Mathematical Analysis II exercises, 2020/21 First semester

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Naive set theory, real numbers

- Prove $a(-b) = -ab$ from the axiom of the real numbers.

  Solution. We use

  - definition of negative: $-x$ is the unique real number such that $x + (-x) = 0$
  - distributive law $(x + y)z = xz + yz$
  - definitions of zero and negative

  Indeed, we have

  $$\begin{aligned} ab + a(-b) &= a(b + (-b)) && \text{(distributive law)} \\ &= a \cdot 0 && \text{(definition of negative)} \\ &= 0 && \text{(definition of zero)} \end{aligned}$$

  and by the definition of negative, $a(-b) = -(ab)$.

- Prove that $a^{-1}b^{-1} = (ab)^{-1}$.

- Prove $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

  Solution. We use

  - commutativity and associativity of product
  - definition of fraction $\frac{x}{y} = xy^{-1}$
  - $a^{-1}b^{-1} = (ab)^{-1}$

  Indeed, we have

  $$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= ab^{-1}cd^{-1} && \text{(definition of fraction)} \\ &= acb^{-1}d^{-1} && \text{(commutativity and associativity)} \\ &= ac(bd)^{-1} && \text{(exercise)} \\ &= \frac{ac}{bd} \end{aligned}$$

- Prove $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

- For real numbers a, b, c, prove that if $a < b$ and $b < c$, then $a < c$.

  Solution. We use

  - if $x < y$, then $x + z < y + z$

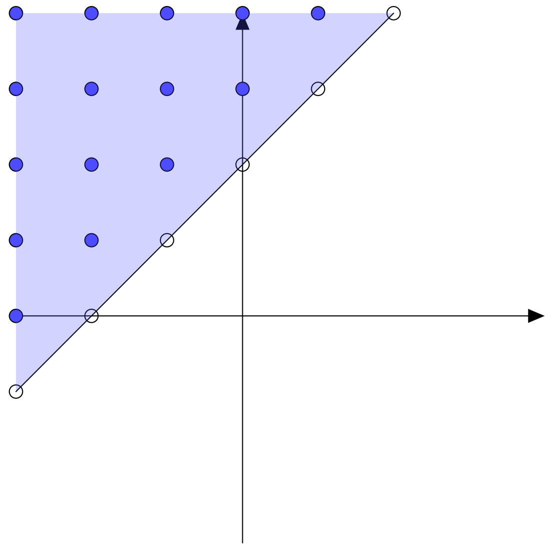


Figure 1: The set of all points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with $y > x + 2$.

- if $0 < x, 0 < y$, then $0 < x + y$
- associativity

Indeed, we have $0 < b - a$ by adding $-a$ to $a < b$. Similarly, $0 < c - b$. By taking the sum, we get $0 < (b - a) + (c - b) = c - a$. Adding a to both side, we get $a < c$.

- Let $A = \{0, 1, 2, 3\}, B = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$. What are $A \cap B$ and $A \cup B$?

  Solution. B is the set of even numbers, hence $A \cap B = \{0, 2\}$. $A \cup B$ does not have a nice representation, but formally it is $A \cup B = \{x \in \mathbb{Z} : x = 1, 3 \text{ or there is } y \in \mathbb{Z} \text{ such that } x = 2y\} = \{0, 1, 2, 3, 4, 6, 8, \cdots\}$.

- Let $A = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 3y\}, B = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = 2y\}$. What is $A \cap B$?

- Let $A_n = \{x \in \mathbb{Z} : \text{ there is } y \in \mathbb{Z} \text{ such that } x = ny\}$. What are $\bigcap_{n \in \mathbb{N}, n \geq 2} A_n$ and $\bigcup_{n \in \mathbb{N}, n \geq 2} A_n$?

  Solution. Let $x \in \mathbb{Z}$. If $x = 0$, then $x \in A_n$ for all n, hence $x \in \bigcap_{n \in \mathbb{N}, n \geq 2} A_n$. On the other hand, if $x \neq 0, -1$, then $x \notin A_{x+1}$, hence $x \notin \bigcap_{n \in \mathbb{N}, n \geq 2} A_n$. If $x = -1$, $x \notin A_2$, hence $x \notin \bigcap_{n \in \mathbb{N}, n \geq 2} A_n$. Altogether, $\bigcap_{n \in \mathbb{N}, n \geq 2} A_n = \{0\}$.

  For $x \not 1, -1$, $x \in A_x$ (if x is potivive) or $x \in A_{-x}$ (if x is negative). On the other hand, $1, -1 \notin A_n$ for any $n \geq 2$. Altogether, $\bigcup_{n \in \mathbb{N}, n \geq 2} A_n = \{x \in \mathbb{Z} : x \neq 1, -1\}$.

- Let $A = \mathbb{Z}$, and $B = \{(x, y) \in A \times A : y > x + 2\}$. Draw (a part of) its graph. What if $A = \mathbb{Q}, \mathbb{R}$?

  Solution. See Figure 1.

- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\}$.

  Solution. See Figure 2.

- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$.

  Solution. Note that

  - if $x = 1$, $y = 1^2 = 1$.

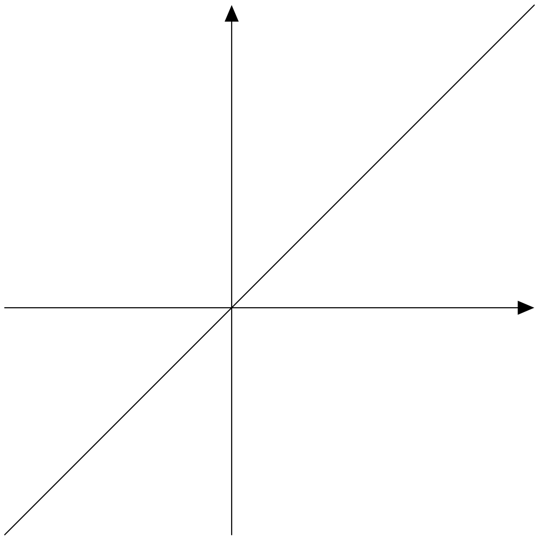


Figure 2: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y = x$.

- if $x = 0.5$, $y = 0.5^2 = 0.25$.
- if $x = 0.1$, $y = 0.1^2 = 0.01$.
- if $x = 2$, $y = 2^2 = 4$.
- if $x = 3$, $y = 3^2 = 9$.
- if $x = -1$, $y = (-1)^2 = 1$.

This is known as a **parabola**. See Figure 3.

- Draw the graph of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x^2 + 1\}$.

  Solution. Note that one has to take the region below the parabola $y = x^2 + 1$. See Figure 4.

- Prove that $2\sqrt{2}$ is irrational.

  Solution 1. Follow the proof of irrationality of $\sqrt{2}$.

  Solution 2. Use the fact that $\sqrt{2}$ is irrational. If $2\sqrt{2}$ were rational, then $2\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$, but this would imply that $\sqrt{2} = \frac{p}{2q}$ is rational, which is a contradiction. Therefore, $2\sqrt{2}$ must be irrational.

- Prove that $\sqrt{3}$ is irrational.

  Solution. Follow the proof of irrationality of $\sqrt{2}$. Use the fact that x^2 is a multiple of 3 if and only if x is a multiple of 3 (why?).

- Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} = \{\frac{1}{n} : n \in \mathbb{N}\}$. Determine $\inf A$ and $\sup A$.

  Solution. 1 is the largest element in A, hence $\sup A = 1$.

  0 is a lower bound of A. On the other hand, for any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (the Archimedean principle). This means that any positive number ϵ cannot be a lower bound. Therefore, 0 is the greatest lower bound: $\inf A = 0$.

- Let $A = \{0.9, 0.99, 0.999, \cdots\}$. Determine $\inf A$ and $\sup A$.

  Solution.

  0.9 is the smallest element in A, hence $\inf A = 0.9$.

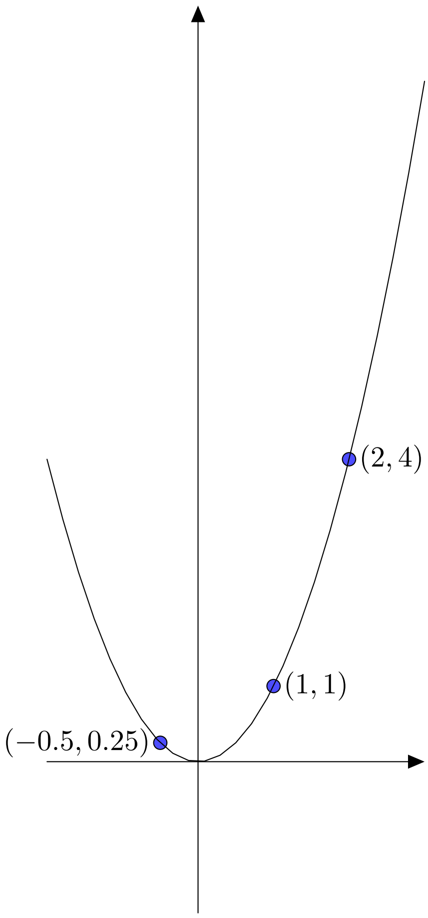


Figure 3: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y = x$.

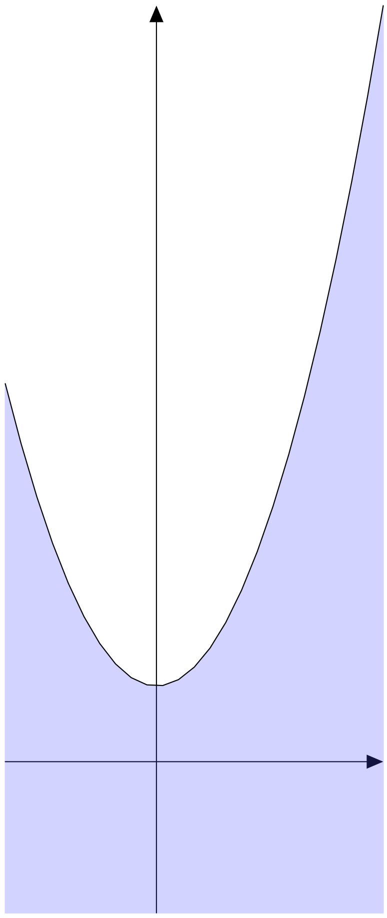


Figure 4: The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $y < x^2 + 1$.

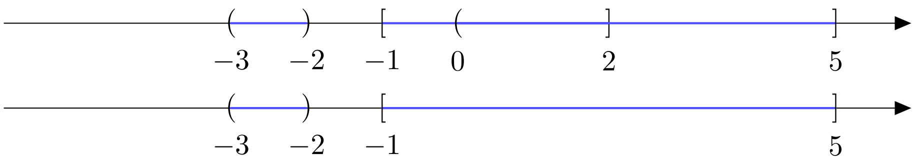


Figure 5: $[-1, 2] \cup (-3, -2) \cup (0, 5] = [-1, 5] \cup (-3, -2)$.

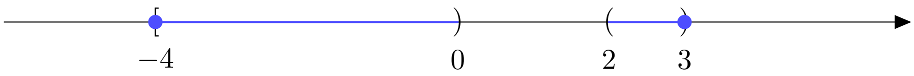


Figure 6: $[-4, 0) \cup (2, 3)$.

1 is an upper bound of A. On the other hand, for any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ (the Archimedean principle). We can take $0.0 \cdots 01 < \frac{1}{n}$, and $1 - 0.0 \cdots 01 = 0.09 \cdots 99$, and the next element in A is larger than it. This means that for any positive number $1 - \epsilon$ cannot be an upper bound Therefore, 1 is the least upper bound: $\sup A = 1$.

- Let $B = \{0.3, 0.33, 0.333, \cdots\}$. Determine $\inf A$ and $\sup A$.

  Solution. We have $B = \frac{1}{3}A$, where A is the set in the previous exercise. It holds that $\sup B = \frac{1}{3} \sup A = \frac{1}{3}, \inf B = \frac{1}{3} \inf A = 0$ (why? See the proof of the theorem $\sup A + \sup B = \sup(A + B)$).

- $x = 0.000001$. For which n does it hold that $\frac{1}{n} < x$?

  Solution. $x = 1/1000000$. So we can take $n = 1000001$.

Intervals, induction, functions.

- Draw the set on the line $[-1, 2] \cup (-3, -2) \cup (0, 5]$.

  Solution. See Figure 5

- Determine the inf and sup of $A = [-4, 0) \cup (2, 3)$.

  Solution. $\inf A = -4, \sup A = 3$. See Figure 6

- Determine the set $(1, 3) + (-2, 2]$.

  Solution. $(-1, 5)$. See Figure 7

- Determine the set $5 \cdot (2, 3)$.

  Solution. $(10, 15)$.

- Represent the set $\{x \in \mathbb{R} : x^2 - 2x < 0\}$ as an interval.

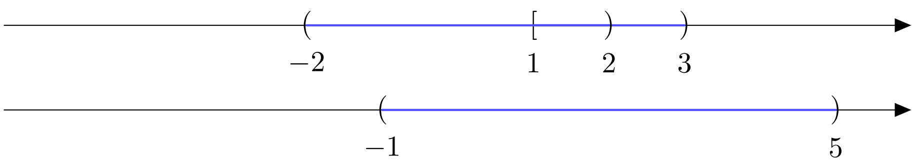


Figure 7: $(1, 3) + (-2, 2] = (-1, 5)$.

Solution. The condition $x^2 - 2x < 0$ can be written equivalently as

$$\begin{aligned} x^2 - 2x < 0 &\Longleftrightarrow x(x-2) < 0 \\ &\Longleftrightarrow (x < 0, x - 2 > 0) \text{ or } (x > 0, x - 2 < 0) \\ &\Longleftrightarrow (x < 0, x > 2) \text{ or } (x > 0, x < 2) \\ &\Longleftrightarrow (0 < x < 2) \end{aligned}$$

hence it is the interval $(0, 2)$.

- Represent the set $\{x \in \mathbb{R} : x^2 - 5x + 6 > 0\}$ as a union of intervals.

  Solution. The condition $x^2 - 5x + 6 > 0$ can be written equivalently as

$$\begin{aligned} x^2 - 5x + 6 > 0 &\Longleftrightarrow (x-2)(x-3) > 0 \\ &\Longleftrightarrow (x - 2 < 0, x - 3 < 0) \text{ or } (x - 2 > 0, x - 3 < 0) \\ &\Longleftrightarrow (x < 2, x < 3) \text{ or } (x > 2, x > 3) \\ &\Longleftrightarrow x < 2 \text{ or } x > 3 \end{aligned}$$

  hence it is the union $(-\infty, 2) \cup (3, \infty)$.

- Determine the decimal representation of $\frac{3}{7}$.

  Solution. Let $\frac{3}{7} = a_0.a_1a_2\cdots$. Note that $\frac{3}{7} < 1$, hence we have $a_0 = 0$. Next, $\frac{3}{7} \times 10 = \frac{30}{7} = 4 + \frac{2}{7}$, hence we have $a_1 = 4$. Next, $\frac{2}{7} \times 10 = \frac{20}{7} = 2 + \frac{6}{7}$, hence we have $a_2 = 2$, and so on.

