1 Taylor expansion and limit

Problem. For various $\alpha, \beta \in \mathbb{R}$, study the limit:

$$\lim_{x\to 0} \frac{\frac{2+x^2}{1-x^2} + x\sin(\frac{x^3}{3}) - \alpha - \beta x^2}{\cos(x^2) - \gamma},$$

- For $\alpha = \beta = 0$.
- For $\gamma = 1$.

Solution. For the case $\alpha = \beta = 0$, the limit $x \to 0$ of the numerator is 2. In this case, the whole limit is divergent if the denominator tends to 0. As $\cos(x^2) \to 1$ when $x \to 0$, this happens when $\gamma = 1$.

Next, put $\gamma = 1$. Then, as $x \to 0$, the denominator tends to 0. For the whole limit to converge, the numerator must also tend to 0, and we need to study the behaviours of the numerator and the denominator as $x \to 0$. We calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 4th order, see below why the 4th order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + o((x-a)^4)$$

as $x \to a$. We take $a = 0$.

- $\frac{1}{1-x^2} = 1 + x^2 + x^4 + o(x^4)$ (this is an identity as a polynomial)
- In general, if $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + o(x^4)$, then we have $x^2g(x) = a_0x^2 + a_1x^3 + a_2x^4 + o(x^4)$, That is, the Taylor formula can be multiplied. This can simplify some calculations.
- $\frac{2+x^2}{1-x^2} = (2+x^2)(1+x^2+x^4+o(x^4)) = 2 + 3x^2 + 3x^4 + o(x^4)$.
- Similarly, as $\sin(y) = y + o(y^2)$, we have $\sin(x^3) = x^3 + o(x^6)$ and hence $x\sin(\frac{x^3}{3}) = \frac{x^4}{3} + o(x^4)$.
- $\cos(x^2) = 1 - \frac{x^4}{2} + o(x^4)$.

Now we have that the denominator is $\cos(x^2) - 1 = -\frac{x^4}{2} + o(x^4)$ as $x \to 0$, and the numerator is

$$2 + 3x^2 + 3x^4 + o(x^4) + \frac{x^4}{3} + o(x^4) - \alpha - \beta x^2 = 2 - \alpha + (3-\beta)x^2 + \frac{10}{3}x^4 + o(x^4)$$

To have a finite limit of $\lim_{x\to 0} \frac{2-\alpha+(3-\beta)x^2+\frac{10}{3}x^4+o(x^4)}{-\frac{x^4}{2}+o(x^4)}$, we must have $2-\alpha = 0, 3-\beta = 0$, because otherwise the limit diverges. Therefore, $\alpha = 2, \beta = 3$, and the given limit is

$$\lim_{x\to 0} \frac{\frac{10}{3}x^4 + o(x^4))}{-\frac{x^4}{2} + o(x^4)} = \lim_{x\to 0} \frac{\frac{10}{3} + \frac{o(x^4)}{x^4})}{-\frac{1}{2} + \frac{o(x^4)}{x^4}} = -\frac{20}{3}$$

Note: $\lim_{x\to 0} \frac{a}{x^4}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x\to 0} \frac{a+bx^2}{x^4}$ converges if and only if $a = b = 0$ (otherwise diverges).

The symbol $g(x) = o(x^4)$ means that $\lim_{x\to 0} \frac{g(x)}{x^4} = 0$. In particular, we can calculate $\lim_{x\to 0} \frac{ax^4+o(x^4)}{x^4} = \lim_{x\to 0} \frac{a+\frac{o(x^4)}{x^4}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = i\sqrt{3}$ in $\sum_{n=0}^{2} \frac{(-1)^n(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n}$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{(-1)^n(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n}$, with various x.

Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. In the case at hand, we have $(i\sqrt{3}+1)^2 = (\sqrt{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^2 = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = -2+2\sqrt{3}i, (i\sqrt{3}+1)^4 = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -8-8\sqrt{3}i$, and

$$
\begin{aligned}
&\sum_{n=0}^{2} \frac{(-1)^n(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n} \\
&= \frac{(-1)^0(4^0-1)}{(0+1)(3^0+1)^2}(i\sqrt{3}+1)^0 + \frac{(-1)^1(4^1-1)}{(1+1)(3^1+1)^2}(-2+2\sqrt{3}i) + \frac{(-1)^2(4^2-1)}{(2+1)(3^2+1)^2}(-8-8\sqrt{3}i) \\
&= 0 + \frac{3-3\sqrt{3}}{16} + \frac{-2-2\sqrt{3}}{5} = -\frac{-17}{80} + i\frac{-47\sqrt{3}}{80}.
\end{aligned}
$$

