Introduction to Algebraic Quantum Field Theory
Lecture 1/6
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Introduction

What is Quantum Field Theory (QFT)?

- QFT is a framework in theoretical physics used to describe particle physics, where there is particle production, and critical phenomena.
- "Standard model" of particle physics, Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD), the Yang-Mills theories are particlar examples of QFT.
- Very successful phenomenologically. Precise predictions in particle physics, universality in condensed matter.
- Some physicists say QFT is well-understood. Others say QFT is not yet defined. What do they mean?
- Can mathematics give answers?

Mathematical theory of physical systems

A physical theory formulates a problem in physics in terms of mathematical objects. Determining the evolution of the system amounts to predicting the change of the physical system.

- Classical mechanics: ODA. The Newton equation $m\frac{d^2}{dt^2}x(t) = -V'(x(t))$, where V is the potential energy.
- Quantum mechanics: Operators on a Hilbert space. Schrödinger equation $i\frac{\partial}{\partial t}\Psi(x,t) = [\frac{1}{2m}\hat{P}^2 + V(\hat{Q})]\Psi(x,t)$, $\hat{P} = i\frac{\partial}{\partial x}$, $\hat{Q} =$ multiplication by x. $[\hat{Q},\hat{P}] = i\mathbb{1}$.
- Classical field theory: PDA. $(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2)\phi(x) = -V'(\phi(x))$
- **Quantum field theory**: ??

Definition of Quantum field $\hat{\phi}(x)$? $\Rightarrow$ Axiomatic approaches (Wightman, Osterwalder-Schrader, Araki-Haag-Kastler...) + Constructive Quantum Field Theory (examples). (cf. **Yang-Mills theory**: a Millenium problem)

Mathematical theories of physical systems

Classical mechanics —"quantization"→ Quantum mechanics

Cannonballs, planets, stars, galaxies... | Atoms, molecules, metals...

Newton's equation of motion | The Schrödinger equation

$m\ddot{x} = F$ | $\frac{d}{dt}\psi = iH\psi$

Classical fields ···quantization?···→ Quantum field theory

Electromagnetism, gravitation... | Elementary particles

Field equations | Quantized field equations?

(e.g. ϕ^4 theory, $V(\phi(x)) = \phi(x)^4$) | ?

$$(\Box + m^2)\phi(x,t) = -\lambda\phi(x,t)^3$$

Quantum Mechanics

- Consider a single particle in the d-dimensional space.
- d degrees of freedom.
- Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^d)$.
- Operators: $Q_j = M_{x_j}$ (multiplication), $P_j = i\frac{\partial}{\partial x_j}$.
- Canonical commutation relations: $[Q_j, P_k] = i\delta_{j,k}$.
- Hamiltonian: $H = \sum_{j=1}^{d} \frac{1}{2m} P_j^2 + V(Q_1, \cdots, Q_d)$, a self-adjoint operator on $\mathcal{H}$.

- Problems: the spectrum of H, the time evolution e^{itH}, asymptotic behaviors (stability of matter, scattering theory...).

Quantum Field Theory

- Classically, one considers functions $\phi(x)$ of the space(time).
- Each point x has a degree of freedom.
- Canonical commutation relations: $[\phi(x), \pi(y)] = i\delta(x-y)$?
- $\phi(x), \pi(y)$ should be operator-valued distributions.

- Hamiltonian: $H \stackrel{?}{=} \int \frac{1}{2}(\pi(y)^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2 + V(\phi(x))dx$
- Need "renormalization".

- **Ultraviolet problem**: more difficult in higher dimensions, higher powers in V.
- **Infrared problem**: integral over $\mathbb{R}^d$.

- Some solutions in $1+d = 2, 3$. Most important $1+d = 4$ is open (cf. "triviality").

Wightman axioms

What should a quantum field theory be?

