1 Taylor expansion and limit

Problem. For various $\alpha, \beta \in \mathbb{R}$, study the limit:

$$\lim_{x \to 0} \frac{e^{-x} + x(1+x)^{\frac{1}{3}} + \alpha + \beta x^2}{x \sin(x^2)},$$

and find α, β such that this converges, and calculate the limit.
Solution. As $x \to 0$, the denominator tends to 0. For the whole limit to converge, the numerator must also tend to 0, and we need to study the behaviours of the numerator and the denominator as $x \to 0$. We calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 3rd order, see below why the 3rd order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + o((x-a)^3) \text{ as } x \to a.$$

We take $a = 0$.

- $e^{-x} = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$ (because $(e^x)' = e^x$)
- In general, if $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3)$, then we have $xg(x) = a_0 x + a_1 x^2 + a_2 x^3 + o(x^3)$, That is, the Taylor formula can be multiplied. This can simplify some calculations.
- As $(1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} - \frac{x^2}{9}$ (because with $f(x) = (1+x)^{\frac{1}{3}}, f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}, f''(x) = -\frac{2}{9}(x+1)^{-\frac{5}{3}}$), we have $x(1+x)^{\frac{1}{3}} = x + \frac{x^2}{3} - \frac{x^3}{9}$.
- As $\sin(y) = y + o(y^2)$, we have $\sin(x^2) = x^2 + o(x^4)$ and hence $x\sin(x^2) = x^3 + o(x^3)$.

Now the numerator is

$$\begin{aligned} &1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3) + x + \frac{x^2}{3} - \frac{x^3}{9} + o(x^3) + \alpha + \beta x^2 \\ &= 1 + \alpha + \left(\frac{5}{6} + \beta\right) x^2 + \frac{-5}{18} x^3 + o(x^3) \end{aligned}$$

To have a finite limit of $\lim_{x\to 0} \frac{1+\alpha+(\frac{5}{6}+\beta)x^2 + \frac{-5}{18}x^3 + o(x^3)}{x^3 + o(x^3)}$, we must have $1 + \alpha = 0, \frac{5}{6} + \beta = 0$, because otherwise the limit diverges. Therefore, $\alpha = -1, \beta = -\frac{5}{6}$, and the given limit is

$$\lim_{x\to 0} \frac{\frac{-5}{18}x^3 + o(x^3)}{x^3 + o(x^3)} = \lim_{x \to 0} \frac{\frac{-5}{18} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} = -\frac{5}{18}$$

Note: $\lim_{x\to 0} \frac{a}{x^3}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x \to 0} \frac{a + bx^2}{x^3}$ converges if and only if $a = b = 0$ (otherwise diverges).

The symbol $g(x) = o(x^3)$ means that $\lim_{x\to 0} \frac{g(x)}{x^3} = 0$. In particular, we can calculate $\lim_{x\to 0} \frac{ax^3 + o(x^3)}{x^3} = \lim_{x \to 0} \frac{a + \frac{o(x^3)}{x^3}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = 1-i$ in $\sum_{n=2}^{\infty} \frac{4^n-2^n}{n^3+2}(x-1)^n$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{4^n-2^n}{n^3+2}(x-1)^n$, with various x.

Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. In the case at hand, we have $1-i+1 = -i$, $(-i)^2 = -1$, and

$$\begin{aligned}
&\sum_{n=0}^{2} \frac{4^n-2^n}{n^3+2}(-i)^n \\
&= \frac{4^0-2^0}{0^3+2}(-i)^0 + \frac{4^1-2^1}{1^3+2}(-i)^1 + \frac{4^2-2^2}{2^3+2}(-i)^2 \\
&= 0 + \frac{-2i}{3} + \frac{-6}{5}.
\end{aligned}$$

As for the convergence, we use the root test. The root test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the root test to our case, for $x \in \mathbb{R}$, we set $a_n = \frac{4^n-2^n}{n^3+2}|x-1|^n$ (need to take the absolute value), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$\begin{aligned}
\lim_{n\to\infty} (a_n)^{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{4^n - 2^n}{n^3+2}|x-1|^n \right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\frac{1 - \frac{2^n}{4^n}}{n^3+2} \right)^{\frac{1}{n}} 4|x-1| \\
&= 4|x-1|
\end{aligned}$$

(recall that $(1+a)^{\frac{1}{n}} \to 0$ for any a, and $n^{\frac{1}{n}} \to 1$).