  Therefore, we have $\frac{3}{7} = 0.428571428571\cdots$.

		0.	4	2	8	5	7	1
7	)	3						
		0						
		3	0					
		2	8					
			2	0				
			1	4				
				6	0			
				5	6			
					4	0		
					3	5		
						5	0	
						4	9	
							1	0
								7
								3

- Give an algorithm to produce a nonrepeating decimal representation.

  Solution. Just an example. Set $0.101001000100001000001\cdots$.

- Compute $\sum_{k=1}^{5}(2k+1)$.

  Solution.

$$\sum_{k=1}^{5}(2k+1) = (2+1) + (4+1) + (6+1) + (8+1) + (10+1) = 3 + 5 + 7 + 9 + 11 = 35.$$

- Compute $\sum_{k=2}^{6}(2(k-1)+1)$.

  Solution.

$$\begin{aligned}\sum_{k=2}^{6}(2(k-1)+1) &= (2+1)+(4+1)+(6+1)+(8+1)+(10+1) \\ &= 3+5+7+9+11 = 35.\end{aligned}$$

- Prove the formula $\sum_{k=1}^{n}(2k-1) = n^2$.

  Solution. By induction. For $n = 1$, we have $\sum_{k=1}^{1}(2k-1) = 1 = 1^2$. Assuming the formula for n, we compute

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \sum_{k=1}^{n}(2k-1) + (2(n+1)-1) \\ &= n^2 + 2n + 1 = (n+1)^2.\end{aligned}$$

- Write $\sum_{k=1}^{n}(2k-1)$ as a sum from $k = 0$ to $n-1$.

  Solution. $\sum_{k=0}^{n-1}(2(k+1)-1) = \sum_{k=0}^{n-1}(2k+1)$.

- Compute the sum $\sum_{k=1}^{n} 10^{-k}$.

  Solution. By the formula for the sum of powers, we have

$$\frac{0.1(1-0.1^n)}{1-0.1} = \frac{0.0\underbrace{9\cdots 9}_{n\text{-times}}}{0.9} = 0.\underbrace{1\cdots 1}_{n\text{-times}}.$$

$$\sum_{k=1}^{n} 10^{-k} = 0.1 + 0.01 + 0.001 \cdots 0.\underbrace{0\cdots 0}_{n-1\text{-times}}1 = 0.\underbrace{1\cdots 1}_{n\text{-times}}.$$

- Compute the sum $\sum_{k=1}^{n} 2^{-1}$.

  Solution. By the formula for the sum of powers, we have

$$\sum_{k=1}^{n} 2^{-1} = \frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n.$$

$$\sum_{k=1}^{1} 2^{-1} = \frac{1}{2}.$$

$$\sum_{k=1}^{2} 2^{-1} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

$$\sum_{k=1}^{3} 2^{-1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

- Expand $(x+y)^5$.

  Solution. By the binominal theorem,

$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\ &= \frac{5!}{0!5!}x^5 + \frac{5!}{1!4!}x^4y + \frac{5!}{2!3!}x^3y^2 + \frac{5!}{3!2!}x^2y^3 + \frac{5!}{4!1!}xy^4 + \frac{5!}{5!0!}y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.\end{aligned}$$

- Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

  Solution. By the binominal theorem, $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$.

- Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

  Solution. By the binominal theorem, $0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$.

- Determine the domains of the following

  - $f(x) = \sqrt{x^2 - 1}$

    Solution. To have the square root, the number must be positive or zero. That is,

    $$\begin{aligned} x^2 - 1 \geq 0 &\iff (x-1)(x+1) \geq 0 \\ &\iff (x - 1 \geq 0, x + 1 \geq 0) \text{ or } (x - 1 \leq 0, x + 1 \leq 0) \\ &\iff (x \geq 1, x \geq -1) \text{ or } (x \leq 1, x \leq -1) \\ &\iff (x \geq 1) \text{ or } (x \leq -1) \end{aligned}$$

    hence the domain is $(-\infty, -1] \cup [1, \infty)$.

  - $f(x) = \frac{1}{x^3 + 2x^2 - x - 2}$

    Solution. To have the division, the denominator must not be zero. That is,

    $$\begin{aligned} x^3 + 2x^2 - x - 2 \neq 0 &\iff (x+2)(x^2 - 1) \neq 0 \\ &\iff (x+2)(x-1)(x+1) \neq 0 \end{aligned}$$

    hence the domain is $(-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty)$.

- Determine the inverse functions of the following.

  - $f(x) = x + 1$

    Solution. The inverse f^{-1} should satisfy $f^{-1}(x+1) = x$. We see that $f^{-1}(x) = x - 1$. Then we indeed have $(x-1) + 1 = x$.

  - $f(x) = \frac{1}{x}$ on $(0, \infty)$.

    Solution. The inverse f^{-1} should satisfy $f^{-1}(\frac{1}{x}) = x$. We see that $f^{-1}(x) = \frac{1}{x}$. Then we indeed have $\dfrac{1}{\frac{1}{x}} = x$.

  Compare the graphs. How can one obtain one from the other?

  - $f(x) = x^2, g(x) = (x-1)^2 + 2$.

    Solution. g can be obtained by shifting f by $(1, 2)$. Indeed, their graphs are

    $$\begin{aligned} &f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \\ &g = \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = (x'-1)^2 + 2\} = \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' - 2 = (x'-1)^2\} \end{aligned}$$

    Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (x+1, y+2)$ is on the graph of g.

  - $f(x) = \frac{1}{2}x^3 - x, g(x) = \frac{x^3}{16} - \frac{x}{2}$.

    Solution. $g(x) = \frac{1}{2}(\frac{x}{2})^3 - \frac{x}{2}$ can be obtained by dilating the x-direction of f by 2. Indeed, their graphs are

    $$\begin{aligned} f &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{2}x^3 - x\} \\ g &= \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = \tfrac{1}{2}(\tfrac{x'}{2})^3 - \tfrac{x'}{2}\} \end{aligned}$$

    Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (2x, y)$ is on the graph of g.

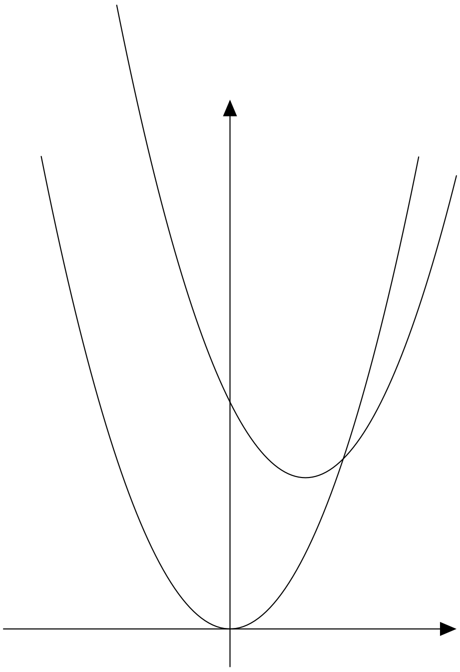


Figure 8: The graphs of $y = x^2$ and $y = (x-1)^2 + 2$.

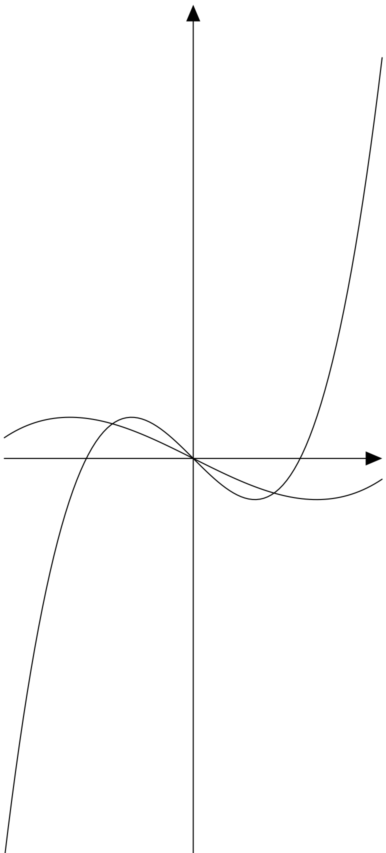


Figure 9: The graphs of $y = \frac{1}{2}x^3 - x$ and $y = \frac{x^3}{16} - \frac{x}{2}$.

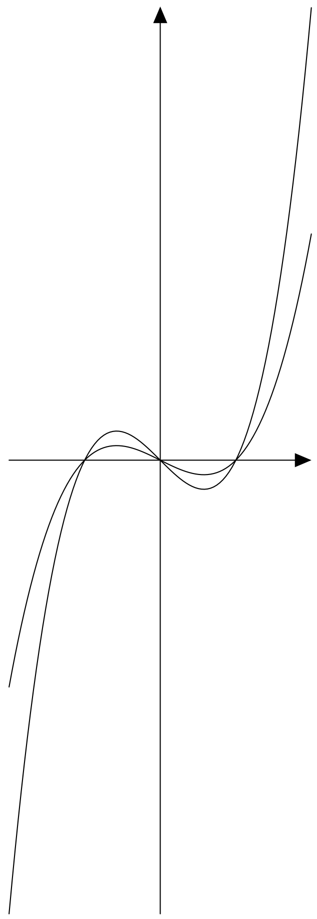


Figure 10: The graphs of $y = x^3 - x$ and $y = \frac{1}{2}(x^3 - x)$.

– $f(x) = x^3 - x, g(x) = \frac{x^3-x}{2}$
  Solution. $g(x)$ can be obtained by dilating the y-direction of f by $\frac{1}{2}$. Indeed, their graphs are

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^3 - x\}$$
$$g = \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = \tfrac{1}{2}(x'^3 - x')\} = \{(x', y') \in \mathbb{R} \times \mathbb{R} : 2y' = x'^3 - x'\}$$

  Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (x, \frac{y}{2})$ is on the graph of g.

– $f(x) = \sqrt{1 - x^2}, g(x) = \frac{1}{3}\sqrt{1 - 4(x + 2)^2}$
  Solution. $g(x)$ can be obtained by dilating the y-direction of f by $\frac{1}{3}$ and by dilating by $\frac{1}{2}$ then shifting the x-direction by -2 . Indeed, their graphs are

$$\begin{aligned} f &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sqrt{1 - x^2}\} \\ g &= \{(x', y') \in \mathbb{R} \times \mathbb{R} : y' = \tfrac{1}{3}\sqrt{1 - 4(x' + 2)^2}\} \\ &= \{(x', y') \in \mathbb{R} \times \mathbb{R} : 3y' = \sqrt{1 - (2(x' + 2))^2}\} \end{aligned}$$

  Therefore, if the point (x, y) is on the graph of f, then the point $(x', y') = (\frac{x}{2} - 2, \frac{y}{3})$ is on the graph of g.

Limit of sequences and functions.

- Let $a_n = \frac{1}{\sqrt{\sqrt{n}}}$ and $\epsilon = 0.01$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

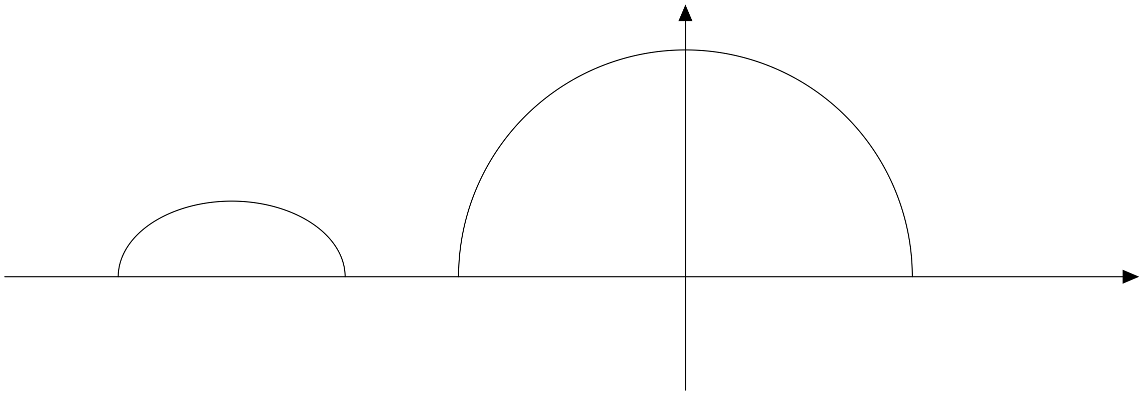


Figure 11: The graphs of $y = \sqrt{1-x^2}$ and $y = \frac{1}{3}\sqrt{1-4(x+2)^2}$.

Solution. Note that $\sqrt{\sqrt{100000000^{-1}}} = \sqrt{\sqrt{0.00000001}} = 0.01$, hence if $n > 100000000$, then $\frac{1}{\sqrt{\sqrt{n}}} < \frac{1}{\sqrt{\sqrt{100000000}}} = 0.01$. We can take $N = 100000000$.

- Let $a_n = \frac{1}{2^n}$ and $\epsilon = 0.00001$. Find N such that for $n > N$ it holds $|a_n| < \epsilon$.

  Solution. Note that $2^{17} = 131072 > 100000$, hence $\frac{1}{2^{17}} < \frac{1}{100000} = 0.00001$. As $\frac{1}{2^n} > \frac{1}{2^{n+1}}$, we can take $N = 17$.

- Show that a constant sequence $a_n = C \in \mathbb{R}$ is convergent.

  Solution. For any given $\epsilon > 0$ we can take $N = 1$ and then for any $n > 1$ we have $|a_n - C| = |C - C| = 0 < \epsilon$.

- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{1}{1+\frac{1}{n}}$.

  Solution. $\frac{1}{n}$ converges to 0, and $1 + \frac{1}{n}$ converges to 1 (sum), and $\frac{1}{1+\frac{1}{n}}$ converges to $\frac{1}{1} = 1$ (quotient with nonzero denominator).

- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n}{1+n}$.

  Solution. Note that $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$, hence this converges to 1 by the previous problem.

- Tell whether $\{a_n\}$ converges, and if it does, compute the limit $a_n = \frac{n^3+n^2+4}{n^3+100}$. *Solution.*

  Note that $\frac{n^3+n^2+4}{n^3+100} = \frac{1+\frac{1}{n}+\frac{4}{n^2}}{1+\frac{100}{n^3}}$. The numerator tends to 1 and the denominator tends to 1 as well, therefore, $a_n \to 1$.

- Let $x = 0.12341234\cdots$. Represent x as a rational number.

  Solution. x is approximated by

$$0.1 + 0.02 + 0.003 + 0.0004 + \cdots = \sum_{k=1}^{n} 1234 \cdot 10000^{-k}$$
$$= \frac{1234(1-10000^{-n}}{1-10000} \to \frac{1234}{10000-1} = \frac{1234}{9999}.$$

- Compute $\lim_{x\to 2} x^2$.