- $\phi(x)$: operator-valued distribution of $\mathbb{R}^{1+d}$.
- $\phi(f)$ is an unbounded operator on a dense domain in a Hilbert space $\mathcal{H}$.
- U: a unitary represenetation of the spacetime symmetry group $\mathcal{P}_+^\uparrow$ (the Poincaré group).
- Ω: the vacuum vector in $\mathcal{H}$.

Wightman axioms

- Covariance: $U(g)\phi(x)U(g)^* = \phi(g \cdot x)$ for $g \in \mathcal{P}_+^\uparrow$.
- Locality (Einstein causality): $[\phi(x), \phi(y)] = 0$ if x, y are spacelike separated.
- Positivity of energy (a spectrum condition on U).
- Properties of the vacuum: $U(g)\Omega = \Omega, \phi(f_1) \cdots \phi(f_n)\Omega$ span $\mathcal{H}$.

Examples of Wightman fields

See Summars' review arXiv:1203.3991

- Free fields in any dimension $1 + d$.
- The $\mathscr{P}(\phi)_2$-models in $1 + 1$ dimensions (Glimm and Jaffe, '72).
- Exponential interactions, the Yukawa model, the Federbush model...
- Euclidean method: construct first a probability theory on $\mathbb{R}^d$ satisfying Reflection Positivity, then reconstruct a QFT by Wick rotation (Osterwalder and Schrader '75).
- The Gross-Neveu model, the sine-Gordon model ($1 + 2$ dimensions), the ϕ^4_3-model ($1 + 2$ dimensions), Abelian Higgs ($\mathrm{U}(1)$-gauge) models ($1 + 1$ and $1 + 2$ dimensions, except uniqueness of the vacuum)...
- Some $d = 1 + 1$ conformal field theories (CFTs, Vertex Operator Algebras), (Carpi-Kawahigashi-Longo-Weiner '18, Raymond-T.-Tener '22 for chiral CFT (unitary vertex operator algebras), Adamo-Giorgetti-T. '23 for some full CFT).

$1+3$ dimensions, triviality

- There is **no known** Wightman fields except the free fields in $1+3$ or higher dimensions.
- The higher the dimensions are, the severer the Ultraviolet (UV) divergence is.
- Triviality of the ϕ^4_4-model by Aizenman and Duminil-Copin '21 (very roughly, one cannot construct interacting scalar fields through lattice approximation in $1+3$ dimensions).
- Yang-Mills theory (one of the Millenium problems)? Quantum Electrodynamics? The standard model?

Current status of QFT

- Physicists often say that the **definition** of QFT is missing...
- Actually, the axioms are there (and are very weak, cf. conformal bootstrap).
- What is missing are **examples in** $1+3$ **dimensions**.

- Apart from constructing examples, one can study some other aspects of QFT (perturbation theory, the formal power series, topological/differential-geometrical properties of the classical theory...)

QFT/operator algebras-related Fields medals

- 1982 Connes (classification of type III factors)
- 1990 Jones (knot invariants and von Neumann algebras)
- 1998 Borcherds (vertex operator algebras, the moonshine conjecture)
- 2006 Werner (critical exponents or two-dimensional percolation)
- 2010 Smirnov (conformal invariance of the planar Ising model)
- 2014 Hairer (stochastic PDE, related with the ϕ^4_3-model)
- 2022 Duminil-Copin (triviality of the ϕ^4_4-model)

- 20?? Construction of the quantum Yang-Mills theory
- 20?? Construction of the standard model

von Neumann algebras

- An operator algebra (C^*, von Neumann) is a $*$-subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$, closed under a topology (norm, weak operator topology).
- If a C^*-algebra $\mathcal{A}$ is commutative, it is isomorphic to $C(X)$ for some topological space X. If a von Neumann algebra $\mathcal{M}$ is commutative, it is isomorphic to $L^\infty(X, d\mu)$ for some measurable space X and a measure μ.
- A noncommutative operator algebra should be regarded as noncommutative geometry/probability theory.
- From a $*$-closed subset S of $\mathcal{B}(\mathcal{H})$, one can take the smallest C^*-/von Neumann algebra S'' containing S.
- Main problems in operator algebras: classification, subalgebras/extensions, (quantum) group actions, representations...

