1 Taylor expansion and limit

Problem. For various $\alpha \in \mathbb{R}$, study the limit:

$$\lim_{x\to 0} \frac{2\cos x - 2 + x^2\sqrt{1+3x} + \alpha x^3}{(\log(1+x^2))^2}$$

and find α such that this converges, and calculate the limit.
Solution. As $x \to 0$, $\log(1+x^2) \to 0$, thus the denominator tends to 0. For the whole limit to converge, the numerator must also tend to 0, and we need to study the behaviours of the numerator and the denominator as $x \to 0$. For this purpose, we calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 4th order, see below why the 4th order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + o((x-a)^4),$$

as $x \to a$. We take $a = 0$.

- Put $f(x) = \cos x$. Then $f'(x) = -\sin x, f''(x) = -\cos x, f^{(3)}(x) = \sin x, f^{(4)}(x) = \cos x$. Applying the general Taylor formula with $a = 0$, we get $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$ as $x \to 0$.
- In general, if $g(x) = a_0 + a_1x + a_2x^2 + o(x^2)$, then we have $x^2g(x) = a_0x^2 + a_1x^3 + a_2x^4 + o(x^4)$, That is, the Taylor formula can be multiplied. This can simplify some calculations.
- For $g(x) = \sqrt{1+3x}$ we have $g'(x) = \frac{3}{2}(1+3x)^{-\frac{1}{2}}, g''(x) = -\frac{9}{4}(1+3x)^{-\frac{3}{2}}$. Thus $\sqrt{1+3x} = 1 + \frac{3}{2}x - \frac{9}{8}x^2 + o(x^2)$ as $x \to 0$.
- By applying the formula for product (see above), $x^2\sqrt{1+3x} = x^2(1 + \frac{3}{2}x - \frac{9}{8}x^2 + o(x^2)) = x^2 + \frac{3}{2}x^3 - \frac{9}{8}x^4 + o(x^4)$.
- As $\log(y) = y + o(y)$, we have $\log(1+x^2) = x^2 + o(x^2)$ and hence $(\log(1+x^2))^2 = x^4 + o(x^4)$.

Now the numerator is

$$\begin{aligned}
&2\cos x - 2 + x^2\sqrt{1+3x} + \alpha x^3 \\
&= 2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) - 2 + \left(x^2 + \frac{3}{2}x^3 - \frac{9}{8}x^4 + o(x^4)\right) + \alpha x^3 \\
&= \left(\frac{3}{2} + \alpha\right)x^3 - \frac{25}{24}x^4 + o(x^4)
\end{aligned}$$

To have a finite limit of $\lim_{x\to 0} \frac{\left(\frac{3}{2}+\alpha\right)x^3 - \frac{25}{24}x^4 + o(x^4)}{x^4 + o(x^4)}$, we must have $\frac{3}{2} + \alpha = 0$, because otherwise the limit diverges. Therefore, $\alpha = -\frac{3}{2}$, and the given limit is

$$\lim_{x\to 0} \frac{-\frac{25}{24}x^4 + o(x^4)}{x^3 + o(x^3)} = -\frac{25}{24}$$

Note: $\lim_{x\to 0} \frac{a}{x^4}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x\to 0} \frac{a+bx^2+cx^3}{x^4}$ converges if and only if $a = b = c = 0$ (otherwise diverges).

The symbol $g(x) = o(x^4)$ means that $\lim_{x\to 0} \frac{g(x)}{x^4} = 0$. In particular, we can calculate $\lim_{x\to 0} \frac{ax^4+o(x^4)}{x^4} = \lim_{x\to 0} \frac{a+\frac{o(x^4)}{x^4}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = i$ in $\sum_{n=0}^{2} \frac{2^n-1}{n!}(x+1)^{2n}$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{2^n-1}{n!}(x+1)^{2n}$, with various x.
Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. Recall that $a^0 = 1$ for all $a \in \mathbb{C}$ and $0! = 1$ by convention. In the case at hand with $x = i$, $(i+1)^0 = 1, (i+1)^2 = 2i, (i+1)^4 = -4$, thus

$$\begin{aligned}
&\sum_{n=0}^{2} \frac{2^n-1}{n!}(x+1)^{2n} \\
&= \frac{2^0-1}{0!}(i+1)^0 + \frac{2^1-1}{1!}(i+1)^2 + \frac{2^2-1}{2!}(i+1)^4 \\
&= 0 + 2i + \frac{3}{2} \cdot (-4) = -6 + 2i.
\end{aligned}$$

As for the convergence, we use the ratio test. The ratio test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the ratio test to our case, for $x \in \mathbb{R}$, we set $a_n = \frac{2^n-1}{n!}(x+1)^{2n}$ (no need to take the absolute value, as $(x+1)^{2n} \geq 0$ and $2^n - 1 \geq 0$), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{n+1}-1}{(n+1)!}(x+1)^{2(n+1)}}{\frac{2^n-1}{n!}(x+1)^{2n}} \\
&= \lim_{n\to\infty} \frac{(2^{n+1}-1)(x+1)^2}{(2^n-1)(n+1)} \\
&\leq \lim_{n\to\infty} \frac{4(x+1)^2}{(n+1)} \\
&= 0
\end{aligned}$$

(we used that $2^{n+1} - 1 < 2^{n+1}$ and $2^n - 1 > 2^{n-1}$, thus $\frac{2^{n+1}-1}{2^n-1} < \frac{2^{n+1}}{2^{n-1}} = 4$). By squeezing, we conclude that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$.