  Solution. We have seen that $f(x) = x$ is continuous, therefore, $\lim_{x\to 2} x = 2$ and with $g(x) = x \cdot x$ we have $\lim_{x\to 2} x^2 = 2 \cdot 2 = 4$.

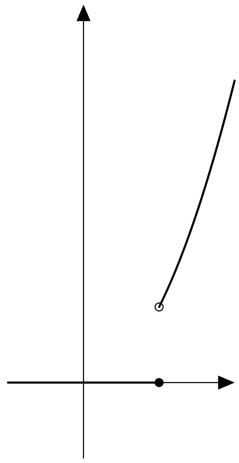


Figure 12: The graphs of $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$.

- Compute $\lim_{x\to 1} \frac{x+2}{x-3}$.

  Solution. It is easy to see that $f(x) = x+2$ and $g(x) = x-3$ are continuous, therefore, the quotient $\frac{x+2}{x-3}$ is continuous as long as $x \neq 3$. That is, $\lim_{x\to 1} \frac{x+2}{x-3} = \frac{\lim_{x\to 1} x+2}{\lim_{x} \to 1 x-3} = \frac{3}{-2} = -\frac{3}{2}$.

- Compute $\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1}$.

  Solution. As it is written, the denominator tends to 0 as $x \to -1$. But actually we have $\frac{x^2+3x+2}{x^2-1} = \frac{(x+2)(x+1)}{(x-1)(x+1)} = \frac{x+2}{x-1}$ for $x \neq -1$. Therefore,

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1} = \lim_{x\to -1} \frac{x+2}{x-1} = \frac{1}{-2} = -\frac{1}{2}.$$

- Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1 \end{cases}$. Is f continuous or not? If not, where is it not continuous?

  Solution. We know that x^2 and 0 are continuous for $x > 1$ and $x < 1$, respectively. The problem is at $x = 1$. If $x_n > 1, x_n \to 1$, then $f(x_n) = x_n^2 \to 1$, but if $x_n < 1, x_n \to 1$, then $f(x_n) = 0 \to 0$, and they do not conincide. Hence f is not continuous at $x = 1$.

- Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x^2-1} & \text{for } x \neq 1, -1 \\ -\frac{1}{2} & \text{for } x = -1 \end{cases}$, defined on $\mathbb{R} \setminus \{1\}$. Is f continuous or not? If not, where is it not continuous?

  Solution. As we saw before, $\frac{x^2+3x+2}{x^2-1} = \frac{x+2}{x-1}$ and $\lim_{x\to -1} \frac{x^2+3x+2}{x^2-1} = -\frac{1}{2}$. As $f(-1) = -\frac{1}{2}$ by definition, f is continuous at $x = -1$. It is also continuous at $x \neq 1$. Therefore, it is continuous on $\mathbb{R} \setminus \{1\}$ (not defined at $x = 1$).

- Let $f(x) = x^4 + 3x^3 - x - 2$. Show that the equation $f(x) = 0$ has at least two solutions.

  Solution. Note that $f(0) = -2, f(1) = 1$. Hence by the intermediate value theorem there is $x_1 \in (-2, 1)$ such that $f(x_1) = 0$. Similarly, $f(0) = -2, f(-3) = 1$. Hence by the intermediate value theorem there is $x_2 \in (-3, 0)$ such that $f(x_2) = 0$.

- Compute $\lim_{x\to 1} \sqrt{x + 3\sqrt{x}}$.

  Solution. We know that $\sqrt{x} = x^{\frac{1}{2}}$ is continuous (on $\mathbb{R}_+ \cup \{0\}$), hence $\lim_{x\to 1} \sqrt{x} = 1$. Further $x + 3\sqrt{x}$ is continuous and $\lim_{x\to 1} x + 3\sqrt{x} = 4$. Finally $\lim_{x\to 1} \sqrt{x + 3\sqrt{x}}$ is continuous (on $\mathbb{R}_+ \cup \{0\}$) and $\lim_{x\to 1} \sqrt{x + 3\sqrt{x}} = \sqrt{4} = 2$.

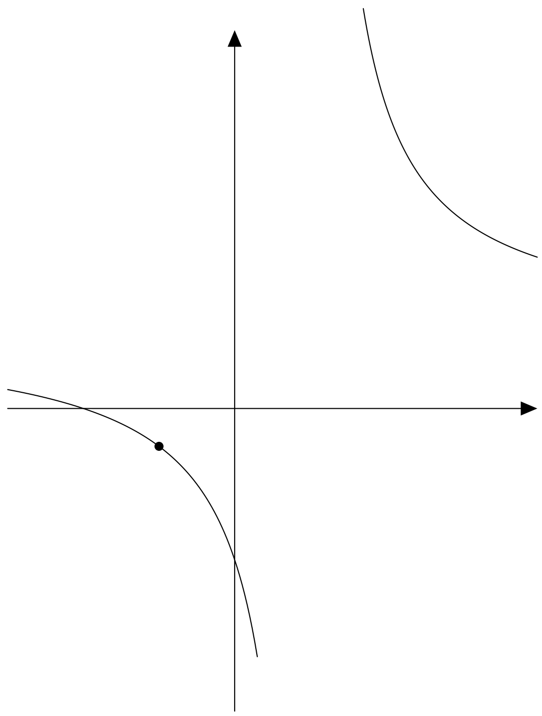


Figure 13: The graph of $f(x) = \frac{x^2+3x+2}{x^2-1}$.

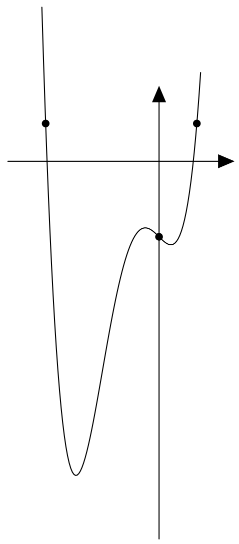


Figure 14: The graphs of $f(x) = x^4 + 3x^3 - x - 2$.

- Compute $\lim_{n\to\infty} \sqrt{n^{\frac{1}{n}} + 3\sqrt{n}^{\frac{1}{n}}}$.

  Solution. We know that $n^{\frac{1}{n}} \to 1$. By combining with the above, the limit is 2.

- Compute $\lim_{n\to\infty}(n + n^2)^{\frac{1}{n}}$.

  Solution. Clearly $1 < (n+n^2)^{\frac{1}{n}} < (2n^2)^{\frac{1}{n}} = 2^{\frac{1}{n}}(n^{\frac{1}{n}})^2 \to 1$, so by squeezing the limit is 1.

- For $a > 1$, compute $\lim_{n\to\infty} \frac{n^{1.5}}{a^n}$.

  Solution. Clearly $0 < \frac{n^{1.5}}{a^n} < \frac{n^2}{a^n} \to 0$, so by squeezing the limit is 0.

- Show that $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$ for $a, b \geq 0$.

  Solution. Note that $(a^{\frac{1}{n}}b^{\frac{1}{n}})^n = (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n = ab$, hence we can take the n-th root of both sides.

- Compute $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

  Solution. At first sight, it would yield $\frac{0}{0}$. However, for $x \neq 0$, we have

  $$\frac{1-\sqrt{1-x^2}}{x^2} = \frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x^2(1+\sqrt{1-x^2})} = \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} = \frac{1}{1+\sqrt{1-x^2}}$$

  Therefore, $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$.

- Compute $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x}$.

  Solution.

  $$\begin{aligned}\frac{\sqrt{1-x}-\sqrt{1+x}}{x} &= \frac{(\sqrt{1-x}-\sqrt{1+x})(\sqrt{1-x}+\sqrt{1+x})}{x(\sqrt{1-x}+\sqrt{1+x})} \\ &= \frac{(1-x)-(1+x)}{x(\sqrt{1-x}+\sqrt{1+x})} \\ &= \frac{-2}{\sqrt{1-x}+\sqrt{1+x}}\end{aligned}$$

  Therefore, $\lim_{x\to 0} \frac{\sqrt{1-x}-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{-2}{\sqrt{1-x}+\sqrt{1+x}} = -1$.

- Consider $f(x) = x^2$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 1$.

  Solution. Note that $(1+y)^2 = 1 + 2y + y^2$. We need that $|2y + y^2| < 0.1$, and this is achieved with $|y| < 0.04$.

- Consider $f(x) = x^{\frac{1}{3}}$. For $\epsilon = 0.1$, find a δ which shows the continuity of f at $x = 0$.

  Solution. We need that $x^{\frac{1}{3}} < 0.1$, hence $x < 0.001$ (and $x \geq 0$).

Exponential, logarithm and their limits.

- Prove that for $p, q, r, s \in \mathbb{N}$, we have $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

  Solution. We have

  $$((a^{\frac{p}{q}})^{\frac{r}{s}})^{qs} = (a^{\frac{p}{q}})^{qr} = ((a^{\frac{p}{q}})^q)^r = a^{pr},$$

  and hence by taking the qs-th root of both sides we obtain the claim.

- Let $a_n = \frac{(-1)^n}{n}$. Determine $\sup\{a_k : k \geq n\}$ and $\inf\{a_k : k \geq n\}$.

  Solution. Note that $a_n \geq 0$ if n is even, and $a_n < 0$ if n is odd. In addition, $|a_n| = \frac{1}{n}$ is monotonically decreasing.

  If n is even, then a_n is the largest in $\{a_k : k \geq n\}$, hence $\sup\{a_k : k \geq n\} = \frac{1}{n}$, while the smallest element is a_{n+1}, hence $\inf\{a_k : k \geq n\} = -\frac{1}{n+1}$.

  Similarly, if n is odd, then $\sup\{a_k : k \geq n\} = \frac{1}{n+1}$, while $\inf\{a_k : k \geq n\} = -\frac{1}{n}$.

- Compute 2^x for $x = 1, 2, 3, 4, \frac{1}{2}, -\frac{3}{2}$.

  Solution. $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^{\frac{1}{2}} = \sqrt{2}, 2^{-\frac{3}{2}} = \frac{1}{2\sqrt{2}}$.

- Compute $(\frac{1}{9})^x$ for $x = 1, 2, -3, -\frac{1}{2}, \frac{3}{2}$.

  Solution. $(\frac{1}{9})^1 = \frac{1}{9}, (\frac{1}{9})^2 = \frac{1}{81}, (\frac{1}{9})^{-3} = 729, (\frac{1}{9})^{-\frac{1}{2}} = 3, (\frac{1}{9})^{\frac{3}{2}} = \frac{1}{27}$.

- Imagine that there is a pond and the leaves of lotus doubles each day. If the pond is completely filled on day 100, when is the pond half filled?

  Solution. It's day 99, because on the next day the pond is filled completely.

- Compute $\log_3(81), \log_{81} 3, \log_2 0.125$.

  Solution. $81 = 3^4$, hence $\log_3 81 = 4$. $3 = 81^{\frac{1}{4}}$, hence $\log_{81} 3 = \frac{1}{4}$.

  $0.125 = \frac{1}{8} = 2^{-3}$, hence $\log_2 0.125 = -3$.

- Compute $(1 + \frac{1}{3})^3$.

  Solution. $(\frac{4}{3})^3 = \frac{64}{27} = 2.370370...$

  $(1 + \frac{1}{5})^5 = 2.48832$.

  $(1 + \frac{1}{10000})^{10000} = 2.718145927$.

  The true value $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.718281828....$

- If $y = Ce^{ax}$, what is the relation between $z = \log y$ and x?

  Solution. We have $e^z = y$, hence $e^z = Ce^{ax}$, and by taking log, we have $z = ax + \log C$.

- If $y = Cx^p$, what is the relation between $z = \log y$ and $w = \log x$?

  Solution. We have $e^z = y, e^w = x$, hence $e^z = Ce^{pw}$, and by taking log, we have $z = pw + \log C$.

- Calculate the integer part of $\log_{10}(232720)$.

  Solution. Note that $10^5 = 100000 = 232720 < 1000000 = 10^6$. As $\log_{10} x$ is monotonically increasing, $5 < \log_{10} 232720 < 6$. Therefore, its integer part is 5.

- Calculate the integer part of $\log_2(13567)$.

  Solution. Note that $2^{13} = 8192 < 13567 < 16384 = 2^{14}$. As $\log_2 x$ is monotonically increasing, $13 < \log_2 13567 < 14$. Therefore, its integer part is 13.

- Compute $\lim_{n\to\infty}(1 + \frac{1}{n})^{n^2}$.

  Solution. We know that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$. In particular, for sufficiently large n, we have $(1 + \frac{1}{n})^n > 2$, and hence $(1 + \frac{1}{n})^{n^2} > 2^n \to \infty$.

- Compute $\lim_{x\to 0} \frac{\log_a(1+x)}{x}$.

  Solution. Use the change of base $\log_a(1 + x) = \log_a e \log(1 + x)$, and hence

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \log_a e \lim_{x\to 0} \frac{\log(1 + x)}{x} = \log_a e.$$

- Compute $\lim_{x\to 0} \frac{a^x-1}{x}$.

  Solution. Use the change of vaiables: $a^x = e^{(\log a)x}$, and if $x \to 0$, then $(\log a)x \to 0$. Therefore,

$$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{x\to 0} \frac{e^{(\log a)x}-1}{x} = \lim_{x\to 0} \frac{e^{(\log a)x}-1}{(\log a)x} \cdot \log a = \lim_{y\to 0} \frac{e^y-1}{y} \cdot \log a = \log a.$$

- Compute $\lim_{x\to 0} \frac{\sinh x}{x}$.

  Solution.

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{e^x-e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x-1+1-e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x-1}{2x} + \frac{e^{-x}-1}{-2x} = \frac{1}{2} + \frac{1}{2} = 1.$$

- Compute $\lim_{x\to \infty} \tanh x$.

  Solution.

$$\lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = 1.$$

- Compute $\lim_{x\to 0} \frac{\sinh x}{e^x-1}$.

  Solution.

$$\lim_{x\to 0} \frac{\sinh x}{e^x-1} = \lim_{x\to 0} \frac{\sinh x}{x} \frac{x}{e^x-1} = 1 \cdot 1 = 1.$$

- Compute $\lim_{x\to 0} \frac{(1+x)^a-1}{x}$.

  Solution.

$$\lim_{x\to 0} \frac{(1+x)^a-1}{x} = \lim_{x\to 0} \frac{e^{a\log(1+x)}-1}{x} = \lim_{x\to 0} \frac{e^{a\log(1+x)}-1}{\log(1+x)} \frac{\log(1+x)}{x}$$
$$= \lim_{y\to 0} \frac{e^{ay}-1}{y} \lim_{x\to 0} \frac{\log(1+x)}{x} = a.$$

Trigonometric functions, open and closed sets, uniform continuity.

- Compute $\cos\frac{5\pi}{4}, \sin\frac{7\pi}{3}, \sin\frac{115\pi}{4}, \sin(-\frac{23\pi}{3})$.

  Solution.

  - $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.
  - $\sin\frac{7\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.
  - $\sin\frac{115\pi}{4} = \sin\frac{3\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.
  - $\sin(-\frac{23\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

- Compute $\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, \sin\frac{\pi}{8}$.

