1 Taylor expansion and limit

Problem. For various $\alpha, \beta \in \mathbb{R}$, study the limit:

$$\lim_{x\to 1} \frac{\log x + \alpha(x-1)\sqrt{x} + \beta(x-1)}{\exp(2(x-1)^3) - 1},$$

and find α, β such that this converges, and calculate the limit.
Solution. As $x \to 1$, as $|\exp(2(x-1)^3)|$ ends to 0. For the whole limit to converge, the numerator must also tend to 0, and we need to study the behaviours of the numerator and the denominator as $x \to 1$. For this purpose, we calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 3rd order, see below why the 3rd order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + o((x-a)^3) \text{ as } x \to a.$$

We take $a = 1$.

- Put $f(x) = \log x$. Then $f'(x) = \frac{1}{x}, f''(x) = -x^{-2}, f^{(3)}(x) = 2x^{-3}$. Applying the general Taylor formula with $a = 1$, we get $\log x = 0 + (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + o((x-1)^3) = 0 + (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + o((x-1)^3)$ as $x \to 1$.
- In general, if $g(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + o((x-1)^3)$, then we have $(x-1)g(x) = a_0(x-1) + a_1(x-1)^2 + a_2(x-1)^3 + o((x-1)^3)$, That is, the Taylor formula can be multiplied. This can simplify some calculations.
- $\sqrt{x} = 1 + \frac{1}{2}(x-1) + \frac{-1}{8}(x-1)^2 + o((x-1)^2)$ (because $(\sqrt{x})' = \frac{1}{2}x^{-\frac{1}{2}}, (\sqrt{x})?? = -\frac{1}{4}x^{-\frac{3}{2}}$)
- By applying the formula for product (see above), $(x-1)\sqrt{x} = (x-1) + \frac{1}{2}(x-1)^2 + \frac{-1}{8}(x-1)^3 + o((x-1)^3)$.
- As $\exp(y) = y + o(y)$, we have $\exp(2(x-1)^3) = 2(x-1)^3 + o((x-1)^3)$.

Now the numerator is

$$\begin{aligned}
\log x &= 0 + (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + o((x-1)^3) \\
&\quad + \alpha((x-1) + \frac{1}{2}(x-1)^2 + \frac{-1}{8}(x-1)^3 + o((x-1)^3)) + \beta(x-1) \\
&= (1+\alpha+\beta)(x-1) + \left(-\frac{1}{2} + \alpha\frac{1}{2}\right)(x-1)^2 + \left(\frac{1}{3} + \alpha\frac{-1}{8}\right)(x-1)^3 + o((x-1)^3)
\end{aligned}$$

To have a finite limit for $\lim_{x\to 1} \frac{\log x + \alpha(x-1)\sqrt{x} + \beta(x-1)}{\exp(2(x-1)^3) - 1}$, we must have $1 + \alpha + \beta = 0, -\frac{1}{2} + \alpha\frac{1}{2} = 0$, because otherwise the limit diverges. Therefore, $\alpha = 1, \beta = -2$, and the given limit is

$$\lim_{x\to 1} \frac{\frac{5}{24}(x-1)^3 + o((x-1)^3)}{2(x-1)^3 + o((x-1)^3)} = \lim_{x\to 1} \frac{\frac{5}{24} + \frac{o((x-1)^3}{(x-1)^3})}{2 + \frac{o((x-1)^3}{(x-1)^3}} = \frac{5}{48}$$

Note: $\lim_{x\to 1} \frac{a}{(x-1)^3}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x\to 1} \frac{a + b(x-1)^2}{(x-1)^3}$ converges if and only if $a = b = 0$ (otherwise diverges).

The symbol $g(x) = o((x-1)^3)$ means that $\lim_{x\to 1} \frac{g(x)}{(x-1)^3} = 0$. In particular, we can calculate $\lim_{x\to 1} \frac{a(x-1)^3 + o((x-1)^3)}{(x-1)^3} = \lim_{x\to 1} \frac{a + \frac{o((x-1)^3)}{(x-1)^3}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = i$ in $\sum_{n=0}^{2} \frac{n}{(1+x)^{2n}}$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{n}{(1+x)^{2n}}$, with various $x \in \mathbb{R}, x \neq -1$.

Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. Recall that $a^0 = 1$ for all $a \in \mathbb{C}$ by convention. In the case at hand with $x = i$, $(i+1)^0 = 1, (i+1)^2 = 2i$, thus

$$\begin{aligned}
&\sum_{n=0}^{2} \frac{n}{(1+x)^{2n}} \\
&= \frac{0}{(1+i)^0} + \frac{1}{(1+i)^2} + \frac{2}{(1+i)^4} \\
&= 0 + \frac{1}{2i} + \frac{2}{-4} = -\frac{1}{2} - \frac{1}{2}i
\end{aligned}$$

As for the convergence, we use the ratio test. The ratio test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the ratio test to our case, for $x \in \mathbb{R}, x \neq -1$, we set $a_n = \frac{n}{(1+x)^{2n}}$ (need to take the absolute value), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{n+1}{(1+x)^{2(n+1)}}}{\frac{n}{(1+x)^{2n}}} \\
&= \lim_{n\to\infty} \frac{n+1}{n(1+x)^2} \\
&= \frac{1}{|x+1|^2}
\end{aligned}$$

Therefore, the ratio test tells that, if $\frac{1}{|x+1|^2} < 1$, the series $\sum_{n=0}^{\infty} \frac{n}{(1+x)^{2n}}$ converges, and as $\frac{n}{(1+x)^{2n}} > 0$, it converges absolutely. The condition is equivalent to $1 < |x+1|^2$, that is, $1 < x+1$ or $x+1 < 1$. This is equivalent to or $0 < x$ or $x < -2$.

For any specific value of x, one has to consider whether $-2 < x < 0$ or not. If $x = -2$, the ratio test does give answer, but the series becomes $\sum_{n=0}^{\infty} \frac{n}{(-1)^{2n}} = \sum_{n=0}^{\infty} n$ and this is divergent.

Problem. Calculate $\sum_{n=0}^{\infty} \frac{5}{3^n}$.

Solution. We have $\sum_{n=0}^{\infty} \frac{5}{3^n} = 5\sum_{n=0}^{\infty} (\frac{1}{3})^n$ and by the formula for geometric series, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$, thus $\sum_{n=0}^{\infty} \frac{5}{3^n} = 5 \cdot \frac{1}{1-\frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$.

Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \cdots$ $(N = 0, 1, 2, 3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \exp\left(\frac{1-x^3}{(x-3)^2}\right)$.
Solution.

- Domain. $\exp y$ is defined for every $y \in \mathbb{R}$. Moreover, we should have $x - 3 \neq 0$ to make sense of $\frac{1-x^3}{(x-3)^2}$. That is, $x \neq 3$. Altogether, the domain is $(-\infty, 3) \cup (3, \infty)$.

- Asymptotes.

  - Vertical asymptotes. As $x \to 3$, the denominator of $\frac{1-x^3}{(x-3)^2}$ tends to $+0$, while the numerator tends to -26, so the whole fraction tends to $-\infty$. Composed with exp, it tends to 0, and no divergence. So there is no vertical asymptote at $x = 3$. and thus $\log \frac{x^2+1}{x+2} \to \infty$. There is a vertical asysmptote at $x = -2$.
  - Horizontal asymptote. As $x \to \infty$, note that $\frac{1-x^3}{(x-3)^2} \to -\infty$, and composed with exp, it tends to 0. So $y = 0$ is a horizontal asymptote for $x \to \infty$. As $x \to -\infty$, note that $\frac{1-x^3}{(x-3)^2} \to \infty$, and compoese with exp, it tends to ∞. So there is no horizontal asymptote for $x \to -\infty$.
  - Oblique asymptote. As $x \to \infty$, we know that $y = 0$ is a horizontal asymptote, so there is no oblique asymptote. As $x \to -\infty$, we saw that $f(x)$ diverges exponentially, so there is no oblique asymptote for $x \to -\infty$.

