|^2+\|\chi_+\|^2.$$

This implies the claimed norm bounds. As for δ^-_ξ, one should use

$$\xi'' = \left(\frac{1}{2}(\mathbb{1}-Z)(\mathbb{1}-E_\pi)+\frac{1}{2}(\mathbb{1}+Z)(\mathbb{1}-E_\varphi)\right)\xi \in i\mathcal{L}'.$$

□

The last paragraph of the following is a variation of [BJ87, Theorem 2.1]

Lemma 6.2. *Assume that $E_\varphi X$ and $E_\pi X$ extend to trace class operators. Then Ξ is nuclear.*

Proof. Let $\{\eta_n\}$ be an orthonormal basis of $\mathcal{F}^-(\mathcal{K})$. We may try to form the decomposition by

$$\Xi(x) = \Gamma(X)x\Omega = \sum\langle\eta_n, \Gamma(X)x\Omega\rangle\eta_n.$$

In order to prove nuclearity, we have to choose an appropriate basis $\{\eta_n\}$.

Note that

$$[a(XZ\xi), A^\pm]_\mp = [a(ZX\xi), A^\pm]_\mp = \frac{1}{2}[\varphi(X\xi)+i\pi(X\xi), A^\pm]_\mp = \frac{1}{2}(\delta^+_{X\xi}+\delta^-_{X\xi})(A^\pm).$$

As X is symmetric, it holds that $a(\eta)\Gamma(X)\Psi = \Gamma(X)a(X\eta)\Psi$. For vectors $\xi_1,\cdots\xi_n \in \mathrm{Dom}(X)$, we have

$$\begin{aligned}\langle a^\dagger(Z\xi_1)\cdots a^\dagger(Z\xi_n)\Omega, \Gamma(X)A^\pm\Omega\rangle &= \langle\Omega, a(XZ\xi_n)\cdots a(XZ\xi_1)A^\pm\Omega\rangle \\ &= 2^{-n}\langle\Omega, (\delta^+_{X\xi_n}+\delta^-_{X\xi_n})\cdots(\delta^+_{X\xi_1}+\delta^-_{X\xi_1})(A^\pm)\Omega\rangle.\end{aligned}$$

Therefore, with $T_\varphi = E_\varphi X, T_\pi = E_\pi X$,

$$\begin{aligned}&|\langle a^\dagger(Z\xi_1)\cdots a^\dagger(Z\xi_n)\Omega, \Gamma(X)A^\pm\Omega\rangle| \\ &\le \frac{1}{2^n}\|A^\pm\|\prod_{j=1}^n((\|(\mathbb{1}-Z)T_\varphi\xi_j\|^2+\|(\mathbb{1}+Z)T_\pi\xi_j\|^2)^{\frac{1}{2}}+(\|(\mathbb{1}+Z)T_\varphi\xi_j\|^2+\|(\mathbb{1}-Z)T_\pi\xi_j\|^2)^{\frac{1}{2}}\end{aligned}$$

By assumption, $T = (|T_\varphi|^2 + |T_\pi|^2)^{\frac{1}{2}}$ is of trace class and commutes with Z. By noting that

$$\|(\mathbb{1} - Z)T_\varphi\xi_j\|^2 + \|(\mathbb{1} + Z)T_\pi\xi_j^2\| \le \|(\mathbb{1} - Z)T\xi_j\|^2 + \|(\mathbb{1} + Z)T\xi_j^2\|) = 4\|T\xi_j\|^2,$$

we see that

$$\begin{aligned} &|\langle a^\dagger(Z\xi_1)\cdots a^\dagger(Z\xi_n)\Omega, \Gamma(X)A^\pm\Omega\rangle| \\ &\le 2^n\|A^\pm\|\prod_{j=1}^n \|T\xi_j\| \end{aligned}$$

This is true for any sequence $\xi_j \in \mathrm{Dom}(X)$, but actually the both sides are continuous in ξ_j, hence for arbitrary sequence $\xi_j \in \mathcal{K}$.