Therefore, the ratio test tells that, for all $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} \frac{2^n-1}{n!}(x+1)^{2n}$ converges.

This means that for any specific value of x, for example $x = -\frac{3}{2}$, the series converges (absolutely).
Problem. Calculate the infinite sum $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$.
Solution. This is a geometric series, that is a series of the form $\sum_{n=0}^{\infty} a^n$ for some $a \in \mathbb{R}$. We know that hthis converges for $-1 < a < 1$ and for such a it holds that $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Therefore, with $a = \frac{3}{5}$, $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$.
Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \cdots$ $(N = 0, 1, 2, 3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \frac{(x-1)^3}{x(x+1)}$.
Solution.

- Domain. $\frac{1}{x-a}$ is defined only for $x \neq a$. We have $f(x) = (x-1)^3 \cdot \frac{1}{x}\frac{1}{x+1}$, therefore, it is not defined for $x = 0, -1$. Altogether, the domain is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

- Asymptotes.

  - Vertical asymptotes. As $x \to 0$, $\frac{(x-1)^3}{x(x+1)} \to \pm\infty$ (the sign depends on whether $x \to 0^+$ or $x \to 0^-$). Similarly, $\lim_{x\to -1} \frac{(x-1)^3}{x(x+1)} = \pm\infty$. There are vertical asysmptotes at $x = 0, -1$.
  - Horizontal asymptote. As $x \to \pm\infty$, note that $\frac{(x-1)^3}{x(x+1)} \to \pm\infty$. Hence there is no horisontal asymptote.
  - Oblique asymptote. We calculate $\lim_{x\to\pm\infty} \frac{(x-1)^3}{x(x+1)}\frac{1}{x} = \lim_{x\to\pm\infty} \frac{x^3-3x^2+3x-1}{x^3+x^2} = 1$. Furthermore,

$$\begin{aligned}
&\lim_{x\to\pm\infty} \frac{(x-1)^3}{x(x+1)} - x = \lim_{x\to\pm\infty} \frac{x^3 - 3x^2 + 3x - 1 - x^2(x+1)}{x(x+1)} - x \\
&= \lim_{x\to\pm\infty} \frac{-4x^2 + 3x - 1}{x^2 + x} = -4,
\end{aligned}$$

    therefore, $y = x - 4$ is an oblique asymptote.

- The derivative. We can use the rule for quotient: $(\frac{g}{h})' = \frac{g'h - gh'}{h^2}$. In our case, $g(x) = (x-1)^3$ and $h(x) = x(x+1)$, therefore, we get

$$\begin{aligned}
f'(x) &= \frac{3(x-1)^2 \cdot x(x+1) - (x-1)^3 \cdot (2x+1)}{x^2(x+1)^2} \\
&= \frac{(x-1)^2(3x(x+1) - (x-1)(2x+1))}{x^2(x+1)^2} = \frac{(x-1)^2(x^2+4x+1)}{x^2(x+1)^2}.
\end{aligned}$$

- In particular, $f'(2) = \frac{13}{36}$.

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $(x-1)^2(x^2+4x+1) = 0, x \neq 0, -1$. Clearly, $x = 1$ is a stationary point. Moreover, the equation $x^2 + 4x + 1 = (x+2)^2 - 3 = 0$ has two (real) solutions $x = -2 \pm \sqrt{3}$. Altogether, there are 3 stationary points.

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  The factor $g(x) = x^2 + 4x + 1$ has a minumum at $x = -2$ and it is monotonically increasing on $x > -2$. Furthermore, $g(3) = 22 > 0$, thus $g(x) > 0$ for $x \in [3, 4]$. This implies that $f'(x) > 0$ for $x \in [3, 4]$, therefore, f is monotonically increasing in that interval.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_1^3 \frac{\log x}{x^2} dx.$$

Solution. Recall the general formula of integration by parts

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x).$$

We can easily find $\int \frac{1}{x^2}dx = -\frac{1}{x}$. Therefore, by integration by parts and $(\log x)' = \frac{1}{x}$, with $f(x) = \log x, g(x) = -\frac{1}{x}, g'(x) = \frac{1}{x^2}$,

$$\begin{aligned}\int_1^3 \frac{\log x}{x^2} dx &= \left[-\frac{1}{x} \cdot \log x\right]_1^3 - \int_1^3 \left(-\frac{1}{x}\right) \cdot \frac{1}{x} dx \\ &= -\frac{\log 3}{3} + \left[-\frac{1}{x}\right]_1^3 \\ &= \frac{2}{3} - \frac{\log 3}{3}.\end{aligned}$$

Note: other useful techniques are substitution (example: $\int xe^{x^2} dx$ by putting $t = x^2$) and change of variables (example: $\int \frac{1}{\sqrt{x^2+1}} dx$ by putting $x = \sin t$, $\frac{dx}{dt} = \cos t$, resulting $\int \frac{1}{\sqrt{x^2+1}} dx = \int \frac{1}{\sqrt{\sin^2 x+1}} \cos t dt = \int dt = t = \arcsin x$).