- The derivative. We can use the chain rule: if $f(x) = g(h(x))$, then $f'(x) = h'(x)g'(h(x))$. In our case, $g(y) = \exp y$ and $h(x) = \frac{1-x^3}{(x-3)^2}$, $g'(y) = \exp y$, $h'(x) = \frac{-3x^2(x-3)^2-(1-x^3)\cdot 2(x-3)}{(x-3)^4} = \frac{-x^3+9x^2-2}{(x-3)^3}$, therefore, we get

$$f'(x) = \exp\left(\frac{1-x^3}{(x-3)^2}\right) \cdot \frac{-x^3+9x^2-2}{(x-3)^3}.$$

- In particular, $f'(1) = \frac{6}{-8} = -\frac{3}{4}$.

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $-x^3 + 9x^2 - 2 = 0$ and $x \neq 3$. The graph of $F(x) = -x^3 + 9x^2 - 2$ crosses the x-axis at least once. Furthermore, $F'(x) = -3x^2 + 18x = -3x(x-6)$ and $x = 0$ is a local minumum of F, while $x = 6$ is a local maximum of F. $F(0) = -2$, while $F(6) = 34$, so F must cross the x-axis 3 times. That is, there are 3 stationary points. (use that $\lim_{x\to-\infty} F(x) = \infty, \lim_{x\to\infty} F(x) = -\infty$ and the intermediate value theorem)

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  It is easy to see that $f'(0) > 0$ while $f'(-1) < 0$. By continuity of $f'(x)$, $f'(x) > 0$ around $x = 0$, so f is increasing there, while $f'(x) < 0$ around $x = -1$, so f is decreasing there.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_1^2 \frac{1}{3^x+2}dx$$

Solution. We change the variables by $3^x = t$, or equivalently, $x \log 3 = \log t$. From this we get $\frac{dx}{dt} = \frac{1}{t\log 3}$. and formally replace dx by $\frac{1}{t\log 3}dt$, therefore, with $3^1 = 3, 3^2 = 9$,

$$\begin{aligned}\int_1^2 \frac{1}{3^x+2}dx &= \int_3^9 \frac{1}{t\log 3(t+2)}dt \\ &= \int_3^9 \frac{1}{\log 3(t^2+2t)}dt.\end{aligned}$$

To carry out this last integral, we need to find the partial fractions: as $t^2+2t = t(t+2)$, we put $\frac{1}{t(t+2)} = \frac{A}{t} + \frac{B}{t+2} = \frac{A(t+2)+Bt}{t^2+2t}$, or $1 = (A+B)t + 2A$. By solving this, $B = -\frac{1}{2}, A = \frac{1}{2}$. Namely, $\frac{1}{t^2+2t} = \frac{1}{2}(\frac{1}{t} - \frac{1}{t+2})$. Altogether,

$$\begin{aligned}\int_1^2 \frac{1}{2^x+3+2(2^{-x})}dx &= \frac{1}{2\log 3}\int_3^9 \left(\frac{1}{t} - \frac{1}{t+2}\right)dt \\ &= \frac{1}{2\log 3}\left[\log t - \log(t+2)\right]_3^9 dt \\ &= \frac{1}{2\log 3}((\log 9 - \log 11) - (\log 3 - \log 5)) \\ &= \frac{\frac{45}{33}}{2\log 3} = \frac{\frac{\log 15}{\log 11}}{\log 9}.\end{aligned}$$

Note: other useful techniques are substitution (example: $\int xe^{x^2}dx$ by putting $t = x^2$) and integration by parts (example: $\int xe^x dx$ by noticing that $(e^x)' = e^x$).

