1 Taylor expansion and limit

Problem. For various $\alpha, \beta \in \mathbb{R}$, study the limit:

$$\lim_{x\to 0} \frac{(x+1)^{\frac{1}{2}} + (x-1)\exp x + \alpha x + \beta x^2}{\sin(x^2)\cdot x},$$

and find α, β such that this converges, and calculate the limit.

Solution. As $x \to 0$, as $|\sin(x^2)| \leq 1$, the denominator tends to 0. For the whole limit to converge, the numerator must also tend to 0, and we need to study the behaviours of the numerator and the denominator as $x \to 0$. For this purpose, we calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 3rd order, see below why the 3rd order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + o((x-a)^3) \text{ as } x \to a.$$

We take $a = 0$.

- Put $f(x) = (x+1)^{\frac{1}{2}}$. Then $f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}, f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}}, f^{(3)}(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}}$. Applying the general Taylor formula with $a = 0$, we get $(x+1)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+o(x^3)$ as $x \to 0$.
- In general, if $g(x) = a_0+a_1x+a_2x^2+a_3x^3+o(x^3)$, then we have $xg(x) = a_0x+a_1x^2+a_2x^3+o(x^3)$, That is, the Taylor formula can be multiplied. This can simplify some calculations.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$ (because $(e^x)' = e^x$)
- By applying the formula for product (see above), $(x-1)e^x = x + x^2 + \frac{x^3}{2!} - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}) + o(x^3) = -1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)$.
- As $\sin(y) = y + o(y^2)$, we have $\sin(x^2) = x^2 + o(x^4)$ and hence $x\sin(x^2) = x^3 + o(x^3)$.

Now the numerator is

$$\begin{aligned} &1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3) - 1 + \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) + \alpha x + \beta x^2 \\ &= \left(\frac{1}{2} + \alpha\right)x + \left(\frac{3}{8} + \beta\right)x^2 + \frac{19}{48}x^3 + o(x^3) \end{aligned}$$

To have a finite limit of $\lim_{x\to 0} \frac{\frac{1}{2}+\alpha+\left(\frac{3}{8}+\beta\right)x^2+\frac{19}{48}x^3+o(x^3)}{x^3+o(x^3)}$, we must have $\frac{1}{2} + \alpha = 0, \frac{3}{8} + \beta = 0$, because otherwise the limit diverges. Therefore, $\alpha = -\frac{1}{2}, \beta = -\frac{3}{8}$, and the given limit is

$$\lim_{x\to 0} \frac{\frac{19}{48}x^3 + o(x^3)}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{19}{48} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} = \frac{19}{48}$$

Note: $\lim_{x\to 0} \frac{a}{x^3}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x\to 0} \frac{a+bx^2}{x^3}$ converges if and only if $a = b = 0$ (otherwise diverges).

The symbol $g(x) = o(x^3)$ means that $\lim_{x\to 0} \frac{g(x)}{x^3} = 0$. In particular, we can calculate $\lim_{x\to 0} \frac{ax^3+o(x^3)}{x^3} = \lim_{x\to 0} \frac{a+\frac{o(x^3)}{x^3}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = i$ in $\sum_{n=0}^{2} \frac{3^n-1}{2^n}(x+1)^n$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{3^n-1}{2^n}(x+1)^n$, with various x.
Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. Recall that $a^0 = 1$ for all $a \in \mathbb{C}$ by convention. In the case at hand with $x = i$, $(i+1)^0 = 1, (i+1)^2 = 2i$, thus

$$\begin{aligned}
&\sum_{n=0}^{2} \frac{3^n-1}{2^n}(i+1)^n \\
&= \frac{3^0-1}{2^0}(i+1)^0 + \frac{3^1-1}{2^1}(i+1)^1 + \frac{3^2-1}{2^2}(i+1)^2 \\
&= 0 + (i+1) + 4i = 1 + 5i.
\end{aligned}$$

As for the convergence, we use the root test. The root test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the root test to our case, for $x \in \mathbb{R}$, we set $a_n = \frac{3^n-1}{2^n}|x+1|^n$ (need to take the absolute value), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$\begin{aligned}
\lim_{n\to\infty}(a_n)^{\frac{1}{n}} &= \lim_{n\to\infty}\left(\frac{3^n-1}{2^n}|x+1|^n\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\frac{(3^n(1-\frac{1}{3^n}))^{\frac{1}{n}}}{(2^n)^{\frac{1}{n}}}|x+1| \\
&= \frac{3}{2}|x+1|
\end{aligned}$$

(recall that $a^{\frac{1}{n}} \to 1$ for any $a > 0$, and $\frac{2}{3} < 1 - \frac{1}{3^n} < 1$, thus we can use the squeezing).