  Solution. Use $\cos^2\theta = \frac{\cos 2\theta+1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$.

  - $\cos\frac{\pi}{12} = \sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}}$.
  - $\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$.
  - $\sin\frac{\pi}{8} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$.

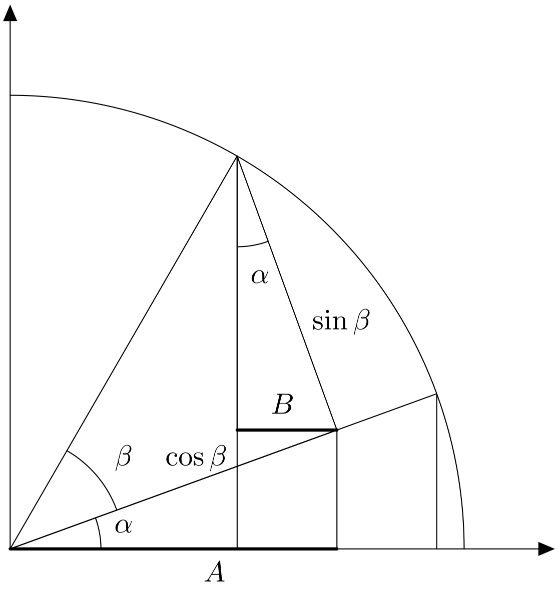


Figure 15: The formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. $A = \cos\beta\cos\alpha, B = \sin\beta\sin\alpha$ and $A - B = \cos(\alpha+\beta)$.

- Compute $\cos\frac{\pi}{4}, \sin\frac{\pi}{4}$ using $\cos\frac{\pi}{2} = 0$ and some of the general formulas.

  Solution. Use $\cos^2\theta = \frac{\cos 2\theta+1}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2}$. $\cos\frac{\pi}{4} = \sqrt{\frac{\cos\frac{\pi}{2}+1}{2}} = \frac{1}{\sqrt{2}}, \sin\frac{\pi}{4} = \sqrt{\frac{1-\cos\frac{\pi}{2}}{2}} = \frac{1}{\sqrt{2}}$.

- What is the domain of $\tan\theta$?

  Solution. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, hence it is defined where $\cos\theta \neq 0$. $\cos\theta = 0$ if and only if $\theta = \frac{(2n+1)\pi}{2}$, hence $\tan\theta$ is defined for $\theta \neq \frac{(2n+1)\pi}{2}$.

- Using the figure, explain the formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

  Solution. See Figure 15.

- Write $\cos 3\theta, \sin 3\theta$ in terms of $\cos\theta, \sin\theta$.

  Solution.

  - $\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (\cos^2\theta - \sin^2\theta)\cos\theta - 2\cos\theta\sin^2\theta$.
  - $\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = 2\cos^2\theta\sin\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$.

- Prove that the union of open sets is open.

  Solution. If $p \in \bigcup_{j\in J} A_j$ and A_j are open, then $p \in A_k$ for some $k \in J$ and there is $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subset A_k \subset \bigcup_{j\in J} A_j$, $\bigcup_{j\in J} A_j$ is open.

- Prove that the intersection of closed sets is closed.

  Solution.

  If $a_n \in \bigcap_{j\in J} A_j$ and A_j are closed, then $a_n \in A_j$ for all $j \in J$ If $a_n \to a$, then $a \in A_j$ for all j because A_j is closed, hence $a \in \bigcap_{j\in J} A_j$ hence $\bigcap_{j\in J} A_j$ is closed.

- Prove that the intersection of two open sets is open.

  Solution. If $p \in A_1 \cap A_2$ and A_1, A_2 are open, then $p \in A_1, A_2$ and there are $\epsilon_1, \epsilon_2 > 0$ such that $(p-\epsilon_1, p+\epsilon_1) \subset A_1, (p-\epsilon_2, p+\epsilon_2) \subset A_2$. Let ϵ be the smallest of the two. Then $(p-\epsilon, p+\epsilon) \subset A_1 \cap A_2$, hence $A_1 \cap A_2$ is open.

- Find an example of intersection of infinitely many open sets which is not open.

  Solution. For example, consider $(-\frac{1}{n}, \frac{1}{n})$. It holds that $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not open.

- Find a subset of $\mathbb{R}$ which is both open and closed.

  Solution. Let A be open and closed (and nonempty). Let $a \in A$. Consider $A^c \cap [a, \infty)$. This is bounded below, hence if it is not empty, there is $\inf(A^c \cap [a, \infty))$. If $x = \inf(A^c \cap [a, \infty)) \notin A^c$, then there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A^c$ because A is closed (hence A^c is closed), hence there are points below x and in $A^c \cap [a, \infty)$, which contradicts that $x = \inf(A^c \cap [a, \infty))$. Hence $x \in A$. But then $(x - \epsilon, x + \epsilon) \subset A$ because A is open, which contradicts that $x = \inf(A^c \cap [a, \infty))$. Therefore, $A^c \cap [a, \infty)$ must be empty. Similarly, $A^c \cap (-\infty, a]$ is empty. That is, $A = \mathbb{R}$. Then indeed A is both open and closed.

- Find a function, continuous defined on $\mathbb{R}$ but bounded.

  Solution. $\sin\theta, \cos\theta, \tanh x$, and so on.

- Find a function, not continuous defined on $\mathbb{R}$ but bounded.

  Solution. $\operatorname{sign} x, x - [x]$, and so on.

- Tell whether $y = \cos x$ admits maxima and minima, and if so, list them up.

  Solution. As $\cos^2 x + \sin^2 x = 1$, it holds that $-1 \leq \cos x \leq 1$. $\cos x = 1$ if and only if $x = 2n\pi, n \in \mathbb{Z}$. $\cos x = -1$ if and only if $x = (2n+1)\pi, n \in \mathbb{Z}$.

- Tell whether $y = \tanh x$ admits maxima and minima, and if so, list them up.

  Solution. As $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, this is monotonically increasing. Indeed,

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

  and if $x > y$, then $1 - e^{-2x} > 1 - e^{-2y}$ while $1 + e^{-2x} < 1 - e^{-2y}$, hence $\tanh x > \tan y$. This means that there is no maxima nor minima.

- Tell whether $y = x$ is uniformly continuous or not, and prove it.

  Solution. For any $x \in \mathbb{R}$ and $\epsilon > 0$, we can take $\delta = \epsilon$, then for y such that $|y - x| < \delta = \epsilon$ we have $|f(y) - f(x)| = |y - x| < \epsilon$. Therefore, this is uniformly continuous.

- Tell whether $y = x^2$ is uniformly continuous or not, and prove it.

  Solution. Let $\epsilon = 1$. For any $\delta > 0$, we can take $x > \frac{1}{\delta}$ then $f(x+\delta) - f(x) = (x+\delta)^2 - x^2 = 2x\delta + \delta^2 > 2 > \epsilon$. Therefore, this is not uniformly continuous.

- Tell whether $y = \sin x$ is uniformly continuous or not, and prove it.

  Solution. By the Heine-Cantor theorem, $y = \sin x$ restricted to $[0, 4\pi]$ is uniformly continuous. That is, for any $\epsilon > 0$ there is $\delta > 0$ such that $|\sin(x) - \sin(y)| < \epsilon$ if $|x - y| < \delta, x, y \in [0, 4\pi]$. Then, for any $x, y \in \mathbb{R}$ such that $|x - y| < \delta$, there is n such that $x + 2n\pi, y + 2n\pi \in [0, 4\pi]$. Therefore, $|f(x) - f(y)| = |f(x + 2n\pi) - f(y + 2n\pi)| < \epsilon$. Therefore, this is uniformly continuous.

- Tell whether $y = \tanh x$ is uniformly continuous or not, and prove it.

  Solution. Let $\epsilon > 0$.

  We know that $\lim_{x\to\infty} \tanh x = 1, \lim_{x\to-\infty} \tanh x = -1$. Therefore, there is $M > 0$ such that $1 - \frac{\epsilon}{2} < \tan x < 1$ for $x > M$. Similarly, $-1 < \tan x < -1 + \frac{\epsilon}{2}$ for $x < -M$. On the other hand, on $[-M,M]$, $\tanh x$ is uniformly continuous, hence there is $\delta>0$ such that if $|x-y|<\delta$ then $|\tanh x-\tanh y|<\frac{\epsilon}{2}$.

Then, for any two points x,y such that $|x-y|<\delta$, $|\tanh x-\tanh y|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$ by possibly taking the point in the middle M or $-M$. Therefore, this is uniformly continuous.

Derivative and some applications.

- Compute the derivative of $f(x)=\begin{cases}x^2 & \text{for } x\geq 0\\ 0 & \text{for } x<0\end{cases}$.

  Solution. For $x\neq 0$ we know $f'(x)=\begin{cases}2x & \text{for } x>0\\ 0 & \text{for } x<0\end{cases}$. For $x=0$, let us compute the left and right derivatives. We compute $\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{0-0}{h}=0$ and $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h^2-0}{h}=\lim_{h\to 0^+}h=0$. So the left and right derivatives coincide, therefore, $f'(0)=0$.

- Tell whether $f(x)=\begin{cases}x\sin\frac{1}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$ is continuous and is differentiable at $x=0$.

  Solution. We have $\frac{f(h)-f(0)}{h}=\frac{h\sin\frac{1}{h}-0}{h}=\sin\frac{1}{h}$ and this does not have the limit $h\to 0$. But it is continuous at $x=0$, because $|\sin\frac{1}{x}|\leq 1$, hence $\lim_{x\to 0}|x\sin\frac{1}{x}|\leq\lim_{x\to 0}|x|=0$.

- Tell whether $f(x)=\begin{cases}x^2\sin\frac{1}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$ is continuous and is differentiable at $x=0$.

  Solution. We have $\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$, hence it is differentiable and in particular continuous.

- Compute the derivative of $f(x)=x^3$ based on the definition.

  Solution.

$$\begin{aligned}\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}&=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\&=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2.\end{aligned}$$

- Compute the derivative of $f(x)=x^2+x$ based on the definition.

  Solution.

$$\begin{aligned}\lim_{h\to 0}\frac{(x+h)^2+(x+h)-x^2-x}{h}&=\lim_{h\to 0}\frac{x^2+2xh+h^2+(x+h)-x^2-x}{h}\\&=\lim_{h\to 0}(2x+h+1)=2x+1.\end{aligned}$$

- Compute the derivative: $f(x)=x^2-\cos(3x)$.

  Solution. By the chain rule and linearity, it is $2x+3\sin(3x)$.

- Compute the derivative: $f(x)=\sqrt{x^2+1}$.

  Solution. By the chain rule with $\sqrt{x^2+1}=(x^2+1)^{\frac{1}{2}}$, it is $2x\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}=\frac{x}{\sqrt{x^2+1}}$.

- Compute the derivative: $f(x)=\sin(\frac{x+2}{e^x})$.

  Solution. By the chain rule, $f'(x)=\frac{e^x-e^x(x+2)}{e^{2x}}\cos(\frac{x+2}{e^x})=-e^{-x}(x+1)\cos(\frac{x+2}{e^x})$.

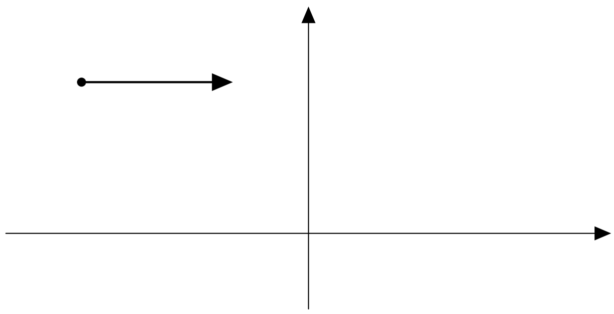


Figure 16: The x-axis is the beach, and the boat sails on the line $y = 4$.

- Compute the derivative: $f(x) = \sin(\cos(x^2))$.

  Solution. By the chain rule, $f'(x) = D(\cos(x^2))\cos(\cos(x^2)) = -2x\sin(x^2)\cos(\cos(x^2))$.

- Compute the derivative: $f(y) = \log y$, using that $\log y$ is the inverse function of e^x.

  Solution. With $y = e^x$, $f'(y) = \frac{1}{e^x} = \frac{1}{y}$.

- Compute the derivative: $f(y) = \sqrt{y}$ using that $\sqrt{y}$ is the inverse function of x^2.

  Solution. With $y = x^2$, $f'(y) = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$.

- Find the stationary points of $y = x^3 - 3x^2 + 3x$.

  Solution. With $f(x) = x^3 - 3x^2 + 3x$, $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$, hence $x = 1$ is the only stationary point.

- Find the stationary points of $y = \sin(x^2)$.

  Solution. With $f(x) = \sin(x^2)$, $f'(x) = 2x\cos(x^2)$, and $f'(x) = 0$ if and only if $x = 0$ or $\cos(x^2) = 0$, hence $x = 0$ or $x = \pm\sqrt{\frac{n\pi}{2}}$ for $n \in \mathbb{N}$ odd.

- Consider the relation $y^2 - x^2 = 1$. Write y as an explicit function of x, and take the derivative. Differentiate it implicitly and find a relation.

  Solution. $y(x)^2 = x^2 + 1$ and hence $y(x) = \pm\sqrt{x^2+1}$ and $y'(x) = \pm\frac{x}{\sqrt{x^2+1}}$, we see the relation $y'(x)y(x) = x$.

  By differentiating the relation, we obtain $2y(x)y'(x) = 2x$, and hence $y(x)y'(x) = x$.

- Consider the relation $y^5 + xy - 2x^3 = 0$. Check that $(x, y) = (1, 1)$ satisfy this equation. Assume that this defines an implicit function $y(x)$, and compute $y'(1)$.

  Solution. $1^5 + 1 \cdot 1 - 2 \cdot 1^3 = 0$. We have $5y'(x)y(x)^4 + y + xy'(x) - 6x^2 = 0$, and hence $y'(1) = \frac{6-1}{5+1} = \frac{6}{5}$.

- A boat sails parallel to a straight beach at a constant speed of 12 miles per hour, staying 4 miles offshore. How fast is it approaching a lighthouse on the shoreline at the instant it is exactly 5 miles from the lighthouse?

  Solution. Let us say that at time t the boat is at the position $(12t, 4)$, and the lighthouse is at $(0, 0)$. The distance between the lighthouse and the boat is $r(t) = \sqrt{(12t)^2 + 4^2} = 4\sqrt{9t^2+1}$, or $r(t)^2 = (12t)^2 + 4^2$.

  The speed with which the boat approaches the lighthouse is $r'(t)$. By differentiating the above relation by t, we have $2r(t)r'(t) = 288t$. Furthermore, When $r(t) = 5$, we have $t = \pm\frac{1}{4}$. Therefore, $2 \cdot 5r'(\pm\frac{1}{4}) = \pm 72$ and $r'(t) = \pm\frac{36}{5}$.

Higher derivatives and curve sketching.