Algebraic QFT

Axioms for QFT in terms of von Neumann algebras (the **Araki-Haag-Kastler axioms**): a family of von Neumann algebras $\{\mathcal{A}(O)\}$, a unitary representation U of the Poincaré group and a vacuum $\Omega \in \mathcal{H}$ satisfying

- Isotony: If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- Locality: If O_1 and O_2 are spacelike separated, then $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$.
- Covariance: $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(g \cdot O)$.
- Positive energy: The spectrum of $U|_{\mathbb{R}^{1+d}}$ is contained in the future lightcone.
- Vacuum: Ω is unique s.t. $U(g)\Omega = \Omega$ and $\bigcup_O \mathcal{A}(O)\Omega$ spans $\mathcal{H}$.
- Weak additivity: If $O \subset \bigcup_j O_j$, then $\mathcal{A}(O) \subset (\bigcup_j \mathcal{A}(O_j))''$.

Wightman + $\alpha \Rightarrow$ Algebraic QFT

- Assume that a Wightman field theory (ϕ, U, Ω) satisfies a technical condition ("linear energy bounds"). For spacetime regions O, define $\mathcal{A}(O) = \{e^{i\phi(f)} : \mathrm{supp}\, f \subset O\}''$.
- Here, for a self-adjoint set M of bounded operators, M' is called the **commutant** of M and it is the set of all bounded operators on $\mathcal{H}$ commuting with all elements of M. M'' is the double commutant, and is the smallest von Neumann algebra including M.
- Then $(\mathcal{A}, U, \Omega)$ satisfy the AHK axioms.
- **Examples**: the $\mathscr{P}(\phi)_2$-models, the Yukawa$_2$ model, the ϕ^4_3-model, CFT$_2$...
- Probably all the other Wightman theories can be associated with AHK theories.

Why AQFT?

Powerful tools of von Neumann algebras (the Tomita-Takesaki modular theory, the nuclearity conditions, the subfactor theory) can be applied to obtain

- Representation theory (states $+$ the GNS construction).
- Extension/classifications of a class of CFT.
- Defining relative entropy/mutual information in QFT.
- Constructing new examples (in $1+1$ dimensions, until now...).
- Studying QFT on curved spacetimes.
- Defining and proving quantum energy inequalities.

Doplicher-Haag-Roberts (DHR) representation theory

- A **representation** of a AHK net $\mathcal{A}$ is a family of representations $\{\rho_O\}$ on a (possibly different) Hilbert space $\mathcal{H}_\rho$ with the compatibility condition

$$\rho_{O_1}|_{\mathcal{A}(O_2)} = \rho_{O_2} \quad \text{for } O_2 \subset O_1.$$

- Consider the class of DHR representation on $\mathcal{H}$ such that, for some O, $\rho_{\mathcal{A}(O')} = \mathrm{id}$. Assuming the Haag duality $\mathcal{A}(O')' = \mathcal{A}(O)$, $\rho_{\mathcal{A}(O)}$ is an **endomorphism** of $\mathcal{A}(O)$. $\Rightarrow$ such ρ's can be composed.
- Consider the category $\mathcal{C}$ of "nice" endomorphisms $\Rightarrow$. In $1+3$ dimensions (or higher), $\mathcal{C}$ is a symmetric tensor category.
- By a generalization of the Tannaka-Krein duality, $\mathcal{C}$ is isomorphic to the category of finite-dimensional representations of some compact group G.
- There is an extension $\mathcal{B}$ of $\mathcal{A}$ and an action of G such that $\mathcal{A}(O) = \mathcal{B}(O)^G$ (the Doplicher-Roberts reconstruction).