Therefore, the root test tells that, if $\frac{3}{2}|x+1| < 1$, the series $\sum_{n=0}^{\infty} \frac{3^n-1}{2^n}|x+1|^n$ converges, or in other words, $\sum_{n=0}^{\infty} \frac{3^n-1}{2^n}(x+1)^n$ converges absolutely. The condition is equivalent to $-\frac{2}{3} < x+1 < \frac{2}{3}$, or $-\frac{5}{3} < x+1 < -\frac{1}{3}$.

For any specific value of x, one has to consider whether $-\frac{5}{3} < x < -\frac{1}{3}$ or not. If $x = -\frac{3}{2}$, as $-\frac{5}{3} < -\frac{3}{2} < -\frac{1}{3}$, the series converges. If $x = -\frac{1}{3}$, the root test does give answer, but the series becomes $\sum_{n=0}^{\infty} \frac{3^n-1}{2^n}(\frac{2}{3})^n$ and as $\frac{3^n-1}{2^n}(\frac{2}{3})^n \to 1$, not $\to 0$, the series is divergent.
Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \cdots$ $(N = 0, 1, 2, 3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \log \frac{x^2+1}{x+2}$.
Solution.

- Domain. $\log y$ is defined only for $y > 0$. In this case, we should have $\frac{x^2+1}{x+2} > 0$. As $x^2 + 1 > 0$ for any $x \in \mathbb{R}$, we should have $x + 2 > 0$. In addition, the denominator should never be 0, so $x \neq -2$. Altogether, the domain is $(-2, \infty)$.

- Asymptotes.

  - Vertical asymptotes. As $x \to -2$ keeping $x > -2$, $\frac{x^2+1}{x+2} \to \infty$ and thus $\log \frac{x^2+1}{x+2} \to \infty$. There is a vertical asysmptote at $x = -2$.
  - Horizontal asymptote. As $x \to \infty$, note that $\frac{x^2+1}{x+2} \to \infty$, and $\log \frac{x^2+1}{x+2} \to \infty$. Hence there is no horisontal asymptote.
  - Oblique asymptote. For $x > \frac{1}{2}$, $\frac{\log \frac{x^2+1}{x+2}}{x} < \frac{\log \frac{x^2+2x}{x+2}}{x} = \frac{\log x}{x} \to 0$ as $x \to \infty$, there is no oblique asymptote.

- The derivative. We can use the chain rule: if $f(x) = g(h(x))$, then $f'(x) = h'(x)g'(h(x))$. In our case, $g(y) = \log y$ and $h(x) = \frac{x^2+1}{x+2}$, $g'(y) = \frac{1}{y}$, $h'(x) = \frac{2x(x+2)-(x^2+1)\cdot 1}{(x+2)^2} = \frac{x^2+4x-1}{(x+2)^2}$, therefore, we get

$$f'(x) = \frac{x+2}{x^2+1} \cdot \frac{x^2+4x-1}{(x+2)^2} = \frac{x^2+4x-1}{(x^2+1)(x+2)}.$$

- In particular, $f'(1) = \frac{4}{6} = \frac{2}{3}$.

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $x^2 + 4x - 1 = 0$, that is $x = -2 \pm \sqrt{5}$. However, $-2 - \sqrt{5} < -2$, and this is not in the domain of the function. Therefore, $-2 + \sqrt{5}$ is the only stationary point.

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  If $x = 0, x^2 + 4x - 1 = -1$, while if $x = 2, x^2 + 4x - 1 = 11 > 0$. From this it is easy to see that $f'(x) < 0$ for $x \in [0, -2 + \sqrt{5})$ and $f'(x) > 0$ for $x \in (-2 + \sqrt{5}, 2]$. Therefore, f is neither decreasing nor increasing in $[0, 2]$.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_1^2 \frac{1}{2^x + 4 + 3(2^{-x})} dx$$

Solution. We change the variables by $2^x = t$, or equivalently, $x = \frac{\log t}{\log 2}$. From this we get $\frac{dt}{dx} = \log 2 \cdot 2^x = t \log 2$. and formally replace dx by $\frac{1}{t \log 2} dt$, therefore, with $2^1 = 2, 2^2 = 4$,

$$\begin{aligned} \int_1^2 \frac{1}{2^x + 3 + 2(2^{-x})} dx &= \int_2^4 \frac{1}{t \log 2(t + 3 + 2\frac{1}{t})} dt \\ &= \int_2^4 \frac{1}{\log 2(t^2 + 3t + 2)} dt. \end{aligned}$$

