1 Taylor expansion and limit

Problem. For various $\alpha, \beta \in \mathbb{R}$, study the limit:

$$\lim_{x \to 1} \frac{x^3 + \alpha(x-1)\log(x^2) - 1 - \beta(x-1)}{\log(x) \cdot (\exp(x-1) - 1) \cdot (x-1)}.$$

Solution. First note that this is a limit of type $\frac{0}{0}$. So we need to study the behaviours of the numerator and the denominator as $x \to 1$. We calculate the Taylor formula of both the numerator and the denominator. The general formula (to the 3rd order, see below why the 3rd order is enough) is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + o((x-a)^3) \text{ as } x \to a.$$

We take $a = 1$.

- $x^3 = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$ (this is an identity as a polynomial)
- In general, if $g(x) = a + b(x-1) + c(x-1)^2 + o((x-1)^2)$, then we have $(x-1)g(x) = a(x-1) + b(x-1)^2 + c(x-1)^3 + o((x-1)^3)$. That is, the Taylor formula can be multiplied. This can simplify some calculations.
- Similarly, if $g(x) = a + b(x-1) + o((x-1)), h(x) = c + d(x-1) + o((x-1))$, then $g(x)h(x) = ac + (ad+bc)(x-1) + o((x-1))$.
- $(x-1)\log(x^2)$ can be calculated either using the general formula, or the above technique. We have $\log(x^2) = 2\log x$ and and $\log(0+1) = 0, (\log(y+1))' = \frac{1}{y+1}, (\log(y+1))'' = -\frac{1}{(y+1)^2}$. Use that $\log(x) = \log((x-1)+1) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + o((x-1)^2)$.

  By multiplying them, we have $(x-1)\log(x^2) = 2(x-1)^2 - (x-1)^3 + o((x-1)^3)$.
- $\log x = 0 + (x-1) + o((x-1))$, $\exp(x-1) = 1 + (x-1) + o((x-1))$ since $\exp(y) = 1 + y + \frac{1}{2!}y^2 + \ldots$. Or $\exp(x-1) - 1 = (x-1) + o((x-1))$. By multiplying them, $\log(x) \cdot (\exp(x-1) - 1) \cdot (x-1) = (x-1)^3 + o((x-1)^3)$.

Now we have that the denominator is $(x-1)^3 + o((x-1)^3)$ as $x \to 1$, and the numerator is

$$(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 - \alpha(2(x-1)^2 - (x-1)^3) - 1 - \beta(x-1) + o((x-1)^3)$$
$$= (3-\beta)(x-1) + (3-2\alpha)(x-1)^2 + (1+\alpha)(x-1)^3 + o((x-1)^3)$$

To have a finite limit of $\lim_{x\to 1} \frac{(3-\beta)(x-1)+(3-2\alpha)(x-1)^2+(1+\alpha)(x-1)^3+o((x-1)^3)}{(x-1)^3+o((x-1)^3)}$, we must have $3-\beta = 0, 3-2\alpha = 0$, because otherwise the limit diverges. Therefore, $\beta = 3, \alpha = \frac{3}{2}$, and the given limit is

$$\lim_{x\to 1} \frac{(1+\frac{3}{2})(x-1)^3 + o((x-1)^3)}{(x-1)^3 + o((x-1)^3)} = \lim_{x \to 1} \frac{\frac{5}{2} + \frac{o((x-1)^3)}{(x-1)^3}}{1 + \frac{o((x-1)^3)}{(x-1)^3}} = \frac{5}{2}.$$

Note: $\lim_{x\to 1} \frac{a}{(x-1)^3}$ converges if and only if $a = 0$ (otherwise diverges). Similarly, we have $\lim_{x\to 1} \frac{a+b(x-1)}{(x-1)^3}$ converges if and only if $a = b = 0$ (otherwise diverges).

The symbol $g(x) = o((x-1)^3)$ means that $\lim_{x\to 1} \frac{g(x)}{(x-1)^3} = 0$. In particular, we have $\lim_{x\to 1} \frac{a(x-1)^3+o((x-1)^3)}{(x-1)^3} = \lim_{x\to 1} \frac{a+\frac{o((x-1)^3)}{(x-1)^3}}{1} = a$.

Examples of Taylor series: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ as $x \to 0$, $\log x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ as $x \to 1$.