- Determine where the function is increasing or decreasing. $f(x) = x^3 - 3x$.

  Solution. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. Hence f is increasing when $f'(x) > 0$, that is $x < -1$ or $x > 1$ and f is decreasing when $f'(x) < 0$, that is $-1 < x < 1$.

- Determine where the function is increasing or decreasing. $f(x) = e^x - x$.

  Solution. $f'(x) = e^x - 1$. Hence f is increasing when $f'(x) > 0$, that is $x > 0$ and f is decreasing when $f'(x) < 0$, that is $x < 0$.

- Determine where the function is increasing or decreasing. $f(x) = x + \frac{1}{x}$, $x > 0$.

  Solution. $f'(x) = 1 - \frac{1}{x^2}$. Hence f is increasing when $f'(x) > 0$, that is $x > 1$ and f is decreasing when $f'(x) < 0$, that is $x < 1$.

- Determine where the function is increasing or decreasing. $f(x) = \frac{x}{x^2+1}$.

  Solution. $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-(x-1)(x+1)}{(x^2+1)^2}$. Hence f is increasing when $f'(x) > 0$, that is $-1 < x < 1$ and f is decreasing when $f'(x) < 0$, that is $x < -1, x > 1$.

- Find the local maxima and minima using the second derivative. $f(x) = 2x^3 - 3x^2$.

  Solution. $f'(x) = 6x^2 - 6x = 6x(x-1)$. Hence $f'(x) = 0$ if and only if $x = 0, 1$. $f''(x) = 12x - 6$, $f''(0) = -6 < 0$ hence $x = 0, f(0) = 0$ is a local maximum, while $f''(1) = 6 > 0$ hence $x = 1, f(1) = -1$ is a local minimum.

- Find the local maxima and minima using the second derivative. $f(x) = xe^x$.

  Solution. $f'(x) = e^x + xe^x = e^x(x+1)$. Hence $f'(x) = 0$ if and only if $x = -1$. $f''(x) = 2e^x + xe^x$, $f''(-1) = \frac{2}{e} - \frac{1}{e} = \frac{1}{e} > 0$ hence $x = -1, f(-1) = -\frac{1}{e}$ is a local minimum.

- Find the asymptotes of $f(x) = \sqrt{x^2+1}$.

  Solution.

  $$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{1}{x^2}} = 1$$

  and

  $$\lim_{x\to\infty} \sqrt{x^2+1} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0.$$

  Hence $y = x$ is an asymptote for $x \to \infty$. Similarly,

  $$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} -\sqrt{1 + \frac{1}{x^2}} = -1$$

  and

  $$\lim_{x\to-\infty} \sqrt{x^2+1} - (-x) = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1} - x} = 0.$$

  Hence $y = -x$ is an asymptote for $x \to -\infty$.

- Find the asymptotes of $f(x) = x + \frac{1}{x}$.

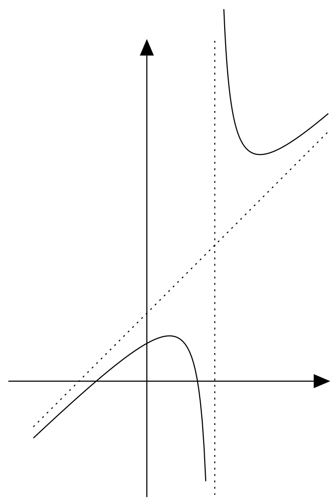
  Solution.

  $$\lim_{x\to\infty} \frac{x + \frac{1}{x}}{x} = \lim_{x\to\infty} 1 + \frac{1}{x^2} = 1,$$

  and

  $$\lim_{x\to\infty} x + \frac{1}{x} - x = \lim_{x\to\infty} \frac{1}{x} = 0.$$

  hence $y = x$ is an asymptote for $x \to \infty$. Similarly, $y = -x$ an asymptote for $x \to -\infty$.



- Sketch the graph of $f(x) = \frac{x^2-5}{x-3}$.

  Solution.

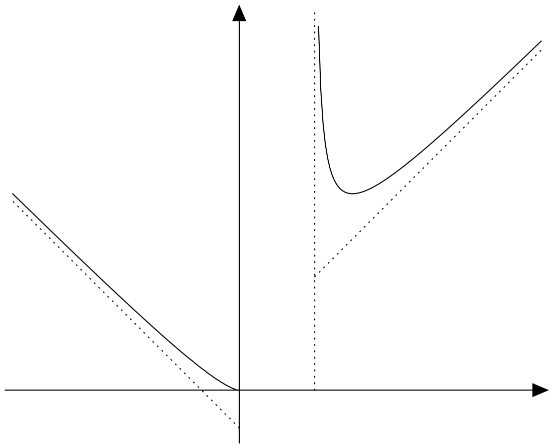
  - Domain: $x \neq 3$.
  - Vertical asymptote: $x = 3$.
    Oblique asymptotes: $\lim_{x\to\infty} \frac{x^2-5}{x(x-3)} = 1$, and
    $$\lim_{x\to\infty} \frac{x^2-5}{x-3} - x = \lim_{x\to\infty} \frac{x^2-5-x(x-3)}{x-3} = \lim_{x\to\infty} \frac{3x-5}{x-3} = 3,$$
    hence $y = x+3$ is an asymptote for $x \to \infty$. Similarly, $y = x+3$ is an asymptote for $x \to -\infty$.
  - $f'(x) = \frac{2x(x-3)-(x^2-5)}{(x-3)^2} = \frac{x^2-6x+5}{(x-3)^2} = \frac{(x-1)(x-5)}{(x-3)^2}$. $f'(x) > 0$ if and only if $x < 1, x > 5$ and $f'(x) < 0$ if and only if $1 < x < 3, 3 < x < 5$.
  - From this, $x = 1$ is a local maximum and $x = 5$ is a local minimum.

x		1		3		5	
$f'(x)$	$+$	0	$-$	nd	$-$	0	$+$
$f''(x)$		$-$		nd		$+$	
$f(x)$	$\nearrow$	2	$\searrow$	nd	$\searrow$	10	$\nearrow$

- Sketch the graph of $f(x) = \sqrt{\frac{x^3}{x-1}}$.

  Solution.

  - Domain: $x \neq 1$, and $\frac{x^3}{x-1} \geq 0$, that is $x > 1$ or $x \leq 0$.
  - Vertical asymptote: $x = 1$.
    Oblique asymptotes: $\lim_{x\to\infty} \sqrt{\frac{x^3}{x-1}}\frac{1}{x} = \lim_{x\to\infty} \sqrt{\frac{x^3}{x^2(x-1)}} = 1$, and
    $$\lim_{x\to\infty} \sqrt{\frac{x^3}{x-1}} - x = \lim_{x\to\infty} x\left(\sqrt{\frac{x}{x-1}} - 1\right)$$
    $$= \lim_{x\to\infty} x\frac{(\sqrt{x}-\sqrt{x-1})(\sqrt{x}+\sqrt{x-1})}{\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})} = \lim_{x\to\infty} x\frac{x-(x-1)}{\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})} = \frac{1}{2}$$



hence $y = x + \frac{1}{2}$ is an asymptote for $x \to \infty$. Similarly, $\lim_{x\to-\infty} \sqrt{\frac{x^3}{x-1}}\frac{1}{x} = \lim_{x\to-\infty} -\sqrt{\frac{x^3}{x^2(x-1)}} = -1$, and

$$\lim_{x\to-\infty} \sqrt{\frac{x^3}{x-1}} - (-x) = \lim_{x\to-\infty} x\left(-\sqrt{\frac{-x}{-x+1}} + 1\right)$$
$$= \lim_{x\to-\infty} x\frac{(\sqrt{-x} - \sqrt{-x+1})(\sqrt{-x} + \sqrt{-x+1})}{\sqrt{-x+1}(\sqrt{-x} + \sqrt{-x+1})}$$
$$= \lim_{x\to-\infty} x\frac{-x-(-x+1)}{\sqrt{-x+1}(\sqrt{-x} + \sqrt{-x+1})} = -\frac{1}{2}$$

hence $y = -x - \frac{1}{2}$ is an asymptote for $x \to -\infty$.

– $f'(x) = \frac{3x^2(x-1)-x^3}{2(x-1)^2 f(x)} = \frac{x^2(2x-3)}{2(x-1)^2 f(x)}$. $f'(x) > 0$ if $x > \frac{3}{2}$ and $f'(x) < 0$ if $x < \frac{3}{2}$. $f(\frac{3}{2}) = \sqrt{\frac{27}{4}}$.

x		0		1		$\frac{3}{2}$	
$f'(x)$	$-$		nd		$-$	0	$+$
$f''(x)$			nd			$+$	
$f(x)$	$\searrow$	0		nd	$\searrow$	$\sqrt{\frac{27}{4}}$	$\nearrow$

- A truck is to be driven 300 miles on a freeway at a constant speed of x miles per hour. Speed laws require $30 < x < 60$. Assume that fuel is consumed at the rate of $2 + x^2/600$ gallons per hour. Which speed should the track driver go to save the fuel cost?

  Solution. The truck has to drive fo $300/x$ hours, and then consumes $f(x) = \frac{300}{x}(2 + \frac{x^2}{600}) = \frac{600}{x} + \frac{x}{2}$ gallons.

  Considering this as a function of x, we find its minimum in $30 < x < 60$. $f'(x) = -\frac{600}{x^2} + \frac{1}{2}$, hence $f(x) = 0$ if $x = \sqrt{1200} \cong 34.6$. $f''(x) = \frac{1200}{x^3}$, hence this is a local minimum.

Bernoulli-de l'Hôpital rule, higher order Taylor formula.

- Compute the limit. $\lim_{x\to0} \frac{\sin^2 x}{x^2}$.

  Solution. We have $D(\sin^2 x) = 2\sin x \cos x, D(x^2) = 2x$, and hence $\lim_{x\to0} \frac{\sin^2 x}{x^2} = \lim_{x\to0} \frac{2\sin x\cos x}{2x} = 1$.

- Compute the limit. $\lim_{x\to0} \frac{\sin x - x}{x^3}$.

Solution. We have $D(\sin x - x) = \cos x - 1, D(x^3) = 3x^2$, and further $D(\cos x - 1) = -\sin x, D(3x^2) = 6x$ hence

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos -1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}.$$

- Compute the limit. $\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x}$.

  Solution. We have $D(\log(x^3+1)) = \frac{3x^2}{x^3+1}, D(\log x) = \frac{1}{x}$ and hence

$$\lim_{x\to\infty} \frac{\log(x^3+1)}{\log x} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3x^3}{x^3+1} = 3.$$

- Compute the limit. $\lim_{x\to 0} x \log x$.

  Solution. We have $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ and $D(\log x) = \frac{1}{x}, D(\frac{1}{x}) = \frac{-1}{x^2}$ and hence $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x) = 0$.

- Find the second order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.

  Solution. We have $f'(x) = 2x\cos(x^2), f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$ and $f(0) = 0, f'(0) = 0, f''(0) = 2$, and hence $f(x) = 0 + 0x + \frac{2x^2}{2!} + o(x^2) = x^2 + o(x^2)$ as $x \to 0$.

- Find the second order Taylor formula. $f(x) = \sqrt{x^2+1}$ as $x \to 1$.

  Solution. We have $f'(x) = \frac{x}{\sqrt{x^2+1}}, f''(x) = \frac{\sqrt{x^2+1} - x\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ and hence $f(1) = \sqrt{2}, f'(1) = \frac{1}{\sqrt{2}}, f''(1) = \frac{1}{2^{\frac{3}{2}}}$. Therefore, $f(x) = \sqrt{2} + \frac{(x-1)}{\sqrt{2}} + \frac{(x-1)^2}{2^{\frac{5}{2}}} + o((x-1)^2)$ as $x \to 1$.

- Find the second order Taylor formula. $f(x) = \sin(x) - 1$ as $x \to \frac{\pi}{2}$.

  Solution. We have $f'(x) = \cos x, f''(x) = -\sin x$ and hence $f(\frac{\pi}{2}) = 0, f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1$. Therefore, $f(x) = -\frac{(x-\frac{\pi}{2})^2}{2} + o((x-\frac{\pi}{2})^2)$ as $x \to \frac{\pi}{2}$.

- Find the second order Taylor formula. $f(x) = \frac{e^x - 1}{\cos x}$ as $x \to 0$.

  Solution. We have $f'(x) = \frac{e^x\cos x + (e^x-1)\sin x}{\cos^2 x}$,

$$\begin{aligned}
&f''(x)\\
&= \frac{(e^x(\cos x - \sin x) + (e^x \sin x + (e^x-1)\cos x))\cos^2 x}{\cos^4 x}\\
&\quad - \frac{(e^x\cos x + (e^x-1)\sin x)(-2\sin x\cos x)}{\cos^4 x}\\
&= \frac{(2e^x-1)\cos^3 x + 2e^x \sin x\cos^2 x + 2e^x\sin^2 x\cos x - 2\sin^2 x\cos x}{\cos^4 x}
\end{aligned}$$

  and hence $f(0) = 0, f'(0) = 1, f''(0) = 1$. Therefore, $f(x) = x + \frac{x^2}{2} + o(x^2)$ as $x \to 0$.

- Find the n-th order Taylor formula. $f(x) = \cos(x)$ as $x \to 0$.

  Solution. $f^{(4n)}(x) = \cos x, f^{(4n+1)}(x) = -\sin x, f^{(4n+2)}(x) = -\cos x, f^{(4n+3)}(x) = \sin x$, and hence $f^{(4n)}(0) = 1, f^{(4n+1)}(0) = 0, f^{(4n+2)}(0) = -1, f^{(4n+3)}(0) = 0$, and

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n})\\
&= \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n}).
\end{aligned}$$

- Find the n-th order Taylor formula. $f(x) = \log(1+x)$ as $x \to 0$.

  Solution. $f^{(2n)}(x) = \frac{-(2n-1)!}{(1+x)^{2n}}, f^{(2n+1)}(x) = \frac{(2n)!}{(1+x)^{2n+1}}$, and $f^{(2n)}(0) = (2n-1)!, f^{(2n+1)}(0) = -(2n)!$, and

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} + o(x^n)$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} + o(x^n).$$

- Find the n-th order Taylor formula. $f(x) = \sin(x^2)$ as $x \to 0$.

  Solution. We know $\sin y = \sum_{k=0}^{n} \frac{(-1)^k y^{2k+1}}{(2k+1)!} + o(y^{2n+1})$ as $y \to 0$ and hence $\sin(x^2) = \sum_{k=0}^{n} \frac{(-1)^k x^{4k+2}}{(2k+1)!} + o(x^{4n+2})$ as $x \to 0$

- Compute the limit.

$$\lim_{x\to 0} \frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}.$$

  Solution. As $x \to x_0 = 0$, we have

  - $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
  - $\cos x = 1 - \frac{x^2}{2} + o(x^3)$
  - $\sin x = x - \frac{x^3}{6} + o(x^3)$
  - $\tan(2x^3) = 2x^3 + o(x^3)$

  Then it holds, as $x \to 0$,

$$\frac{e^x + \cos(x) - \sin(x) - 2}{\tan(2x^3)}$$
$$= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + 1 - \frac{x^2}{2} - x + \frac{x^3}{6} - 2 + o(x^3)}{2x^3 + o(x^3)}$$
$$= \frac{\frac{x^3}{3} + o(x^3)}{2x^3 + o(x^3)}$$

  hence $\lim_{x\to 0} \frac{x - \ln(1-x) - 2x\sqrt{1+x}}{\sin(x) - xe^x} = \frac{1}{6}$.