To carry out this last integral, we need to find the partial fractions: as $t^2 + 3t + 2 = (t+1)(t+2)$, we put $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} = \frac{A(t+2)+B(t+1)}{t^2+3t+2}$, or $1 = (A+B)t + 2A + B$. By solving this, $B = -1, A = 1$. Namely, $\frac{1}{t^2+3t+2} = \frac{1}{2}(\frac{1}{t+1} - \frac{1}{t+2})$. Altogether,

$$\begin{aligned} \int_1^2 \frac{1}{2^x + 3 + 2(2^{-x})} dx &= \frac{1}{\log 2} \int_2^4 \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt \\ &= \frac{1}{\log 2} \left[\log(t + 1 - \log(t+2)\right]_2^4 dt \\ &= \frac{1}{\log 2}((\log 5 - \log 6) - (\log 3 - \log 4)) \\ &= \frac{\frac{20}{18}}{\log 2} = \frac{\frac{\log 10}{\log 9}}{\log 2}. \end{aligned}$$

Note: other useful techniques are substitution (example: $\int x e^{x^2} dx$ by putting $t = x^2$) and integration by parts (example: $\int x e^x dx$ by noticing that $(e^x)' = e^x$).

5 Differential equations

Problem. Find the general solution of

$$y'(x) = -y(x)^2 \cos(x^2)x.$$

and a special solution with $y(0) = \frac{1}{2}$.
Solution. This is a separable differential equation, because the right-hand side is a product of a function of x $(\cos(x^2)x)$ and a function of y $(-y(x)^2)$. The solution is given by integrating separately $-\frac{1}{y(x)^2}$ and $x\cos(x^2)$, that is

$$\int -\frac{1}{y(x)^2} y'(x)dx = \int x\cos(x^2)\exp(x^3)dx + C,$$
$$\frac{1}{y(x)} = \frac{1}{2}\sin(x^2) + C,$$

and this is equivalent to $y(x) = \frac{1}{\frac{1}{2}\sin(x^2)+C}$.

If $y(0) = \frac{1}{2}$, then $\frac{1}{2} = \frac{1}{0+C}$, hence $C = 2$.
Problem. Find the general solution of the following differential equation.

$$y''(x) + 2y'(x) - 8y(x) = 0$$

and a special solution such that $y(0) = 3$ and $\lim_{x\to\infty} y(x) = 0$.
Solution. This second order linear differential equation with constant coefficients can be solved by finding the solutions of the equation $z^2 + 2z - 8 = 0$, that are $z = 2, -4$. With them, the general solution is $y(x) = C_1\exp(2x) + C_2\exp(-4x)$. The term $C_1\exp(-4x)$ diverges as $x \to -\infty$, while the term $C_2\exp(2x)$ diverges as $x \to \infty$. With the condition $y(0) = 3$, we have $C_1 + C_2 = 3$, while if $C_1 \neq 0$, we would have $y(x) \to (\operatorname{sign} C_1)\infty$, and this does not satisfy the given condition, therefore, it must be that $C_1 = 0$ and $C_2 = 5$.
Note: the meaning that y is the solution of the differential equation is that, if we take $y(x) = C_1e^{2x} + C_2e^{-4x}$, then it holds that $y''(x) + 2y'(x) - 8y(x) = 0$. Indeed, $y'(x) = 2C_1e^{2x} - 4C_2e^{-4x}, y''(x) = 4C_1e^{2x} + 16e^{-4x}$ and hence

$$\begin{aligned}&y''(x) + 2y'(x) - 8y(x)\\&= 4C_1e^{2x} + 16C_2e^{-4x} + 2(2C_1e^{2x} - 4C_2e^{-4x}) - 8(C_1e^{2x} + C_2e^{-4x})\\&= 0.\end{aligned}$$

Similarly, if we take $y(x) = \frac{1}{\frac{1}{2}\sin(x^2)+C}$, it satisfies $y'(x) = -y(x)^2x\cos(x^2)$. Indeed,

$$\begin{aligned}y'(x) &= \frac{-x\cos(x^2)}{(\frac{1}{2}\sin(x^2) + C)^2}\\&= -y(x)^2x\cos(x^2).\end{aligned}$$