2 Series

Problem. Calculate the finite sum for $x = i$ in $\sum_{n=0}^{2} \frac{n^3}{8^n+n^2}(x+1)^{3n}$ and study the convergence of the infinite series $\sum_{n=0}^{\infty} \frac{n^3}{8^n+n^2}(x+1)^{3n}$, with various x.

Solution. The finite sum is $\sum_{n=0}^{2} a_n = a_0 + a_1 + a_2$. In the case at hand, we have $(i+1)^3 = (\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^3 = 2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = -2+2i, (i+1)^6 = 8(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}) = -8i$, and

$$\begin{aligned}\sum_{n=0}^{2} \frac{n^3}{8^n+n^2}(x+1)^{3n} &= \frac{0^3}{8^0+0^2}(i+1)^0 + \frac{1^3}{8^1+1^2}(i+1)^3 + \frac{2^3}{8^2+2^2}(i+1)^6 \\ &= 0 + \frac{-2+i}{9} + \frac{-8i\cdot 8}{68} = -\frac{2}{9} + i\frac{-177}{153}.\end{aligned}$$

As for the convergence, we use the ratio test. The ratio test tells, for a series $\sum_{n=0}^{\infty} a_n$ with $a_n > 0$, that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges, and if $L > 1$, then the series diverges.

To apply the ratio test to our case, for $x \in \mathbb{R}$, we set $a_n = \frac{n^3}{8^n+n^2}|x+1|^{3n}$ (need to take the absolute value), and see if $L > 1$ or $L < 1$, depending on x.

To calculate the limit,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)^3}{8^{n+1}+(n+1)^2}|x+1|^{3(n+1)}}{\frac{n^3}{8^n+n^2}|x+1|^{3n}} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3}\frac{8^n+n^2}{8^{n+1}+(n+1)^2}|x+1|^3 \\ &= \lim_{n\to\infty} \frac{1}{8}\frac{1+\frac{n^2}{8^n}}{1+\frac{(n+1)^2}{8^{n+1}}}|x+1|^3 = \frac{1}{8}|x+1|^3.\end{aligned}$$

Therefore, the ratio test tells that, if $\frac{1}{8}|x+1|^3 < 1$, the series $\sum_{n=0}^{\infty} \frac{n^3}{8^n+n^2}|x+1|^{3n}$ converges, or in other words, $\sum_{n=0}^{\infty} \frac{n^3}{8^n+n^2}(x+1)^{3n}$ converges absolutely. The condition is equivalent to $-8 < (x+1)^3 < 8$, or $-2 < x+1 < 2$, or $-3 < x < 1$.

For any specific value of x, one has to consider whether $-3 < x < 1$ or not. If $x = 1$, the ratio test does not apply, but the series becomes $\sum_{n=0}^{\infty} \frac{n^3}{8^n+n^2}2^{3n} = \sum_{n=0}^{\infty} \frac{n^3}{1+\frac{n^2}{8^n}}$, and as $\frac{n^3}{1+\frac{n^2}{8^n}}$ diverges, also the sum diverges.

Note: a series $\sum a_n$ is a new sequence obtained from the sequence a_n by $a_0, a_0+a_1, a_0+a_1+a_2, \cdots$. For example, if $a_n = \frac{1}{2^n}$, then $\sum_{n=0}^{N} a_n$ are $\frac{1}{1} = 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \cdots$ $(N = 0,1,2,3)$.

3 Graph of functions

Problem. Study the graph of the function $f(x) = \log \frac{x^3+1}{x^2-4}$.
Solution.

- Domain. $\log y$ is defined only for $y > 0$. In this case, we should have $\frac{x^3+1}{x^2-4} > 0$. $x^2 - 4 \neq 0$ should hold. A fraction is positive if and only if both the denominator and the numerator have the same sign.
  - $x^3 + 1 > 0$ and $x^2 - 4 > 0$. This is equivalent to $x^3 > -1, x^2 > 4$, or $-1 < x$ and $x < -2, 2 < x$. The intersection is $2 < x$.
  - $x^3 + 1 < 0$ and $x^2 - 4 < 0$. This is equivalent to $x^3 < -1, x^2 < 4$, or $-1 > x$ and $-2 < x < 2$. The intersection is $-2 < x < -1$.

  The domain is $(-2, -1) \cup (2, \infty)$.