The classification theory of chiral CFT

- For chiral CFT (QFT on S^1), one can consider a similar representation theory.
- For a "completely rational" net $\mathcal{A}$, the category $\mathcal{C}$ is a modular tensor category.
- Assume that there is an extension $\mathcal{A} \subset \mathcal{B}$. Consider all representations ρ of $\mathcal{A}$ and make the "induced" representations $\hat{\rho}^{\pm}$, there are actually two such "solitonic" representations.
- The matrix $Z_{\rho,\sigma} = \dim(\hat{\rho}^{+}, \hat{\sigma}^{-})$ gives a modular invariant.
- A classification of extensions $\mathcal{A} \subset \mathcal{B}$ can be reduced to the classification of modular invariants.
- Carried out for the Virasoro nets $\mathrm{Vir}_c, c < 1$ (Kawahigashi-Longo '04).

The Tomita-Takesaki modular theory

- Let $\mathcal{M}$ be a von Neumann algebra, Ω be cyclic ($\overline{\mathcal{M}\Omega} = \mathcal{H}$) and separatiang (for $x \in \mathcal{M}, x \neq 0$, $x\Omega \neq 0$). The densely defined antilinear operator

$$\mathcal{M}\Omega \ni x\Omega \longmapsto x^*\Omega$$

  is closable. Its closure has the polar decomposition $S = J\Delta^{\frac{1}{2}}$.
- Δ is called the modular operator, J the modular conjugation.
- $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for $t \in \mathbb{R}$, $J\mathcal{M}J = \mathcal{M}'$ (Tomita).

- In a AHK net, take $\mathcal{M} = \mathcal{A}(W_{\mathrm{L}}), \Omega$ the vacuum vector. Then Δ^{it} are the Lorentz boosts, J the TCP operator (the Bisognano-Wichmann property).

Araki's relative entropy

- In Quantum Mechanics ($\mathcal{B}(\mathcal{H})$ for some finite-dimensional $\mathcal{H}$), any state σ is given by $x \mapsto \mathrm{tr}(xM_\sigma)$, where $M_\sigma \in \mathcal{B}(\mathcal{H})_+, \mathrm{tr}\, M_\sigma = 1$. For two states σ_1, σ_2, the relative entropy is given by $S(\sigma_1|\sigma_2) = \mathrm{tr}(\sigma_1(\log\sigma_1 - \log\sigma_2))$.
- Relative entropy in QFT plays important role in theoretical physics. But there is no tr for local algebras $\mathcal{A}(O)$ in a AHK net.
- Let Ω, Ψ be vectors cyclic and separating for $\mathcal{A}(O)$. Let $\omega = \langle\Omega, \cdot\Omega\rangle, \varphi = \langle\Phi, \cdot\Phi\rangle$. The relative modular objects is given by $S_{\omega,\varphi} = J_{\Omega,\varphi}\Delta^{\frac{1}{2}}_{\Omega,\varphi}$, where

$$S_{\omega,\varphi} x\Psi = x^*\Omega.$$

- The relative entropy of $\mathcal{M}$ with respect to ω, φ is

$$S(\omega,\varphi) = \langle\Omega, \log\Delta_{\omega,\varphi}\Omega\rangle.$$

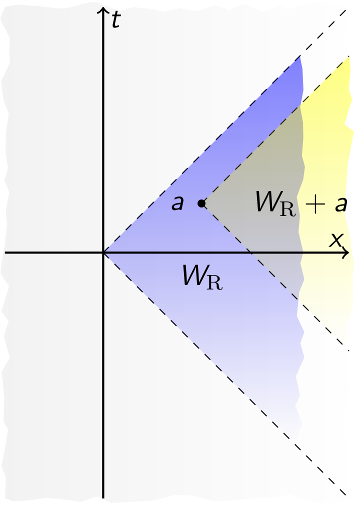
- In some models and some states, $S(\omega,\varphi)$ can be calculated.