Now, as the basis for $\mathcal{K}$, we take the eigenvectors of T. As T commutes with Z, we can choose the basis in such a way that they are eigenvectors of Z with eigenvalue ± 1. Then the vectors of the form

$$a^\dagger(Z\xi_{k_1})\cdots a^\dagger(Z\xi_{k_n})\Omega = a^\dagger(\xi_{k_1})\cdots a^\dagger(\xi_{k_n})\Omega, \quad k_1 < k_2 < \cdots < k_n.$$

span $\mathcal{F}^-(\mathcal{K})$.

Now we calculate

$$\langle a^\dagger(Z\xi_{k_1})\cdots a^\dagger(Z\xi_{k_n})\Omega, \Gamma(X)A^\pm\Omega\rangle \le 2^n\|A^\pm\|\prod_{j=1}^n \|T\xi_j\| = \|A^\pm\|\prod_{j=1}^n 2t_{k_j},$$

therefore,

$$\|\Xi\|_1 \le \sum_{k_1<k_2<\cdots<k_n} 2\prod_{j=1}^n 2t_{k_j} = 2\det(\mathbb{1} + 2T) \le 2e^{2\|T\|_1}.$$

Finally, we show that the same estimates holds for $A \in \mathcal{N}$. By Kaplansky's density theorem, there are $A_n \in \mathcal{P}(\mathcal{L})$, $A_n \to A$ in SOT with $\|A_n\| \le \|A\|$. Then $A_n\Omega \to A\Omega$ and $\Gamma(X)A_n\Omega$ is weakly bounded because for any $\eta \in \mathcal{K}$ and the basis $\{\eta_n\}$ constructed above, we have

$$|\langle \eta, \Gamma(X)A_n\Omega\rangle| \le \sum_m |\langle \eta, \eta_m\rangle||\langle \eta_m, \Gamma(X)A_n\Omega\rangle| \le \|A_n\||\eta|\sum_n \|\Xi\|_1,$$

thus by the uniformly boundedness principle $\Gamma(X)A_n\Omega$ is bounded. We take a subsequence, which we denote again by A_n, such that $\Gamma(X)A_n\Omega$ is weakly convergent. In this case, as X is a closed operator, $(A_n\Omega, \Gamma(X)A_n\Omega) \in \mathcal{F}^-(\mathcal{K}) \oplus \mathcal{F}^-(\mathcal{K})$ has a limit in the graph, that is, the limit $A\Omega \in \mathrm{Dom}(\Gamma(X))$, and $\Gamma(X)A_n\Omega$ is weakly convergent, therefore, the bound $|\langle \eta_m, \Gamma(X)A_n\Omega\rangle| \le c_m\|A_n\|$ of the functionals remain the same. This implies that the nuclearity bound extends to $\mathcal{N}$. □

6.3 Modular nuclearity for wedges in the free fermion net

Let us prepare a general lemma.

Lemma 6.3. *Let $a > 0, b \neq 0, b \in \mathbb{R}$. Then the integral operator $T_{a,b}$ on $L^2(\mathbb{R}, d\theta)$, defined by*

$$(T_{a,b}\Psi)(\theta) = \int \frac{e^{-a\cosh\theta}}{\theta - \theta' + ib}\Psi(\theta')d\theta'$$

is of trace class.

Proof. We may assume $b > 0$. By considering the Fourier transforms, we can write $T_{a,b} = R_a S_b$, where R_a, S_b are integral operators with the kernels

$$R_a(\theta, \theta') = e^{-a\cosh\theta}\frac{i + \theta'}{(i + \theta - \theta')^2},$$
$$S_b(\theta, \theta') = -i\frac{(i + \theta - \theta')^2}{i + \theta}\Theta(\theta - \theta')e^{-b(\theta-\theta')}.$$

Indeed, in terms of X the multiplication operator by θ and P the differential operator $i\frac{\partial}{\partial\theta}$,

$$R_a = e^{-a\cosh X}(i + P)^{-2}(i + X),$$
$$S_b = (i + X)^{-1}(i + P)^2\Theta(P)e^{-bP}.$$

Each of their kernels is L^2-functions, therefore, they define integral operators of trace class. □