- Asymptotes.
  - Vertical asymptotes. As $x \to -2$, the denominator tends to 0, while the numerator tends to 9, and $\log y$ diverges as $y \to \infty$, so there is a vertical asymptote there. As $x \to -1$, the numerator tends to 0 (negative), while the denominator tends to -3, and $\log y$ diverges as $y \to 0$, so there is a vertical asymptote there. As $x \to 2$, the denominator tends to 0, while the numerator tends to 9, and $\log y$ diverges as $y \to \infty$, so there is a vertical asymptote there.
  - Horizontal asymptote. As $x \to \infty$, note that, if $x > 3$, then $\log \frac{x^3+1}{x^2-4} > \log x^3 = 3 \log x$, and this diverges, hence $f(x)$ diverses. There is no horizontal asymptote.
  - Oblique asymptote. The limit $\frac{\log \frac{x^3+1}{x^2-4}}{x} < \frac{\log(2x^3)}{x} = \frac{\log 2 + 3 \log x}{x} \to 0$. There is no oblique asymptote.

- The derivative. We can use the chain rule: if $f(x) = g(h(x))$, then $f'(x) = h'(x)g'(h(x))$. In our case, $g(y) = \log y$ and $h(x) = \frac{x^3+1}{x^2-4}$, and $g'(y) = \frac{1}{y}$, $h'(x) = \frac{3x^2(x^2-4)-2x(x^3+1)}{(x^2-4)^2} = \frac{x(x^3-12x-2)}{(x^2-4)^2}$, and

$$f'(x) = \frac{x^2-4}{x^3+1} \frac{x(x^3-12x-2)}{(x^2-4)^2} = \frac{x(x^3-12x-2)}{(x^3+1)(x^2-4)}.$$

- Stationary points. They are points x in the domain where $f'(x) = 0$ holds. As we have computed $f'(x)$, the condition is that $x(x^3-12x-2) = 0$, and x is in the domain. First, $x = 0$ does satisfy the condition, but is not in the domain. The equation $g(x) = x^3 - 12x - 2 = 0$ has three solutions: It has one solution in $[3, 4]$ because $g(3) < 0, g(4) > 0$. Other solutions are again outside the domain (they are in $x < -2$ and in $[-1, 0]$, by the same reasoning as above). Therefore, there is only one stationary point.

- Behaviour of the graph. Recall that the function f is monotonically increasing in an interval if $f'(x) > 0$ there, and is monotonically decreasing in an interval if $f'(x) < 0$.

  If $x \in [4, 5]$, $f'(x)$ is positive: $g(4) = 64 - 48 - 2 > 0$, and $g'(x) = 3x^3 - 12 > 0$ in $x \in [4, 5]$, therefore, $g(x)$ is monotonically increasing in $[4, 5]$. Therefore, f is monotonically increasing in $[4, 5]$.

Note: the graph of a function $f(x)$ is the collection of points $(x, f(x))$ where x is in the domain of f.

4 Integral

Problem. Calculate the integral

$$\int_2^3 \frac{1}{2^x + 4 + 3(2^{-x})} dx.$$

Solution. We change the variables by $2^x = t$. This is equivalent to $x = \frac{\log t}{\log 2}$, or $\frac{dx}{dt} = \frac{1}{t \log 2}$. and formally replace dx by $\frac{1}{t \log 2} dt$, and substitute 2^x by t, while the limits of the integral becomes $2^2 = 4$ and $2^3 = 8$, therefore,

$$\begin{aligned}
\int_2^3 \frac{1}{2^x + 4 + 3(2^{-x})} dx &= \int_4^8 \frac{1}{t + 4 + 3t^{-1}} \frac{1}{t \log 2} dt \\
&= \int_4^8 \frac{1}{\log 2(t^2 + 4t + 3)} dt.
\end{aligned}$$