5 Improper integrals

Problem. For various $\alpha \in \mathbb{R}$, study the improper integral $\int_0^\infty x \exp(\alpha x^2) dx$.
Solution. The function $f(x) = x\exp(\alpha x^2)$ is bounded on any finite interval, but the integration region $[0, \infty)$ is infinite. Therefore, we need to take $\beta > 0$ and calculate the integral on $[0, \beta]$. Noting that $(\alpha x^2)' = 2\alpha x$, thus by substitution, for $\alpha \neq 0$,

$$\begin{aligned}\int_0^\beta x\exp(\alpha x^2)dx &= \frac{1}{2\alpha}\int_0^\beta 2\alpha x\exp(\alpha x^2)dx \\ &= \frac{1}{2\alpha}\left[\exp(\alpha x^2)\right]_0^\beta \\ &= \frac{1}{2\alpha}(\exp(\alpha\beta^2) - \exp 0).\end{aligned}$$

As $\beta \to \infty$, this is convergent only if $\alpha < 0$, and in that case, the limit is $-\frac{1}{2\alpha}$.

If $\alpha = 0$, then the integral is

$$\int_0^\beta x dx = \frac{1}{2}\left[x^2\right]_0^\beta = \frac{1}{2}(\beta^2 - 0),$$

and this diverges as $\beta \to \infty$.

Altogether, the improper integral converges only for $\alpha < 0$. For the specific value $\alpha = -5$, the integral is $-\frac{1}{2\alpha} = \frac{1}{10}$.

Note: Actually, it is not necessary to divide into two parts as suggested in the Moodle question, but the results are the same.
Problem. Choose all improper integrals that are convergent.

- $\int_0^\infty \log x/x dx$
- $\int_0^\infty \log x/x^2 dx$
- $\int_1^\infty \log x/x dx$
- $\int_1^\infty \log x/x^2 dx$
- $\int_0^\infty \exp(-x)/x dx$
- $\int_0^\infty \exp(x)/x dx$
- $\int_1^\infty \exp(-x)/x^2 dx$
- $\int_1^\infty \exp(x)/x^2 dx$

Solution.

Each of the functions is integrated over an infinite interval, and is possibly unbounded somewhere. Therefore, we need to look at whether the integral converges for large x and whether it converges when the function diverges.

- $\int_0^\infty \log x/x dx$. As $x \to \infty$, $\log x/x > \frac{1}{x}$, and we know that $\int_1^\infty \frac{1}{x} dx$ diverges, thus this diverges as well.
- $\int_0^\infty \log x/x^2 dx$. As $x \to 0$, $|\log x/x^2| > 1/x^2$ and we know that $\int_0^1 \frac{1}{x^2} dx$ diverges, thus this diverges as well.
- $\int_1^\infty \log x/x dx$. As $x \to \infty$, $\log x/x > \frac{1}{x}$, and we know that $\int_1^\infty \frac{1}{x} dx$ diverges, thus this diverges as well.

- $\int_1^\infty \log x/x^2 dx$. The function is bounded. As $x \to \infty$, $|\log x/x^2| < 1/x^{1.5}$ and we know that $\int_0^1 \frac{1}{x^{1.5}} dx$ converges, thus this integral converges as well.
- $\int_0^\infty \exp(-x)/x dx$. As $x \to 0$, $|\exp(-x)/x^2| > 1/2x^2$ and we know that $\int_0^1 \frac{1}{2x^2} dx$ diverges, thus this diverges as well.
- $\int_0^\infty \exp(x)/x dx$. As $x \to \infty$, the function diverges, thus the integral diverges as well.
- $\int_1^\infty \exp(-x)/x^2 dx$. The function is bounded. As $x \to \infty$, $\exp(-x)/x^2 < \exp(-x)$ and we know that $\int_1^\infty \exp(-x) dx$ is convergent, thus this integral is convergent as well.
- $\int_1^\infty \exp(x)/x^2 dx$. As $x \to \infty$, the function diverges, thus the integral diverges as well.

Note: An integral is improper if the interval is unbounded or the function is unbounded. In that case, we define

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^c f(x)dx + \lim_{\epsilon \to 0} \int_c^{b-\epsilon} f(x)dx,$$

where $c \in (a, b)$.