We set $\mathbb{S}_{0,\pi} = \mathbb{R} + i(0, \pi)$ and

$$H^2(\mathbb{S}_{0,\pi}) = \{\Psi \in L^2(\mathbb{R}, d\theta) : \Psi \text{ continues analytically to } \mathbb{S}_{0,\pi}, \|\Psi(\cdot + i\lambda)\| \leq C\}.$$

Any $\Psi \in H^2(\mathbb{S}_{0,\pi})$ has the L^2 boundary values as $\lambda \to 0, \pi$ and for $\Psi \in H^2(\mathbb{S}_{0,\pi})$, we identify $\Psi(\theta)$ with $\lim_{\lambda\to 0}\Psi(\theta + i\lambda)$. In this sense, $(\Delta^{\frac{1}{2}}\Psi)(\theta) = \Psi(\theta + i\pi)$ and $\mathrm{Dom}(\Delta^{\frac{1}{2}}) = H^2(\mathbb{S}_{0,\pi})$ [Tan15].

For our concrete application, we take

- $\mathcal{K} = \mathcal{H}_m = L^2(\mathbb{R}, d\theta)$
- $\mathcal{L}_{\varphi,0} = \{\Psi \in H^2(\mathbb{S}_{0,\pi}) : \Psi(\theta + i\pi) = \Psi(-\theta)\}$
- $\mathcal{L}_{\pi,0} = \{\Psi \in H^2(\mathbb{S}_{0,\pi}) : \Psi(\theta + i\pi) = -\Psi(-\theta)\}$
- $\mathcal{L}_\varphi = U((0, -1), 0)\mathcal{L}_{\varphi,0}$
- $\mathcal{L}_\pi = U((0, -1), 0)\mathcal{L}_{\pi,0}$
- $X = \Delta^{\frac{1}{2}}$.
- $(Z\Psi)(\theta) = \overline{\Psi(-\theta)}$

Then $\mathcal{L} = (\mathbb{1}+Z)\mathcal{L}_\varphi+(\mathbb{1}-Z)\mathcal{L}_\pi$ contains f^+ for $\operatorname{supp} f \subset W_{\mathrm{L}}$. Indeed, for any $\Psi \in \mathcal{L}$, it holds that $\Psi(\theta+i\pi) = \overline{\Psi(-\theta)}$ and conversely, if Ψ satisfies this condition, then so does $Z\Psi$ and with $\Psi = \Psi_+ + \Psi_-$ where $Z\Psi_\pm = \pm\Psi_\pm$, $\Psi_+ \in \mathcal{L}_\varphi, \Psi_- \in \mathcal{L}_\pi$ and $\Psi = \frac{1}{2}((\mathbb{1}+Z)\Psi_+ + (\mathbb{1}-Z)\Psi_-)$.

Recall that, if $\operatorname{supp} f \subset W_{\mathrm{L}}$, then $f^+ \in \mathcal{L}$. Therefore, $\mathcal{A}(W_{\mathrm{L}}) \subset \mathcal{A}(\mathcal{L})$, and it is enough to show the nuclearity for $U((1,-1),0)\mathcal{A}(\mathcal{L}) \supset \mathcal{A}(W_{\mathrm{L}} + (1,-1))$ with the operator $X = \Delta^{\frac{1}{4}}$, where Δ is the modular operator of $\mathcal{A}(W_{\mathrm{L}})$.

On the other hand, we have $X\Psi(\theta) = \Delta^{\frac{1}{4}}\Psi(\theta) = \Psi(\theta + \frac{i\pi}{2})$. For $\Psi \in \mathcal{L}_{\varphi,0}$, we can write this as the Cauchy integral:

$$\begin{aligned}\Psi(\theta + \tfrac{i\pi}{2}) &= \frac{1}{2\pi i}\int d\theta' \left(\frac{\Psi(\theta')}{\theta' - \theta - \frac{i\pi}{2}} - \frac{\Psi(\theta' + i\pi)}{\theta' - \theta + \frac{i\pi}{2}}\right) \\ &= \frac{1}{2\pi i}\int d\theta' \left(\frac{1}{\theta' - \theta - \frac{i\pi}{2}} - \frac{1}{\theta' - \theta + \frac{i\pi}{2}}\right)\Psi(\theta')\end{aligned}$$