Therefore, the root test tells that, if $4|x-1| < 1$, the series $\sum_{n=0}^{\infty} \frac{4^n-2^n}{n^3+2}|x-1|^n$ converges, or in other words, $\sum_{n=0}^{\infty} \frac{4^n-2^n}{n^3+2}(x-1)^n$ converges absolutely. The condition is equivalent to $-\frac{1}{4} < x-1 < \frac{1}{4}$, or $\frac{3}{4} < x+1 < \frac{5}{4}$.

For any specific value of x, one has to consider whether $\frac{3}{4} < x < \frac{5}{4}$ or not. If $x = \frac{5}{4}$, the root test does not apply, but the series becomes $\sum_{n=0}^{\infty} \frac{4^n-2^n}{(n^3+2)4^n}$ and as $\sum_{n=0}^{\infty} \frac{1}{n^3+2}$ converges absolytely, this series converges absolutely as well.

Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \cdots$ $(N = 0, 1, 2, 3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \arctan\left(\frac{x}{1+\log(x)}\right)$.
Solution.

- Domain. $\log y$ is defined only for $y > 0$, while $\arctan y$ is defined for all $y \in \mathbb{R}$. In this case, we should have $x > 0$. In addition, the denominator should never be 0, so $\log x \neq -1$, or $x \neq \frac{1}{e}$. Altogether, the domain is $(0, \frac{1}{e}) \cup (\frac{1}{e}, \infty)$.

- Asymptotes.

  - Vertical asymptotes. As $x \to 0$, $\frac{x}{1+\log(x)} \to 0$ and hence there is no vertical asymptote at $x = 0$. As for $x \to \frac{1}{e}$, we have that $\frac{x}{1+\log(x)} \to \pm\infty$, but $y \to \pm\infty$, $\arctan(y) \to \pm\frac{\pi}{2}$ and there is no vertical asymptote $x = \frac{1}{e}$.
  - Horizontal asymptote. As $x \to \infty$, note that $\frac{x}{1+\log(x)} \to \infty$, and $\arctan\left(\frac{x}{1+\log(x)}\right) \to \frac{\pi}{2}$. Hence $y = \frac{\pi}{2}$ is a horizontal asymptote.
  - Oblique asymptote. As both $x \to \infty$ have a horizontal asymptote, there is no oblique asymptote.

- The derivative. We can use the chain rule: if $f(x) = g(h(x))$, then $f'(x) = h'(x)g'(h(x))$. In our case, $g(y) = \arctan y$ and $h(x) = \frac{x}{1+\log(x)}$, $g'(y) = \frac{1}{y^2+1}$, $h'(x) = \frac{(1+\log(x)-x(\frac{1}{x})}{(1+\log(x))^2} = \frac{\log(x)}{(1+\log(x))^2}$, therefore, we get

$$f'(x) = \frac{\log(x)}{(\frac{x^2}{(1+\log x)^2} + 1)(1+\log(x))^2} = \frac{\log(x)}{x^2 + (1+\log x)^2}.$$

- In particular, $f'(e) = \frac{1}{e^2+4}$.

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $\log x = 1$, that is $x = 1$ is the only stationary point.

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  If $x \in [1, 2]$, $f'(x)$ is positive. Therefore, f is monotonically increasing there.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta} d\theta.$$

Solution. We change the variables by $\sin\theta = t$. From this we get $\frac{dt}{d\theta} = \cos\theta$. and formally replace $d\theta$ by $\frac{1}{\cos\theta} dt$, therefore, with $\sin 0 = 0, \sin\frac{\pi}{6} = \frac{1}{2}$,

$$\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta} d\theta. = \int_0^{\frac{1}{2}} \frac{1}{\cos^2\theta} dt. = \int_0^{\frac{1}{2}} \frac{1}{1-\sin^2\theta} dt. = \int_0^{\frac{1}{2}} \frac{1}{1-t^2} dt.$$