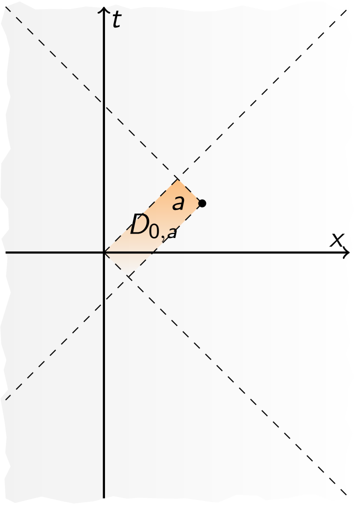
New examples

- Isotony: $O_1 \subset O_2 \Longrightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- **Larger** regions contain **more** observables.
- It is difficult to construct local interacting quantum fields $\phi(x)$.
- Maybe simpler operators in larger regions?
- **Wedge**: $W_{\mathrm{R}} := \{(t,x) : x > |t|\}$.

Construct first the algebras $\mathcal{A}(W_{\mathrm{R}})$ of observables in wedges. Local observables are obtained by $\mathcal{A}(O) = \mathcal{A}(W_{\mathrm{R}} + a) \cap \mathcal{A}(W_{\mathrm{L}} + b)$.

Standard wedge and double cone





New examples

- Fix a nice analytic function ("S-matrix") $S : \mathbb{R} + i(0, \pi) \to \mathbb{C}$,
- S-symmetric Fock space: $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$, $\mathcal{H}_n = P_n \mathcal{H}_1^{\otimes n}$, where P_n is the projection onto S-symmetric functions: $\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n)$.
- Zamolodchikov-Faddeev algebra: S-symmetrized creation and annihilation operators $z^\dagger(\xi) = Pa^\dagger(\xi)P, z(\xi) = Pa(\xi)P, P = \bigoplus_n P_n$.
- **Wedge-local field**: $\phi(f) = z^\dagger(f^+) + z(f^+)$.
- $\mathcal{A}(W_\mathrm{R}) = \overline{\{e^{i\phi(f)} : \mathrm{supp}\, f \subset W_\mathrm{R}\}}^{\mathrm{vN}}$.
- Need to prove that $\mathcal{A}(O) = \mathcal{A}(W_\mathrm{R} + a) \cap \mathcal{A}(W_\mathrm{L} + b)$ is large.
- This can be reduced to proving that $\mathcal{A}(W_\mathrm{R} + a) \subset \mathcal{A}(W_\mathrm{R})$ is **split**.
- The massive Ising model, Sinh-model with CDD factors (Lechner), more algebraic construction of the Federbush-like model (T.), the Bullough-Dodd model (Bostelmann-Cadamuro-T., in progress).

Summary

- Mathematical definitions of relataivistic quantum fields (Wightman) and algebras of observables (Araki-Haag-Kastler).
- Examples from Constructive QFT, 2d CFT
- Representation of AHK nets, extension and classification of 2d CFTs
- Tomita-Takesaki theory, relative entropy
- the wedge construction, new 2d massive examples

Overview of the lectures

- Lecture 2: basics
  - von Neumann algebras, the Tomita-Takesaki modular theory
  - the Minkowski space, the Poincaré group
- Lecture 3: the Araki-Haag-Kastler axioms
  - local net of von Neumann algebras
  - the commutator theorem, strong locality
  - consequences of the axioms (the Reeh-Schlieder property)
- Lecture 4: the free field net
  - the Fock space, field operators
  - construction of the free field nets
- Lecture 5: further properties of AHK nets
  - the Bisognano-Wichmann property
  - nuclearity conditions, the split property for the fermionic free field
  - wedge construction, twisting
- Lecture 6: modular nuclearity/advanced topics
  - proof of modular nuclearity for the fermionic free field
  - advanced topics