As for the convergence, we use the root test. The root test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the ratio test to our case, for $x \in \mathbb{R}$, we set $a_n = \frac{(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n}$ (need to take the absolute value), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$
\begin{aligned}
\lim_{n\to\infty}(a_n)^{\frac{1}{n}} &= \lim_{n\to\infty}(\frac{(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n})^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\left(\frac{4^n}{3^{2n}}\frac{1-\frac{1}{4^n}}{(n+1)(1+\frac{1}{3^n})^2}\right)^{\frac{1}{n}}(x+1)^2 \\
&= \frac{4}{9}\lim_{n\to\infty}\left(\frac{1-\frac{1}{4^n}}{(n+1)(1+\frac{1}{3^n})^2}\right)^{\frac{1}{n}}(x+1)^2 \\
&= \frac{4}{9}(x+1)^2
\end{aligned}
$$

Therefore, the root test tells that, if $\frac{4}{9}(x+1)^2 < 1$, the series $\sum_{n=0}^{\infty} \frac{(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n}$ converges, or in other words, $\sum_{n=0}^{\infty} \frac{(-1)^n(4^n-1)}{(n+1)(3^n+1)^2}(x+1)^{2n}$ converges absolutely. The condition is equivalent to $-\frac{9}{4} < (x+1)^2 < \frac{9}{4}$, or $-\frac{3}{2} < x+1 < \frac{3}{2}$, or $-\frac{5}{2} < x < \frac{1}{2}$.

For any specific value of x, one has to consider whether $-\frac{5}{2} < x < \frac{1}{2}$ or not. If $x = \frac{5}{2}$, the root test does not apply, but the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n(4^n-1)}{(n+1)(3^n+1)^2}(\frac{3}{2})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n(1-\frac{1}{4^n})}{(n+1)(1+\frac{1}{9^n})^2}$ and as $\frac{(-1)^n(1-\frac{1}{4^n})}{(n+1)(1+\frac{1}{9^n})^2}$ is asymptotically equivalent to $\frac{(-1)^n}{(n+1)}$, and the sum $\sum_n \frac{(-1)^n}{(n+1)}$ converges by the Leibniz criterion, our series converges as well. But it does not converge absolutely, because $\sum_n \frac{1}{(n+1)}$ diverges.

Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0+a_1, a_0+a_1+a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \cdots$ $(N = 0, 1, 2, 3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \log \frac{x^2+2}{x^2-3x+2}$.
Solution.

- Domain. $\log y$ is defined only for $y > 0$. In this case, we should have $\frac{x^2+2}{x^2-3x+2} > 0$. As $x^2 + 2 > 0$ for any x, this is equivalent to $x^2 - 3x + 2 > 0$. As we have $x^2 - 3x + 2 = (x-1)(x-2) > 0$, the domain is $(-\infty, 1) \cup (2, \infty)$.

- Asymptotes.
  - Vertical asymptotes. As $x \to 1, 2$, the denominator tends to 0, while the numerator tends to a non-zero number, and $\log y$ diverges as $y \to \infty$, so there is a vertical asymptote there.
  - Horizontal asymptote. As $x \to \pm\infty$, note that $\frac{x^2+2}{x^2-3x+2} \to 1$, and hence $\log \frac{x^2+2}{x^2-3x+2} \to 0$. Hence $y = 0$ is a horizontal asymptote.
  - Oblique asymptote. As both $x \to \pm\infty$ have a horizontal asymptote, there is no oblique asymptote.

- The derivative. We can use the chain rule: if $f(x) = g(h(x))$, then $f'(x) = h'(x)g'(h(x))$. In our case, $g(y) = \log y$ and $h(x) = \frac{x^2+2}{x^2-3x+2}$, $g'(y) = \frac{1}{y}$, $h'(x) = \frac{2x(x^2-3x+2)-(x^2+2)(2x-3)}{(x^2-3x+2)^2} = \frac{-3x^2+6}{(x^2-3x+2)^2}$, therefore, we get

$$f'(x) = \frac{-3x^2+6}{(x^2-3x+2)^2} \frac{x^2-3x+2}{x^2+2} = \frac{-3x^2+6}{(x^2-3x+2)(x^2+2)}.$$

- In particular, $f'(-1) = \frac{1}{6}$.