To carry out this last integral, we need to find the partial fractions: as $1 - t^2 = (1-t)(1+t)$, we put $\frac{1}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t)+B(1-t)}{(1-t^2}$, or $1 = (A-B)t + A + B$. By solving this, $B = \frac{1}{2}, A = \frac{1}{2}$. Namely, $\frac{1}{1-t^2} = \frac{1}{2}(\frac{1}{1-t} + \frac{1}{1+t})$. Altogether,

$$\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta} d\theta. = \frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt. = \frac{1}{2}\left[-\log(1-t) + \log(1+t)\right]_0^{\frac{1}{2}} = \frac{\log 3}{2}.$$

Note: other useful techniques are substitution (example: $\int xe^{x^2}dx$ by putting $t = x^2$) and integration by parts (example: $\int xe^x dx$ by noticing that $(e^x)' = e^x$).

5 Differential equations

Problem. Find the general solution of

$$y'(x) = (y(x) + 1)^{\frac{1}{2}} x \cos(x^2).$$

and a special solution with $y(0) = 0$.
Solution. This is a separable differential equation, because the right-hand side is a product of a function of x ($x\cos(x^2)$) and a function of y ($(y(x)+1)^{\frac{1}{2}}$). The solution is given by integrating separately $(y(x)+1)^{-\frac{1}{2}}$ and $x\cos(x^2)$, that is

$$\int (y+1)^{-\frac{1}{2}} y'(x) dx = \int x\cos(x^2) dx + C,$$
$$2(y+1)^{\frac{1}{2}} = \frac{1}{2}\sin(x^2) + C,$$

and this is equivalent to $y(x) = (\frac{\sin(x^2)}{4} + \frac{C}{2})^2 - 1$.

If $y(0) = 0$, then $0 = (0 + \frac{C}{2})^2 - 1$, hence $C = \pm 2$.
Problem. Find the general solution of the following differential equation.

$$y''(x) + 2y'(x) - 3y(x) = 0$$

and a special solution such that $y(0) = 5$ and $\lim_{x\to\infty} y(x) = 0$.
Solution. This second order linear differential equation with constant coefficients can be solved by finding the solutions of the equation $z^2 + 2z - 3 = 0$, that are $z = 1, -3$. With them, the general solution is $y(x) = C_1 \exp(-3x) + C_2 \exp(x)$. The term $C_1 \exp(-3x)$ diverges as $x \to -\infty$, while the term $C_2 \exp(x)$ diverges as $x \to \infty$. With the condition $y(0) = 5$, we have $C_1 + C_2 = 5$, while if $C_2 \neq 0$, we would have $y(x) \to (\operatorname{sign} C_2)\infty$, and this does not satisfy the given condition, therefore, it must be that $C_2 = 0$ and $C_1 = 5$.
Note: the meaning that y is the solution of the differential equation is that, if we take $y(x) = C_1 e^{-3x} + C_2 e^x$, then it holds that $y''(x) + 2y'(x) - 3y(x) = 0$. Indeed, $y'(x) = -3C_1 e^{-3x} + C_2 e^x, y''(x) = 9C_1 e^{-3x} + C_2 e^x$ and hence

$$\begin{aligned} & y''(x) + 2y'(x) - 3y(x) \\ & = 9C_1 e^{-3x} + C_2 e^x + 2(-3C_1 e^{-3x} + C_2 e^{2x}) - 3(C_1 e^{-3x} + C_2 e^x) \\ & = 0. \end{aligned}$$

Similarly, if we take $y(x) = (\frac{\sin(x^2)}{4} + \frac{C}{2})^2 - 1$, it satisfies $y'(x) = (y(x)+1)^{\frac{1}{2}} x\cos(x^2)$. Indeed,

$$\begin{aligned} y'(x) &= 2\frac{2x\cos(x^2)}{x}\left(\frac{\sin(x^2)}{4} + \frac{C}{2}\right) \\ &= x\cos(x^2)(y(x)+1)^{\frac{1}{2}}. \end{aligned}$$