Let us put $V = U((1,-1),0)$, the multiplication by $e^{im\sinh\theta}$. If $V\Psi \in \mathcal{L}_\varphi$ with $\Psi \in \mathcal{L}_{\varphi,0}$,

$$(\Delta^{\frac{1}{4}}V\Psi)(\theta) = \frac{1}{2\pi i}\int d\theta' \left(\frac{e^{-m\cosh\theta}}{\theta' - \theta - \frac{i\pi}{2}} - \frac{e^{-m\cosh\theta}}{\theta' - \theta + \frac{i\pi}{2}}\right)\Psi(\theta')$$

and this integral operator is of trace class by the above Lemma. Similarly, $\Delta^{\frac{1}{4}}E_\pi$ is of trace class, as well. Altogether, we have the following.

Theorem 6.4. *The two-dimensional massive fermion satisfies the wedge split property:* $\mathcal{A}(W_{\mathrm{L}} + a) \subset \mathcal{A}(W_{\mathrm{L}})$ *for any* $a \in W_{\mathrm{L}}$ *is split*

Corollary 6.5. *By defining* $\mathcal{A}((W_{\mathrm{L}} + a) \cap W_{\mathrm{L}}) := \mathcal{A}(W_{\mathrm{L}} + a)' \cap \mathcal{A}(W_{\mathrm{L}})$ *and extending it by covariance, we obtain a AHK net* $(\mathcal{A}, U, \Omega)$.

This is called the massive Ising model.

6.4 More comments

6.4.1 Integrable models

More interacting model can be constructing by first specifying the algebras for wedges and then taking their intersections, as we did for the Ising model. See [Lec06] for those called "integrable models", and [Tan14] for some purely operator-algebraic construction. See [Lec15].

6.4.2 CFT

From two-dimensional net, one can restrict the AHK net to one of the lightrays and obtain a net associated with the intervals in the lightray. Such a net can be extended to S^1 and satisfy an analogue of the AHK axioms for S^1. They are called conformal nets on S^1. These nets have additional, diffeomorphism covariance and have very rich structures. See [Reh15].

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A Miscellany

A.1 GNS representation

A.1.1 $*$-algebras case

Let $\mathcal{A}$ be a $*$-algebra, ω a symmetric ($\omega(x^*) = \overline{\omega(x)}$) linear functional on $\mathcal{A}$ such that $\omega(x^*x) \geq 0$ for all $x \in \mathcal{A}$. We note that the GNS construction works for ω.

First, we have the Cauchy-Schwarz inequality: $|\omega(x^*y)| \leq \omega(x^*x)\omega(y^*y)$. There are two cases. Assume first that $\omega(y^*y) = 0$. Then, note that if $0 \leq \omega((x-\lambda y)^*(x-\lambda y)) = \omega(x^*x) - \bar{\lambda}\omega(y^*x) - \lambda\omega(x^*y) + |\lambda|^2\omega(y^*y)$ for all $\lambda \in \mathbb{C}$. In particular, for $\lambda \in \mathbb{R}$, we obtain $0 \leq \omega(x^*x) - 2\lambda\Re\omega(y^*x)$ and we must have $\Re\omega(y^*x) = 0$. With $\lambda \in i\mathbb{R}$, we have $0 \leq \omega(x^*x) + 2\lambda\Im\omega(y^*x)$ and we must have $\Im\omega(y^*x) = 0$. Altogether, $\omega(y^*x) = \omega(x^*y) = 0$ if $\omega(y^*y) = 0$. If $\omega(y^*y) \neq 0$, then

$$0 \leq \omega((x - \tfrac{\omega(y^*x)}{\omega(y^*y)}y)^*(x - \tfrac{\omega(y^*x)}{\omega(y^*y)}y)) = \omega(x^*x) - \tfrac{\omega(y^*x)\omega(x^*y)}{\omega(y^*y)},$$

from which it follows that $|\omega(y^*x)|^2 \leq \omega(x^*x)\omega(y^*y)$.