- For which α does the following limit exist?

$$\lim_{x\to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)}$$

  Solution. As $x \to x_0 = 0$, we have

  - $\frac{1+y}{1-y} = 1 + 2y + o(y)$ and $\frac{1+x^2}{1-x^2} = 1 + 2x^2 + o(x^2)$
  - $\sin x = x + o(x^2)$
  - $1 - \cos x = \frac{x^2}{2} + o(x^2)$

  Then it holds, as $x \to 0$,

$$\frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x)}{1 - \cos(x)} = \frac{1 + 2x^2 - \alpha x - 1 + o(x^2)}{\frac{x^2}{2} + o(x^2)}$$

  hence $\lim_{x\to 0} \lim_{x\to 0} \frac{\left(\frac{1+x^2}{1-x^2}\right) - \alpha \sin(x) - 1}{1 - \cos(x)}$ exists if and only if $\alpha = 0$, and in that case, the limit is 4.

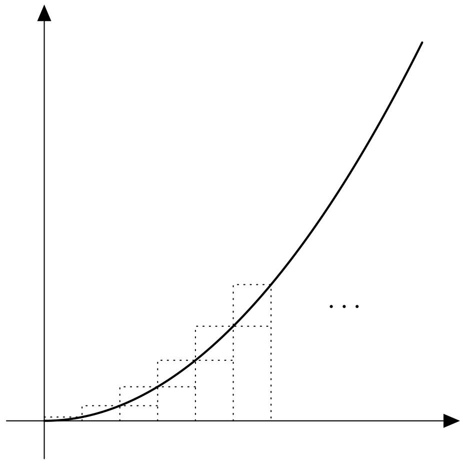


Figure 17: The upper and lower sum for $f(x) = x^2$.

Integral, primitive (and Taylor formula).

- Compute the integral $\int_0^1 x^2 dx$ based on the definition (using the upper and lower sums).
  Solution. Let us take a partition $P_n = \{[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots, [\frac{n-1}{n}, 1]\}$ of $[0, 1]$.
  - Let $f(x) = x^2$.

$$\begin{aligned}\overline{S}_I(f, P_n) &= \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \cdot \sum_{j=1}^{n} j^2 \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

  Analogously,

$$\begin{aligned}\underline{S}_I(f, P_n) = \sum_{j=1}^{n} \left(\frac{j-1}{n}\right)^2 \cdot \frac{1}{n} &= \frac{1}{n^3} \cdot \sum_{j=1}^{n} (j-1)^2 \\ &= \frac{1}{n^3} \cdot \sum_{j=0}^{n-1} j^2 \\ &= \frac{1}{n^3} \cdot \frac{(n-1)n(2(n-1)+1)}{6}\end{aligned}$$

  Therefore, by taking $n \to \infty$, we obtain $\lim_{n\to\infty} \overline{S}_I(f, P_n) = \lim_{n\to\infty} \underline{S}_I(f, P_n) = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$ while $\lim_{n\to\infty} \underline{S}_I(f, P_n) = \lim_{n\to\infty} \underline{S}_I(f, P_n) \frac{1}{n^3} \cdot \frac{(n-1)n(2(n-1)+1)}{6} = \frac{1}{3}$
  .

- Compute $\int_{-1}^{1} (x^4 + (x-2)^3 + x(x-1)) dx$.

Solution. We have

$$\begin{aligned}
&\int_{-1}^{1}(x^4+(x-2)^3+x(x-1))dx \\
&= [\frac{x^5}{5}+\frac{(x-2)^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}]_{-1}^{1} \\
&= (\frac{1}{5}+\frac{1}{4}+\frac{1}{3}-\frac{1}{2})-(\frac{-1}{5}+\frac{81}{4}+\frac{-1}{3}-\frac{1}{2}) \\
&= \frac{2}{5}-20+\frac{2}{3}=\frac{6-300+10}{15}=-\frac{284}{15}.
\end{aligned}$$

- Compute $\int_0^{\frac{\pi}{2}}\sin(2(x+\frac{\pi}{6}))dx$. We have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\sin(2(x+\frac{\pi}{6}))dx \\
&= [-\frac{1}{2}\cos(2(x+\frac{\pi}{6}))]_0^{\frac{\pi}{2}} \\
&= -\frac{1}{2}(\cos\frac{2\pi}{3}-\cos\frac{\pi}{3}) \\
&= -\frac{1}{2}(-\frac{1}{2}-\frac{1}{2})=\frac{1}{2}.
\end{aligned}$$

  Solution.

- Compute $\int_{-1}^{1}e^{2(x-1)}dx$.

  Solution. We have

$$\begin{aligned}
&\int_{-1}^{1}e^{2(x-1)}dx \\
&= [\frac{1}{2}e^{2(x-1)}]_{-1}^{1} \\
&= \frac{1}{2}(e^0-e^{-4}) \\
&= \frac{1}{2}(1-e^{-4})
\end{aligned}$$

- Compute $\int_1^2\frac{x^2+3x+1}{x}dx$.

  Solution.

$$\begin{aligned}
&\int_1^2\frac{x^2+3x+1}{x}dx \\
&= \int_1^2(x+3+\frac{1}{x})dx \\
&= [\frac{x^2}{2}+3x+\log x]_1^2 \\
&= (\frac{4}{2}+6+\log 2)-(\frac{1}{2}+3+\log 1) \\
&= \frac{9}{2}+\log 2
\end{aligned}$$

- Compute $\int_0^{\pi}\sin^2 xdx$.

Solution. Note that $\cos 2x = 1 - 2\sin^2 x$, hence $\sin^2 x = \frac{1-\cos 2x}{2}$ and

$$\begin{aligned}
&\int_0^\pi \sin^2 x dx \\
&= \int_0^\pi \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{2}[x - \frac{1}{2}\sin 2x]_0^\pi \\
&= \frac{1}{2}((\pi - 0) - (0-0)) = \frac{\pi}{2}.
\end{aligned}$$

- Compute $\int_0^1 \frac{x^2}{1+x^2}dx$.

  Solution. Note that $D(\arctan x) = \frac{1}{x^2+1}$.

$$\begin{aligned}
&\int_0^1 \frac{x^2}{1+x^2}dx \\
&= \int_0^1 \frac{x^2+1-1}{1+x^2}dx \\
&= [x - \arctan x]_0^1 \\
&= (1 - \frac{\pi}{4}) - (0-0) = 1 - \frac{\pi}{4}
\end{aligned}$$

- Find the 2nd order Taylor formula for $f(x) = \sqrt{1+2x}$ around $x = 0$.

  Solution. We have $f'(x) = 2 \cdot \frac{1}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$, $f''(x) = 2 \cdot (-\frac{1}{2}\frac{1}{(1+2x)^{\frac{3}{2}}}) = -\frac{1}{(1+2x)^{\frac{3}{2}}}$.

  Therefore, $f(x) = 1 + x + \frac{-x^2}{2} + o(x^2)$.

- Find the 3nd order Taylor formula for $f(x) = \log(x+1)$ around $x = 2$.

  Solution. We have $f'(x) = \frac{1}{x+1}$, $f''(x) = -\frac{1}{(x+1)^2}$.

  Therefore, $f(x) = \log 3 + \frac{(x-2)}{3} - \frac{(x-2)^2}{18} + o((x-2)^2)$.

- Compute the limit. $\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}}$.

  Solution. We have $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$, and hence

$$\lim_{x\to 0} \frac{x^4}{\cos x - 1 + \frac{x^2}{2}} = \lim_{x\to 0} \frac{x^4}{\frac{x^4}{24} + o(x^4)} = 24.$$

- Determine $\alpha \in \mathbb{R}$ for which the limit $\lim_{x\to 0} \frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6}-1}$ exists, and in that case, compute the limit.

  Solution. We have

  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$
  - $\sin^2 x = x^2 - \frac{x^4}{3} + (\frac{2}{5!} + (\frac{1}{3!})^2)x^6 + o(x^6)$
  - $e^y = 1 + y + o(y)$
  - $e^{x^6} = 1 + x^6 + o(x^6)$

  and hence

$$\frac{(\sin x)^2 - x^2 + \alpha x^4}{e^{x^6} - 1} = \frac{(\alpha - \frac{1}{3})x^4 + \frac{4}{45}x^6 + o(x^6)}{x^6 + o(x^6)}$$

  The limit $x \to 0$ exists if and only if $\alpha - \frac{1}{3} = 0$, that is, $\alpha = \frac{1}{3}$ and in that case, the limit is $\frac{4}{45}$.

Integral calculus.

- Calculate the indefinite integral. $\int xe^x dx$.

  Solution. By integration by parts,

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx + C \\ &= xe^x - e^x + C.\end{aligned}$$

- Calculate the indefinite integral. $\int e^x \sin x dx$.

  Solution. By integration by parts,

$$\begin{aligned}\int e^x \sin x dx &= e^x \sin x - \int e^x \cos x dx + C \\ &= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x) dx\right) + C,\end{aligned}$$

  hence $\int e^x \sin x dx = \frac{1}{2}(e^x(\sin x - \cos x)) + C$.

- Calculate the definite integral. $\int_0^1 x^2 e^{-x} dx$.

  Solution. By integration by parts,

$$\begin{aligned}\int_0^1 x^2 e^{-x} dx &= [-x^2 e^{-x}]_0^1 + \int_0^1 2xe^{-x} dx \\ &= -\frac{1}{e} - [2xe^{-x}]_0^1 + \int_0^1 2e^{-x} dx \\ &= -\frac{3}{e} - [2e^{-x}]_0^1 = 2 - \frac{5}{e}\end{aligned}$$

- Calculate the indefinite integral. $\int x\sqrt{1-x^2} dx$.

  Solution. By substitution $\varphi(x) = -x^2, \varphi'(x) = -2x$,

$$\begin{aligned}\int x\sqrt{1-x^2} dx &= -\frac{1}{2}\int (-2x)\sqrt{1-x^2} dx = -\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}} + C \\ &= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

- Calculate the indefinite integral. $\int xe^{x^2} dx$.

  Solution. By substitution $\varphi(x) = x^2, \varphi'(x) = 2x$,

$$\int xe^{x^2} dx = \frac{1}{2}\int 2xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

- Calculate the definite integral. $\int_0^1 x^3 e^{x^2} dx$.

  Solution. By integration by parts and substitution $\varphi(x) = x^2, \varphi'(x) = 2x$,

$$\begin{aligned}\int_0^1 x^3 e^{x^2} dx &= \frac{1}{2}\int_0^1 x^2 \cdot 2xe^{x^2} dx = \frac{1}{2}[x^2 e^{x^2}]_0^1 - \frac{1}{2}\int_0^1 2xe^{x^2} dx \\ &= \frac{e}{2} - \frac{1}{2}[e^{x^2}]_0^1 = \frac{1}{2}.\end{aligned}$$

- Calculate the indefinite integral. $\int \frac{x}{x^2-1}dx$.

  Solution 1. By substitution $\varphi(x) = x^2-1, \varphi'(x) = 2x$, $\int \frac{x}{x^2-1}dx = \frac{1}{2}\log|x^2-1|+C$.

  Solution 2. Let us find the partial fractions. $x^2-1 = (x-1)(x+1)$.

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)},$$

  and from this we have $x = A(x+1)+B(x-1) = (A+B)x+A-B$, therefore, $A-B = 0, A+B = 1$, and $A = \frac{1}{2}, B = \frac{1}{2}$.

$$\int \frac{x}{x^2-1}dx = \int \left(\frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right)dx = \frac{1}{2}(\log|x-1|+\log|x+1|)+C.$$

- Calculate the definite integral. $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx$.

  Solution. Let us find the partial fractions. $x^3-2x^2+x-2 = (x^2+1)(x-2)$.

$$\frac{1}{x^3-2x^2+x-2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2} = \frac{(Ax+B)(x-2)+C(x^2+1)}{(x^2+1)(x-2)}$$

  and from this we have $1 = (A+C)x^2+(B-2A)x+(C-2B)$, and hence $A+C = 0, B-2A = 0, C-2B = 1$. By solving this, $C = \frac{1}{5}, A = -\frac{1}{5}, B = -\frac{2}{5}$. That is,

$$\begin{aligned}\int \frac{1}{x^3-2x^2+x-2}dx &= \frac{1}{5}\int\left(\frac{-x-2}{x^2+1} + \frac{1}{x-2}\right)dx \\ &= \frac{1}{10}(-\log(x^2+1) - 4\arctan x + 2\log|x-2|).\end{aligned}$$

  Therefore, $\int_0^1 \frac{1}{x^3-2x^2+x-2}dx = -\frac{\pi}{10} - \frac{3\log 2}{10}$.

- Calculate the indefinite integral. $\int \frac{1}{\cos x}dx$.

  Solution 1. By substitution $t = \varphi(x) = \sin x, \varphi'(x) = \cos t$,

$$\begin{aligned}\int \frac{1}{\cos x}dx &= \int \frac{1}{\cos^2 x}\cos x dx \\ &= \int \frac{1}{1-\sin^2 x}\varphi'(x)dx = \int \frac{1}{1-t^2}dt \\ &= \frac{1}{2}\int(\frac{1}{1+t} + \frac{1}{1-t})dt = \frac{1}{2}\log\frac{|1+t|}{|1-t|}.\end{aligned}$$

  That is, $\int \frac{1}{\cos x}dx = \frac{1}{2}\log\frac{|\sin x+1|}{|\sin x-1|}$.

  Solution 2. By change of variables $x = \varphi(t) = 2\arctan t$, or $t = \tan\frac{x}{2}$, $\varphi'(t) = \frac{2}{t^2+1}, \cos x = \frac{1-t^2}{t^2+1}$,

$$\int \frac{1}{\cos x}dx = \int \frac{2}{1-t^2}dt = \int(\frac{1}{t+1} - \frac{1}{t-1})dt = \log\frac{|t+1|}{|t-1|}.$$

  That is, $\int \frac{1}{\cos x}dx = \log\frac{|\tan\frac{x}{2}+1|}{|\tan\frac{x}{2}-1|}$.

- Calculate the indefinite integral. $\int \frac{1}{\cos x+\sin x}dx$.