To carry out this last integral, we need to find the partial fractions: as $t^2 + 4t + 3 = (t+1)(t+3)$, we put $\frac{1}{t^2+4t+3} = \frac{A}{t+1} + \frac{B}{t+3} = \frac{A(t+3)+B(t+1)}{t^2+4t+3}$, or $1 = (A + B)t + 3A + B$. By solving this, $-A = B, 1 = 2A$ and $A = \frac{1}{2}, B = -\frac{1}{2}$. Namely, $\frac{1}{t^2+4t+3} = \frac{1}{2}(\frac{1}{t+1} - \frac{1}{t+3})$. Altogether,

$$\begin{aligned}
\int_2^3 \frac{1}{2^x + 4 + 3(2^{-x})} dx &= \int_4^8 \frac{1}{\log 2(t^2 + 4t + 3)} dt \\
&= \int_4^8 \frac{1}{2 \log 2} \left(\frac{1}{t+1} - \frac{1}{t+3} \right) dt \\
&= \frac{1}{2 \log 2} \left[\log(t+1) - \log(t+3) \right]_4^8 \\
&= \frac{1}{2 \log 2} \left(\log \frac{9}{11} - \log \frac{5}{7} \right) = \frac{\log \frac{63}{55}}{2 \log 2}.
\end{aligned}$$

Note: other useful techniques are substitution (example: $\int x e^{x^2} dx$ by putting $t = x^2$) and integration by parts (example: $\int x e^x dx$ by noticing that $(e^x)' = e^x$).

5 Differential equations

Problem. Find the general solution of

$$y'(x) = 6x^2 \exp(x^3) y^{\frac{1}{2}}.$$

and a special solution with $y((\log 2)^{\frac{1}{3}}) = 9$.
Solution. This is a separable differential equation, because the right-hand side is a product of a function of x ($6x^2 \exp(x^3)$) and a function of y ($y^{\frac{1}{2}}$). The solution is given by integrating separately $y^{-\frac{1}{2}}$ and $6x^2 \exp(x^3)$, that is

$$\int y^{-\frac{1}{2}} dy = \int 6x^2 \exp(x^3) dx + C, \qquad 2y^{\frac{1}{2}} = 2\exp(x^3) + C,$$

and this is equivalent to $y(x) = (\exp(x^3) + \frac{C}{2})^2$.

If $y((\log 2)^{\frac{1}{3}}) = 9$, then $9 = (\exp((\log 2)^{\frac{1}{3}})^3 + \frac{C}{2})^2 = (\exp \log 2 + \frac{C}{2})^2 = (2 + \frac{C}{2})^2$, and $3 = 2 + \frac{C}{2}$, hence $C = 2, -10$.
Problem. Find the general solution of the following differential equation.

$$y''(x) + y'(x) - 6y(x) = 0$$

and a special solution such that $y(0) = 4$ and $\lim_{x\to\infty} y(x) = 0$.
Solution. This second order linear differential equation with constant coefficients can be solved by finding the solutions of the equation $z^2 + z - 6 = 0$, that are $z = 2, -3$. With them, the general solution is $y(x) = C_1 \exp(-3x) + C_2 \exp(2x)$. The term $C_1 \exp(-3x)$ diverges as $x \to -\infty$, while the term $C_2 \exp(3x)$ diverges as $x \to \infty$. The only possible value of a such that $\frac{e^{3x}}{e^{ax}}$ tends to a non-zero limit as $x \to -\infty$ is $a = 3$ (for other values, it is either ∞ or 0).
Note: the meaning that y is the solution of the differential equation is that, if we take $y(x) = C_1 e^{-3x} + C_2 e^{2x}$, then it holds that $y''(x) + y'(x) - 6y(x) = 0$. Indeed, $y'(x) = -3C_1 e^{-3x} + 2C_2 e^{2x}, y''(x) = 9C_1 e^{-3x} + 4C_2 e^{2x}$ and hence

$$\begin{aligned}
&y''(x) + y'(x) - 6y(x) \\
&= 9C_1 e^{-3x} + 4C_2 e^{2x} - 3C_1 e^{-3x} + 2C_2 e^{2x} - 6(C_1 e^{-3x} + C_2 e^{2x}) \\
&= 0.
\end{aligned}$$

Similarly, if we take $y(x) = (\exp(x^3) + \frac{C}{2})^2$, it satisfies $y'(x) = 6x^2 \exp(x^3) y^{\frac{1}{2}}$. Indeed,

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(\exp(x^3) + \frac{C}{2}\right)^2 \\
&= 3x^2 \exp(x^3) \cdot 2\left(\exp(x^3) + \frac{C}{2}\right) \\
&= 6x^3 \exp(x^3) \cdot y^{\frac{1}{2}}.
\end{aligned}$$