5 Improper integral

Problem. Study the improper integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx.$$

Solution. The function $f(x) = \frac{x^2}{x^6+1}$ is bounded on $\mathbb{R}$, but the integration region $(-\infty, \infty)$ is infinite. Therefore, we need to take $\alpha < \beta$ and calculate the integral on $[\alpha, 0]$ and $[0, \beta]$. Noting that $(x^3)' = 3x^2$, thus by substitution,

$$\begin{aligned}\int_0^\beta \frac{x^2}{x^6+1} dx &= \frac{1}{3}\int_0^\beta \frac{3x^2}{x^6+1} dx \\ &= \frac{1}{3}[\arctan(x^3)]_0^\beta \\ &= \frac{1}{3}(\arctan(\beta^3) - \arctan 0] \\ &= \frac{1}{3}\left(\arctan(\beta^3)\right)\end{aligned}$$

As $\beta \to \infty$, this is convergent to $\frac{1}{3}(\frac{\pi}{2}) = \frac{\pi}{6}$. That is, $\int_0^\infty \frac{x^2}{x^6+1} dx = \lim_{\beta\to\infty}\int_0^\beta \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$.

Similarly, one has $\int_{-\infty}^0 \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$.

Altogether, the improper integral converges absolutely.

Problem. Determine for which $s > 0$ the integral $\int_0^\infty \frac{1}{(x^2+1)^s} dx$ converges.
Solution. The function $\frac{1}{(x^2+1)^s}$ is bounded around $x = 0$, so we only have to consider the integral as $x \to \infty$. Let us cut the integral at $x = 1$, thus $\int_1^\infty \frac{1}{(x^2+1)^s} dx$. Compare this with $\int_1^\infty \frac{1}{(x^2)^s} dx$. As $\lim_{x\to\infty} \frac{(x^2)^s}{(x^2+1)^s} = 1$, the original integral converge if and only if the latter converges. The latter is $\int_1^\infty \frac{1}{(x^2)^s} dx = \lim_{\beta\to\infty} \frac{1}{2s+1}[x^{-2s+1}]_1^\beta$ (if $s = \frac{1}{2}$, it is $\log x$) and this converges if and only if $-2s + 1 < 0$, or $\frac{1}{2} < s$.

Problem. Determine for which $s > 0$ the series $\sum_{n=1}^\infty \frac{1}{(n^2+1)^s} dx$ converges.
Solution. By the integral test, it converges $\frac{1}{2} < s$.

Problem. Consider the following three improper integrals and determine which is larger.

$$(1)\int_0^\infty x^{101}e^{-x}dx, \quad (2)\int_1^\infty x^{100}e^{-x}dx, \quad (3)\int_0^\infty e^{-x}dx, \quad (4)\int_0^\infty e^{-100x}dx.$$

Compare (1) and (2). The functions are positive, and for $x \geq 1$ we have $x^{101}e^{-x} > x^{100}e^{-x}$, so $\int_0^\infty x^{101}e^{-x}dx > \int_1^\infty x^{101}e^{-x}dx > \int_1^\infty x^{100}e^{-x}dx$.

Compare (3) and (4). As $e^{-x} > e^{-100x}$ for $x \geq 0$, we have $\int_0^\infty e^{-x}dx > \int_0^\infty e^{-100x}dx$.

Compare (2) and (3). We have $\int_0^\infty e^{-x}dx = \int_0^2 e^{-x}dx + \int_2^\infty e^{-x}dx$ and $\int_0^2 e^{-x}dx < 2$ because $e^{-x} < 1$. On the other hand, $\int_1^\infty x^{100}e^{-x}dx = \int_1^2 x^{100}e^{-x}dx + \int_2^\infty x^{100}e^{-x}dx$ and $\int_1^2 x^{100}e^{-x}dx > \int_{1.5}^2 x^{100}e^{-x}dx > 100e^{-2} > 10$ (because $(1.5)^{100} > 100$ and $e < 3, e^2 < 9 < 10$). Furthermore, $\int_2^\infty x^{100}e^{-x}dx > \int_2^\infty e^{-x}dx$. Thus altogether $\int_1^\infty x^{100}e^{-x}dx > \int_0^\infty e^{-x}dx$.
Note: An integral is improper if the interval is unbounded or the function is unbounded. In that case, we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^c f(x)dx + \lim_{\epsilon\to 0}\int_c^{b-\epsilon} f(x)dx,$$

where $c \in (a, b)$.