Next, $\mathcal{N} = \{x \in \mathcal{A} : \omega(x^*x) = 0\}$ is a left ideal of $\mathcal{A}$. Indeed, if $x \in \mathcal{N}$, then for any $y \in \mathcal{A}$ we have $\omega((yx)^*yx) = 0$ by the Cauchy-Schwarz inequality. Then we can define an inner product on $\mathcal{A}/\mathcal{N}$ by $\langle x, y\rangle = \omega(x^*y)$, where $\mathcal{A}/\mathcal{N}$ is the linear quotient space. This is well-defined, because $\omega((x+n)^*(y+m)) = \omega(x^*y) + \omega(n^*y) + \omega(x^*m) + \omega(n^*m) = \omega(x^*y)$, where we used the Cauchy-Schwarz inequality and $n, m \in \mathcal{N}$. By definition, $\langle\cdot,\cdot\rangle$ is positive-definite, therefore, is an innter product. We complete $\mathcal{A}/\mathcal{N}$ by the norm defined by $\langle\cdot,\cdot\rangle$ and denote it by $\mathcal{H}$. Denote by ι the natural map $\mathcal{A} \to \mathcal{A}/\mathcal{N} \to \mathcal{H}$.

Define, for $x \in \mathcal{A}$, an operator on $\mathcal{A}/\mathcal{N}$ by $\rho(x)\iota(y) = \iota(xy)$. It is immediate that this is well-defined. One can prove straightforwardly that ρ is a $*$-representation.

A.1.2 C^*-algebras case

Now we specialize to the case where $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. We denote $\overline{\mathcal{A}}$ the norm closure, which is a C^*-algebra. Assume further that ω extends to $\overline{\mathcal{A}}$ by norm. Then ω remains positive on $\overline{\mathcal{A}}$. Indeed, let $0 \leq a \in \overline{\mathcal{A}}$. Then we can write $a = x^*x, x \in \mathcal{A}$ and there is a sequence $x_n \in A, x_n \to A$. Then $a = \lim_n x_n^*x_n$ and $\omega(a) = \lim_n \omega(x_n^*x_n) \geq 0$.

This applies, in particular, if $\mathcal{N} \subset \mathcal{M}$ are two von Neumann algebras in $\mathcal{B}(\mathcal{H})$ and there is a linear functional on $\mathcal{N} \odot \mathcal{M}'$ given by $\omega(x \otimes y) = \langle\Omega, xy\Omega\rangle$. Then ω satisfies $\omega(z^*z) \geq 0$ for $z \in \mathcal{N} \odot \mathcal{M}'$.

If ω extends to $\mathcal{N}' \otimes \mathcal{M}$ by continuity, its extension is positive: it is positive on the norm closure by the previous paragraph, and remains positive by Kaplansky's density theorem. Therefore, one can apply the GNS construction. As the extension is normal, then the GNS representation is normal as well.

A.2 On analytic functions on the strip

Lemma A.1. *Let ξ be ananalytic function on $\mathbb{S}_{0,\pi}$ bounded by C_ξ, continuous on the closure and it is L^2 on the boundary. Then $\xi \in H^2(\mathbb{S}_{0,\pi})$.*

Proof. Let $\eta \in L^2(\mathbb{R}, d\theta)$, bounded by C with $\operatorname{supp}\eta \subset [-n, n]$. We define

$$\xi_\eta(z) = \int \xi(\theta + z)\eta(\theta)d\theta.$$

Then by Morera's theorem, $\xi_\eta(z)$ is analytic on $\mathbb{S}_{0,\pi}$, and continuous on the closure. As ξ is bounded and $\eta \in L^1(\mathbb{R}, d\theta)$ (because of the bound C and the support), ξ_η is bounded by $C_\xi\|\eta\|_{L^1}$. By the three line theorem [Rud87, Theorem 12.9], $|\xi_\eta(z)|$ is bounded by its values on the boundaries, and in this case by $\max\{\|\xi\|_{L^2}, \|\xi(\cdot + i\pi)\|_{L^2}\}\|\eta\|_{L^2}$. As such η span a dense subspace, $\|\xi(\cdot + i\zeta)\| \le \max\{\|\xi\|_{L^2}, \|\xi(\cdot + i\pi)\|_{L^2}\}$ for all $\zeta \in (0, \pi)$. □

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