Solution. By change of variables By change of variables $x = \varphi(t) = 2\arctan t$, or $t = \tan\frac{x}{2}$, $\varphi'(t) = \frac{2}{t^2+1}, \sin x = \frac{2t}{t^2+1}$,

$$\begin{aligned}
\int \frac{1}{\cos x + \sin x} dx &= \int \frac{1}{\frac{1-t^2}{t^2+1} + \frac{2t}{t^2+1}} \cdot \frac{2}{t^2+1} dt \\
&= -\int \frac{2}{t^2 - 2t - 1} dt \\
&= -\frac{1}{\sqrt{2}} \int \left(\frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}} \right) dt \\
&= -\frac{1}{\sqrt{2}} \log \frac{|t-1-\sqrt{2}|}{|t-1+\sqrt{2}|}.
\end{aligned}$$

That is, $\int \frac{1}{\cos x+\sin x} dx = -\frac{1}{\sqrt{2}} \log \frac{|\tan\frac{x}{2}-1-\sqrt{2}|}{|\tan\frac{x}{2}-1+\sqrt{2}|}$.

- Calculate the definite integral. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2}dx$.

  Solution. Change of variables $x = 2\sin t$. Note that $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, with $\frac{dx}{dt} = 2\cos t$,

$$\begin{aligned}
\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4-x^2}dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{4 - 4\sin^2 x} \cdot 2\cos t dt \\
&= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 t dt \\
&= 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos(2t)+1}{2} dt \\
&= 4\left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2 + \pi
\end{aligned}$$

- Calculate the indefinite integral. $\int_0^2 \sqrt{8-x^2}dx$.

  Solution. Change of variables $x = 2\sqrt{2}\sin t$. Note that $\sin 0 = 0, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, with $\frac{dx}{dt} = 2\sqrt{2}\cos t$,

$$\begin{aligned}
\int_0^2 \sqrt{4-x^2}dx &= \int_0^{\frac{\pi}{4}} \sqrt{8 - 8\sin^2 x} \cdot 2\sqrt{2}\cos t dt \\
&= 8\int_0^{\frac{\pi}{4}} \cos^2 t dt \\
&= 8\int_0^{\frac{\pi}{4}} \frac{\cos(2t)+1}{2} dt \\
&= 8\left[\frac{\sin(2t)}{4} + \frac{t}{2}\right]_0^{\frac{\pi}{4}} = 2 + \pi
\end{aligned}$$

- Calculate the integral. $\int_{-1}^{1} \sin(\sin x)dx$.

  Solution. Note that $\sin(\sin(-x)) = \sin(-\sin(x)) = -\sin(\sin x)$, and the interval $[-1, 1]$ is symmetric, hence this is 0.

- Calculate the inproper integral. $\int_0^\infty xe^{-x}dx$.

Solution. Note that with $F(x) = -e^{-x} - xe^{-x}$, $F'(x) = xe^{-x}$. Therefore,

$$\int_0^\beta xe^{-x}dx = [-e^{-x} - xe^{-x}]_0^\beta = -e^{-\beta} - \beta e^{-\beta} + 1$$

and as $\beta \to \infty$, this tends to 1. Hence $\int_0^\infty xe^{-x}dx = 1$.

Improper integrals.

- Calculate the following improper integral. $\int_0^\infty x^{\frac{1}{3}}e^{-x^{\frac{4}{3}}}dx$

  Solution. The function $x^{\frac{1}{3}}e^{x^{\frac{4}{3}}}$ is bounded on any bounded interval. The integral is improper only as $x \to \infty$. Let us compute $\int_0^\beta x^{\frac{1}{3}}e^{-x^{\frac{4}{3}}}dx$:

  $$\int_0^\beta x^{\frac{1}{3}}e^{-x^{\frac{4}{3}}}dx = -\frac{3}{4}\left[e^{-x^{\frac{4}{3}}}\right]_0^\beta = -\frac{3}{4}(e^{-\beta^{\frac{4}{3}}} - e^0)$$

  By taking the limit $\beta \to \infty$, we have $\int_0^\infty x^{\frac{1}{3}}e^{-x^{\frac{4}{3}}}dx = \frac{3}{4}$.

- Calculate the following improper integral. $\int_1^\infty \frac{\log x}{x^2}dx$

  Solution. The function $\frac{\log x}{x^2}$ is bounded on any interval of the form $[1, \beta]$. The integral is improper only as $x \to \infty$. Let us compute $\int_1^\beta \frac{\log x}{x^2}dx$:

  $$\begin{aligned}\int_1^\beta \frac{\log x}{x^2}dx &= \left[-\frac{\log x}{x}\right]_1^\beta + \int_1^\beta \frac{1}{x^2}dx \\ &= -\frac{\log\beta}{\beta} + 0 + \left[-\frac{1}{x}\right]_1^\beta = -\frac{\log\beta}{\beta} + (-\frac{1}{\beta} - (-1))\end{aligned}$$

  By taking the limit $\beta \to \infty$, we have $\int_1^\beta \frac{\log x}{x^2}dx = 1$.

- Determine whether the following improper integral converges. $\int_1^\infty \frac{x^3}{x^4+1}dx$.

  Solution 1. The function $\frac{x^3}{x^4+1}$ is bounded on any interval of the form $[1, \beta]$. The integral is improper only as $x \to \infty$. Furthermore, $\frac{x^3}{x^4+1}$ is asymptotically equal to $\frac{1}{x}$ as $x \to \infty$, that is,

  $$\frac{\frac{x^3}{x^4+1}}{\frac{1}{x}} = \frac{x^4}{x^4+1} \to 1 \text{ as } x \to \infty.$$

  On the other hand, we know that $\int_1^\beta \frac{1}{x}dx = [\log x]_0^\beta = \log\beta$ diverges as $\beta \to \infty$. Therefore, the integral $\int_1^\infty \frac{x^3}{x^4+1}dx$ diverges as well.

  Solution 2. $\int_1^\infty \frac{x^3}{x^4+1}dx = \frac{1}{4}[\log(x^4+1)]_1^\beta = \frac{1}{4}(\log(\beta^4+1) - \log 2)$ and this diverges as $\beta \to \infty$.

- Determine whether the following improper integral converges. $\int_1^2 \frac{x^2}{(x-1)^{\frac{1}{2}}}dx$.

  Solution. The function $\frac{x^2}{(x-1)^{\frac{1}{2}}}$ is bounded on any interval of the form $[1+\epsilon, 2]$ for $\epsilon > 0$. The integral is improper only as $x \to 1$. Furthermore, $\frac{x^2}{(x-1)^{\frac{1}{2}}}$ is asymptotically equal to $\frac{1}{(x-1)^{\frac{1}{2}}}$ as $x \to 1$, that is,

  $$\frac{\frac{x^2}{(x-1)^{\frac{1}{2}}}}{\frac{1}{(x-1)^{\frac{1}{2}}}} = x^2 \to 1 \text{ as } x \to 1.$$

On the other hand, we know that $\int_{1+\epsilon}^{2} \frac{1}{(x-1)^{\frac{1}{2}}} dx = [2(x-1)^{\frac{1}{2}}]_{1+\epsilon}^{2} = 2 - 2\epsilon^{\frac{1}{2}}$ converges (to 2 as $\epsilon \to 0$) as $\epsilon \to 0$. Therefore, the improper integral $\frac{x^2}{(x-1)^{\frac{1}{2}}}$ is also convegent.

- Determine whether the following improper integral converges. $\int_0^1 \frac{x^2}{\sin x - x} dx$.

  Solution. The function $\frac{x^2}{\sin x - x}$ is bounded on any interval of the form $[\epsilon, 1]$ for $\epsilon > 0$. The integral is improper only as $x \to 0$. Furthermore, as $\sin x - x = -\frac{x^3}{6} + o(x^3)$, $\frac{x^2}{\sin x - x}$ is asymptotically equal to $\frac{x^2}{\frac{x^3}{6}}$ as $x \to 0$, that is,

$$\frac{\frac{x^2}{\sin x - x}}{\frac{x^2}{-\frac{x^3}{6}}} \to 1 \text{ as } x \to 0.$$

  On the other hand, we know that $\int_\epsilon^1 \frac{x^2}{-\frac{x^3}{6}} dx = -6[\log x]_\epsilon^1 = 6 \log \epsilon$ diverges as $\epsilon \to 0$. Therefore, the improper integral $\frac{x^2}{\sin x - x}$ is also divergent.

- Calculate the area of the region surrounded by $y = x^2 - 1$ and the x-axis.

  Solution. The function $y = x^2 - 1$ and the x-axis intersects when $x^2 - 1 = 0$, that is, at $x = -1, 1$. Therefore, the region is given by $D = \{(x, y) : -1 \le x \le 1, x^2 - 1 \le y \le 0\}$. Its area is by definition

$$\int_{-1}^{1} 0 - (x^2 - 1) dx = [x - \frac{x^3}{3}]_{-1}^{1} = \frac{4}{3}.$$

- Calculate the area of the region surrounded by $y = x^2$ and $y = 5x + 6$.

  Solution. The function $y = x^2$ and $y = 5x + 6$ intersects when $x^2 = 5x + 6$, that is, at $x = -1, 6$. Therefore, the region is given by $D = \{(x, y) : -1 \le x \le 6, x^2 \le y \le 5x + 6\}$. Its area is by definition

$$\int_{-1}^{6} (5x + 6 - x^2) dx = [\frac{5x^2}{2} + 6x - \frac{x^3}{3}]_{-1}^{6} = (90 + 36 - 72) - (\frac{5}{2} - 6 - \frac{1}{3}) = \frac{343}{6}.$$

- Calculate the area of the region given by $\{(x, y) : a^2x^2 + b^2y^2 \le 1\}$.

  Solution. The condiction can be written equivalently as $b^2y^2 \le 1 - a^2x^2$,

$$-\frac{1}{b}\sqrt{1 - a^2x^2} \le y \le \frac{1}{b}\sqrt{1 - a^2x^2}.$$

  Furthermore, there is such x if and only if $1 - a^2x^2 \ge 0$, that is, $-\frac{1}{a} \le x \le \frac{1}{a}$.

  This region can be written as

$$D = \left\{(x, y) : -\frac{1}{a} \le x \le \frac{1}{a}, -\frac{1}{b}\sqrt{1 - a^2x^2} \le y \le \frac{1}{b}\sqrt{1 - a^2x^2}\right\}.$$

  The area is given, with the change of variables $x = \frac{t}{a}$ and $\frac{dx}{dt} = \frac{1}{a}$, $t = \sin\theta$, $\frac{dt}{d\theta} = \cos\theta$, by

$$\begin{aligned}
&\int_{-\frac{1}{a}}^{\frac{1}{a}} \frac{1}{b}\sqrt{1 - a^2x^2} - (-\frac{1}{b}\sqrt{1 - a^2x^2}) dx \\
&= \frac{2}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{1 - a^2x^2} dx = \frac{2}{b}\int_{-1}^{1} \sqrt{1 - t^2}\frac{1}{a} dt = \frac{2}{ab}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= \frac{2}{ab}[\frac{\cos 2\theta + 1}{2}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2\pi}{ab}.
\end{aligned}$$

- Calculate the length of the curve given by $f(x) = \frac{x^2}{2}$ from $x = -\frac{e-\frac{1}{e}}{2}$ to $x = \frac{e-\frac{1}{e}}{2}$.

  Solution. By definition, we need to compute $f'(x) = x$ and the length is

$$\int_{-\frac{e-\frac{1}{e}}{2}}^{\frac{e-\frac{1}{e}}{2}} \sqrt{1+x^2}dx.$$

  By the change of variables $x = \sinh t = \frac{e^t - e^{-t}}{2}$, $\frac{dx}{dt} = \cosh t$ and $\sqrt{1+\sinh^2 t} = \cosh t$, hence

$$\begin{aligned} &\int_{-\frac{e-\frac{1}{e}}{2}}^{\frac{e-\frac{1}{e}}{2}} \sqrt{1+x^2}dx \\ &= \int_{-1}^{1} \cosh^2 t dt = \int_{-1}^{1} \frac{e^{2x}+2+e^{-2x}}{2} dt \\ &= \frac{1}{4}[e^{2x}+2x-e^{2x}]_{-1}^{1} = 1 + \frac{e^2 - e^{-2}}{2}. \end{aligned}$$

Series.

- Compute the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

  Solution. We have $\sum_{k=1}^{n} a^k = a + a^2 + a^3 + \cdots + a^n = \frac{a-a^{n+1}}{1-a}$.

$$\sum_{n=1}^{\infty} \frac{3}{4^n} = 3\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = 3\frac{\frac{1}{4}}{1-\frac{1}{4}} = 3\frac{\frac{1}{4}}{\frac{3}{4}} = 1$$

  Note: We have $\sum_{k=0}^{n} a^k = 1 + a + a^2 + a^3 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$.

- Compute the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

  Solution. Note that

$$\begin{aligned} \frac{1}{n(n+1)(n+2)} &= \frac{1}{2(n+1)} \frac{(n+2)-n}{n(n+2)} = \frac{1}{2(n+1)}\left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \end{aligned}$$

  So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ is a telescopic series with $b_n = \frac{1}{2n(n+1)}$, and $\lim_{n\to\infty} b_n = 0$, therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1 = \frac{1}{4}$$

- Compute the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

  Solution. Note that

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{1}{2}\frac{(n+1)-(n-1)}{(n+1)(n-1)} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

  Therefore,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{1}{n^2-1} \\ &= \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \frac{1}{5^2-1} + \cdots \\ &= \frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10} + \frac{1}{8} - \frac{1}{12} + \cdots = \frac{1}{2} + \frac{1}{4} = \frac{3}{8} \end{aligned}$$

- Compute the series $\sum_{n=1}^{\infty} \frac{1+n}{n!}$.

  Solution. Recall that
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
  hence $e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$, and $\sum_{n=1}^{\infty} \frac{1}{n!} = (1 + \frac{1}{2} + \frac{1}{6} + \cdots) = e - 1$. Similarly, $\sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

  Therefore, the sum is $e - 1 + e = 2e - 1$.

- Determine whether $\sum_n \frac{n}{n^2+1}$ converges.

  Solution. This diverges because $\sum_n \frac{1}{n}$ diverges and
$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

- Determine whether $\sum_n \frac{(n!)^2}{(2n)!}$ converges.

  Solution.

  Put $a_n = \frac{(n!)^2}{(2n)!}$. Then
$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{\frac{(n+1)!^2}{2(n+1)!}}{\frac{(n!)^2}{(2n)!}} = \lim_n \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$
  Hence by ratio test this converges.

- Determine whether $\sum_n \frac{2^n+3^n}{5^n}$ converges.

  Hint. Use root test, and $2^n + 3^n = 3^n((\frac{2}{3})^n + 1)$.

  Or, compute two terms separately (geometric series).

- Determine whether $\sum_n \frac{\log n}{n^2}$ converges.

  Solution.

  We know that $\sum_n \frac{1}{n^{\frac{3}{2}}}$ converges, and
$$\frac{\frac{\log n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}} = \frac{\log n}{n^{\frac{1}{2}}} \to 0.$$
  Hence this converges by comparison.

  Or one can use also the condensation principle.

- Determine whether $\sum_n \frac{1}{(\log n)^2}$ converges.