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $-3x^2 + 6 = -3(x^2 - 2) = -3(x - \sqrt{2})(x + \sqrt{2}) = 0$, and only $x = -\sqrt{2}$ is in the domain. Therefore, there is only one stationary point.

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  If $x \in [3, 5]$, $f'(x)$ is negative. Therefore, f is monotonically decreasing in $[3, 5]$.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_{\log 2}^{\log 3} \frac{1+e^{2x}}{e^x - 1} dx.$$

Solution. We change the variables by $e^x = t$. This is equivalent to $x = \log t$, or $\frac{dx}{dt} = \frac{1}{t}$. and formally replace dx by $\frac{1}{t}dt$, and substitute e^x by t, while the limits of the integral becomes $e^{\log 2} = 2$ and $e^{\log 3} = 3$, therefore,

$$\int_{\log 2}^{\log 3} \frac{1+e^{2x}}{e^x - 1} dx. = \int_2^3 \frac{1+t^2}{(t-1)t} dt$$

To carry out this last integral, we need to find the partial fractions: as $1 + t^2 = (t-1)t + t + 1$, we have $\frac{1+t^2}{(t-1)t} = 1 + \frac{t+1}{(t-1)t}$ and we put $\frac{t+1}{(t-1)t} = \frac{A}{t-1} + \frac{B}{t} = \frac{At+B(t-1)}{(t-1)t}$, or $t + 1 = (A+B)t - B$. By solving this, $B = -1, A = 2$. Namely, $\frac{1+t^2}{(t-1)t} = 1 + \frac{2}{t-1} - \frac{1}{t}$. Altogether,

$$\begin{aligned}\int_{\log 2}^{\log 3} \frac{1+e^{2x}}{e^x - 1} dx. &= \int_2^3 (1 + \frac{2}{t-1} - \frac{1}{t}) dt \\ &= [t + 2\log(t-1) - \log t]_2^3 = 1 + 3\log 2 - \log 3\end{aligned}$$

Note: other useful techniques are substitution (example: $\int xe^{x^2} dx$ by putting $t = x^2$) and integration by parts (example: $\int xe^x dx$ by noticing that $(e^x)' = e^x$).

5 Improper integrals

Problem. Consider the following improper integral for various $\alpha \in \mathbb{R}$.

$$\int_0^\infty \frac{\sin x}{x^\alpha (x+1)} dx$$

.

Solution. Depending on the value of α this integral is improper as $x \to \infty$ and possibly $x \to 0$ (when $\alpha > 1$).

To study the behaviour as $x \to 0$, we consider

$$\frac{\sin x}{x+1} = (x + o(x^2))(1 - x + x^2 + o(x^2)) = x - x^2 + o(x^2).$$

Therefore,

$$\frac{\sin x}{x^\alpha x + 1} = (x + o(x^2))(1 - x + x^2 + o(x^2)) = x^{1-\alpha} + o(x^{2-\alpha}).$$

and as we know that $\int_0^1 x^\gamma dx$ converges for $\gamma > -1$, we should have $1 - \alpha > -1$, or $2 > \alpha$.

On the other hand, as for the integral on $(1, \infty)$, note that $|\sin x| \leq 1$, hence $\left|\int_1^\infty \frac{\sin x}{x^\alpha (x+1)} dx\right| \leq \int_1^\infty \frac{1}{x^\alpha (x+1)} dx$ and the latter is convergent if $\alpha > 0$, because then $\frac{1}{x^\alpha (x+1)} < \frac{1}{(x+1)^{1+\alpha}}$ and we know that $\int_1^\infty x^\gamma dx$ converges if $\gamma < -1$. As for the case $-1 < \alpha \leq 0$, this is similar to the case $\int \frac{\sin x}{x}$ which we studied in the lecture, and this is convergent (but not absolutely). For $\alpha \leq -1$, the function $\frac{\sin}{x^\alpha (x+1)}$ do not decay, thus the integral does not converge.

We can calculate

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1} dx &= \lim_{\beta \to \infty} \int_0^\beta \frac{1}{x^2+1} dx \\ &= \lim_{\beta \to \infty} [\arctan x]_0^\beta = \lim_{\beta \to \infty} \arctan \beta - 0 = \frac{\pi}{2}.\end{aligned}$$

Note: An integral is improper if the interval is unbounded or the function is unbounded. In that case, we define

$$\int_a^b f(x) dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^c f(x) dx + \lim_{\epsilon \to 0} \int_c^{b-\epsilon} f(x) dx,$$

where $c \in (a, b)$.