  Solution. Note that $(\log n)^2 < n$ for sufficiently large n, hence $\frac{1}{n} < \frac{1}{(\log n)^2}$, and we know that $\sum_n \frac{1}{n}$ diverges, hence also the given series diverges.

  Or use the condensation principle.

- Determine whether $\sum_n \frac{(-1)^n n}{n^2+1}$ converges.

  Solution. This is an alternating series and with $a_n = \frac{n}{n^2+1}$, $a_n \to 0$ and a_n is decreasing (because $f(x) = \frac{x}{x^2+1}$ is decreasing for sufficiently large x, by computing the derivative).

  Hence by Leibniz criterion this is convergent.

- Determine whether $\sum_n \frac{(-1)^n}{n \log n}$ converges.

  Solution. Use the Leibniz criterion with $a_n = \frac{1}{n \log n}$ and this is convergent.

- Determine for which x, $\sum_n \frac{x^{2n}}{x^{2n}+1}$ converges.

  Solution.

  Put $a_n = \frac{x^{2n}}{x^{2n}+1}$.

  If $|x| > 1$, then $a_n = \frac{1}{1+\frac{1}{x^{2n}}} \to 1$, hence the series does not converge.

  If $|x| = 1$, then $a_n = \frac{1}{2}$ hence the series does not converge.

  If $|x| < 1$, then $a_n < x^{2n}$ and $\sum_n x^{2n}$ converges (geometric series), hence also the given series converges.

- Determine for which x, $\sum_n \frac{n^2 x^{2n}}{(2n)!}$ converges.

  Solution. Converges for all x. Indeed, if we put $a_n = \frac{n^2 x^{2n}}{(2n)!}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2 x^{2(n+1)}}{(2(n+1))!}}{\frac{n^2 x^{2n}}{(2n)!}} = \frac{(n+1)^2 x^2}{n^2(2n+2)(2n+1)} \to 0$$

  for all $x \in \mathbb{R}$. Therefore, by the ratio test, this converges for all $x \in \mathbb{R}$.

- Determine for which x, $\sum_n \frac{(-1)^n x^n}{n}$ converges.

  Solution. Let us put $a_n = \frac{|x|^n}{n}$. This is a positive sequence, and $\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1} n}{|x|^n (n+1)} \frac{|x| n}{n+1}$ to $|x|$. Therefore, if $|x| < 1$, then the series converges absolutely.

  On the other hand, if $|x| > 1$, then a_n diverges and in particular the series does not converge.

  Finally, if $x = 1$, the series is $\sum_n \frac{(-1)^n}{n}$ which converges by Leibniz criterion. If $x = -1$, then the series is $\sum_n \frac{(-1)^n (-1)^n}{n} = \sum_n \frac{1}{n}$ which diverges.

  Altogether, the series converges if and only if $x \in (-1, 1]$.

Differential equations.

- Solve the following differential equation. $y' = 2y$ with $y(0) = 2$.

  Solution. The general solution is $y(x) = Ce^{2x}$. Indeed, $y'(x) = 2Ce^{2x} = 2y(x)$. With the initial condition $y(0) = 2$, and $y(0) = Ce^0 = 2$, hence $C = 2$, and $y(x) = 2e^{2x}$.

- Solve the following differential equation. $y' = -3y$ with $y(1) = -1$.

  Solution. The general solution is $y(x) = Ce^{-3x}$. Indeed, $y'(x) = -3Ce^{-3x} = -3y(x)$.

  With the initial condition $y(1) = -1$, and $y(1) = Ce^{-3} = -1$, hence $C = -e^3$, therefore, $y(x) = -e^3 e^{-3x}$.

- Solve the following differential equation. $y' = x^3$ with $y(0) = 2$.

  Solution. More precisely, we have $y'(x) = x^3$. Therefore, $y(x) = \frac{x^4}{4} + C$. With the initial condition $y(0) = 2$, we obtain $y(0) = \frac{0^4}{4} + C = 2$, therefore, $C = 2$, and $y(x) = \frac{x^4}{4} + 2$.

- Solve the following differential equation. $y' = e^{2x}$ with $y(1) = -1$.

  Solution. More precisely, we have $y'(x) = e^{2x}$. The general solution is $y(x) = \frac{1}{2}e^{2x} + C$. With the initial condition $y(1) = -1$, $y(1) = \frac{1}{2}e^2 + C = -1$, $C = -1 - \frac{1}{2}e^2$ and altogether $y(x) = \frac{1}{2}e^{2x} - 1 - \frac{1}{2}e^2$.

- Solve the following differential equation. $y' + 2x^2y = 0$ with $y(0) = 2$.

  Solution. We can rewrite this as $y' = -2x^2y$ and $D(\log y) = \frac{y'}{y} = -2x^2$.

  Therefore,

  $$\log y = \int (-2x^2)dx + C = -\frac{2x^3}{3} + C$$

  and $y(x) = e^{-\frac{2x^3}{3}+C}$. With the given initial condition, $y(0) = e^C = 2$ and hence $C = \log 2$, $y(x) = 2e^{-\frac{2x^3}{3}}$.

- Solve the following differential equation. $y' + xe^xy = 0$ with $y(1) = 1$.

  Solution. The general solution is $\log y = \int -xe^x dx + C = -xe^x + e^x + C$. With the initial condition $0 = \log y(1) = -e^1 + e^1 + C$, hence $C = 0$ and

  $$y(x) = e^{-xe^x+e^x}.$$

- Solve the following differential equation. $xy' - 3y = x^5$ with $y(1) = 1$.

  Solution. The differential equation can be written as $y' - \frac{3}{x}y = x^4$. With $P(x) = -\frac{3}{x}$ and $Q(x) = x^4$, we have $A(x) = -\int_1^x \frac{3}{t}dt = -3\log x$. Furthermore,

  $$\int_1^x Q(t)e^{A(t)}dt = \int_1^x t^4e^{-3\log t}dt = \int_1^x tdx = \frac{x^2}{2} - \frac{1}{2}.$$

  Hence the general solution is

  $$y(x) = Ce^{3\log x} + e^{3\log x}(\frac{x^2}{2} - \frac{1}{2}) = Cx^3 + x^3(\frac{x^2}{2} - \frac{1}{2})$$

  With the initial condition $y(1) = 1$, we have $C = 1$ and $y(x) = x^3 + x^3(\frac{x^2}{2} - \frac{1}{2})$.

- Solve the following differential equation. $y' + xy = x$ with $y(0) = 2$.

  Hint. With $P(x) = x$ and $Q(x) = x$, we have $A(x) = \int_0^x \frac{x}{d}x = \frac{x^2}{2}$. Furthermore,

  $$\int_0^x Q(t)e^{A(t)}dt = \int_0^x te^{\frac{t^2}{2}}dt = e^{\frac{x^2}{2}} - 1.$$

  Hence the general solution is

  $$y(x) = Ce^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \cdot (e^{\frac{x^2}{2}} - 1) = (C-1)e^{-\frac{x^2}{2}} + 1$$

- A thermometer is stored in a room whose temperature is 35°C. Five minutes after being taken outdoor is 25°C. After another five minutes, it reads 20°C. Compute the outdoor temperature.

  Solution. With T the outside temperature, we know that the temperature $y(x)$ of the thermometer obeys

  $$y(x) = T + (35 - T)e^{-kx}$$

  Since $y(0) = 30, y(5) = 25$ and $y(10) = 20$ we have that $(35 - T)(1 - e^{-5k}) = 10$ and $(35 - T)(e^{-5k} - e^{-10k}) = 5$, hence $e^{-5k} = \frac{1}{2}$ and $35 - T = 20$, that is, $T = 15$.

- The half-life for Caesium-137 is about 30 years. Compute the percentage of a given quantity of Caesium that disintegrates in 10 years.

  Solution. Let the initial quantity C, then the quantity at time x (years) is

  $$y(x) = C2^{-x/30}.$$

  With $x = 10$, $y(x) = C2^{-10/30} = C2^{-1/3} \cong 0.79C$ hence the quantity that disintegrates in the meantime is 21%.

- Solve the following differential equation. $y'' + 4y = 0$ with $y(0) = 1, y'(0) = 1$.

  Solution. The general solution is

$$y(x) = C_1 \sin(2x) + C_2 \cos(2x).$$

  With the initial condition $y(0) = C_2 = 1$, $y'(0) = 2C_1 = 1$, hence $C_1 = \frac{1}{2}$. Altogether,

$$y(x) = \frac{1}{2}\sin(2x) + \cos(2x).$$

- Solve the following differential equation. $y'' - 4y' + 3y = 0$ with $y(0) = 1, y'(0) = 1$.

  Hint. The general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

  With this, it is straightforward to determine C_1 and C_2.

- Solve the following differential equation. $y'' - y = x$ with $y(1) = 1, y'(1) = 1$.

  Hint. There is a solution $y(x) = -x$ to the differential equation. A general solution is given as the sum of $y(x) = -x$ and and a general solution of $y'' - y = 0$, that is, $C_1 e^x + C_2 e^{-x}$, hence the general solution of $y'' - y = x$ is

$$y(x) = -x + C_1 e^x + C_2 e^{-x}.$$

- Solve the following differential equation. $y'' - y = x^2$ with $y(0) = 1, y'(0) = 1$.

  Hint. There is a solution $y(x) = -x^2 - 2$ to the differential equation. A general solution is given as the sum of $y(x) = -x^2 - 2$ and and a general solution of $y'' - y = 0$, that is, $C_1 e^x + C_2 e^{-x}$, hence the general solution of $y'' - y = x^2$ is

$$y(x) = -x^2 - 2 + C_1 e^x + C_2 e^{-x}.$$

- Solve the following differential equation. $y'' + y = e^x$ with $y(0) = 1, y'(0) = 1$.

  Hint. There is a solution $y(x) = \frac{1}{2}e^x$ to the differential equation. A general solution is given as the sum of $y(x) = \frac{1}{2}e^x$ and and a general solution of $y'' + y = 0$, that is, $C_1 \sin x + C_2 \cos x$, hence the general solution of $y'' - y = x^2$ is

$$y(x) = \frac{1}{2}e^x + C_1 \sin x + C_2 \cos x.$$

Differential equations and complex numbers.

- Find a relation between y and x for the following differential equation. $y' = \frac{x^3}{y^2}$.

  Solution.

  This equation is a separable equation with $Q(x) = x^3, R(y) = \frac{1}{y^2}$.

  This is equivalent to

$$y^2 y' = x^3.$$

  By integrathing these by y and x respectively,

$$\frac{y^3}{3} = \frac{x^4}{4} + C.$$

  Explicitly, we have $y = (\frac{3}{4}x^4 + C)^{\frac{1}{3}}$.

  (A formal way to remember this is to see $y' = \frac{dy}{dx}$, and

$$\frac{dy}{R(y)} = Q(x)dx.)$$

- Find a relation between y and x for the following differential equation. $y' = (y-1)(y-2)$.

  Solution. This is a separable equation with $Q(x) = 1, R(y) = (y-1)(y-2)$, and hence

$$\int \frac{1}{(y-1)(y-2)} dy = \int 1dx + C$$

  By $\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2} = \frac{-1}{y-1} + \frac{1}{y-2}$, we have $\int (\frac{-1}{y-1} + \frac{1}{y-2}) dy = \log(y-2) - \log(y-1) = \log \frac{y-2}{y-1} = x + C$.

  This can be solved explicitly as $\frac{y-2}{y-1} = C'e^x$ and solving this with respect to y: $y - 2 = (y-1)C'e^x$ and hence $y(1 - C'e^x) = -C'e^x + 2$, or $y = \frac{-C'e^x+2}{1-C'e^x}$.

- Find a relation between y and x for the following differential equation. $y' = \frac{x^2+y^2}{xy}$.

  Solution.

  The right-hand side is a homogeneous function of x, y, therefore, by introducing $v = \frac{y}{x}$, or $y = xv$ and $y' = v + xv'$ and

$$v + xv' = \frac{1 + (\frac{y}{x})^2}{\frac{y}{x}} = \frac{1+v^2}{v},$$

  or $v' = (\frac{1+v^2}{v} - v)\frac{1}{x} = \frac{1}{v}\frac{1}{x}$. Hence $vv' = \frac{1}{x}$ and $\frac{v^2}{2} = \log|x| + C$, or by $v = \frac{y}{x}$ we get $\frac{y^2}{2x^2} = \log|x| + C$

$$y^2 = 2x^2(\log|x| + C).$$

  or $y = \pm|x|\sqrt{2(\log|x| + C)}$.

- Find a relation between y and x for the following differential equation. $y' = 1 + \frac{y}{x}$.

  Solution.

  The right-hand side is a homogeneous function of x, y, therefore, by introducing $v = \frac{y}{x}$, or $y = xv$ and $y' = v + xv'$ and

$$v + xv' = 1 + v,$$

  or $v' = \frac{1}{x}$. This is separable, hence $v = \log x + C$, or $\frac{y}{x} = \log x + C$, $y = x \log x + Cx$.

- Calculate the product $3 + 2i$ and $1 - 2i$.

  Solution.

$$\begin{aligned}(3+2i)(1-2i) &= 3 \cdot 1 + 2i \cdot 1 - 3 \cdot 2i - 2i \cdot 2i = 3 + 2i - 6i - (-1) \cdot 2 \\ &= 7 - 4i\end{aligned}$$

- Calculate the inverse of $2 + i$.

  Solution.

$$\frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4+2i-2i-i^2} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i.$$

  (In general,

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+abi-abi-b^2i^2} = \frac{a-bi}{a^2+b^2}.)$$

- Calculate the 3rd root of i.

  Solution.

  We have $i = (0, 1) = (1 \cos \frac{\pi}{2}, 1 \sin \frac{\pi}{2})$ and hence $i^{\frac{1}{3}} = (1 \cos \frac{\pi}{6}, 1 \sin \frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

- Calculate the 4th root of -1.

  Solution.

  We have $-1 = (-1, 0) = (1 \cos \pi, 1 \sin \pi)$ and hence $(-1)^{\frac{1}{4}} = (1 \cos \frac{\pi}{4}, 1 \sin \frac{\pi}{4}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

- Solve the equation $x^2 + 2x + 5 = 0$.

  Solution.

  $(x + 1)^2 + 4 = 0$, or $(x + 1) = \pm\sqrt{-4} = \pm 2i$, hence $x = -1 \pm 2i$.

- Solve the equation $x^3 + 1 = 0$.

  Hint.

  There is one solution $x = -1$, indeed, $(-1)^3 + 1 = -1 + 1 = 0$. Then we can divide $x^3 + 1$ by $x + 1$ and get $x^2 - x + 1$, hence we only have to solve $x^2 - x + 1 = 0$.

- Represent $e^{\frac{\pi i}{2}}$ in the form of $a + ib$.

  Hint. Use the fact that $e^{i\theta} = \cos\theta + i \sin\theta$.

- Find $z \in \mathbb{C}$ such that $e^z = 1$.

  Hint. If we take $z = i\theta$, then $e^z = \cos\theta + i \sin\theta$, and this is equal to 1 if $\theta = 2\pi n$, where $n \in \mathbb{Z}$.