Mathematical Analysis I: Lecture 60

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15/01/2020

Start recording...

Annoucements

- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request
- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

Detailed answers to the simulation questions

(watch also the video, where I explain how to insert answers on Moodle)

Differential equations

A differential equation is an equation about a function involving its derivatives.

- First-order equation: $y' = f(x, y)$ (more precisely $y'(x) = f(x, y(x))$). General solution is determined by an initial condition $y(a) = b$.
- Second-order equation: $y'' = f(x, y, y')$. General solution is determined by an initial condition $y(a) = b_1, y'(a) = b_2$.

Some types of differential equations.

- $f(x)$ depends only on x. $y' = f(x)$. $y(x) = \int f(x)dx + C$.
- First-order linear homogeneous equations with constant coefficients.
  $y' + ay = 0$ (equivalently, $y' = -ay$).
  $y = Ce^{-ax}$.
- Second-order linear homogeneous equations with constant coefficients. $y'' + ay' + by = 0$ **with** a, b **real**.
  If the equation $z^2 + az + b = 0$ has two different (possibly complex) solutions z_1, z_2, then $y(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x}$.
  If $z_1, z_2 \neq 0$ are real, this formula gives the general real solution. If $z_2 = 0$, the $y(x) = C_1 e^{z_1 x} + C_2$.
  If z_1, z_2 are complex, this can also be written as
  $y(x) = C_1 e^{\mathrm{Re}\, z_1 x} \cos(\mathrm{Im}\, z_1 x) + C_2 e^{\mathrm{Re}\, z_1 x} \sin(\mathrm{Im}\, z_1 x)$ (note that $z_2 = \mathrm{Re}\, z_1 - \mathrm{Im}\, z_2$).
  If $z_1 = z_2$, then $y(x) = C_1 e^{z_1 x} + C_2 x e^{z_1 x}$.

Some types of differential equations.

- Separable equation $y' = Q(y)R(x)$. This is implicitly solved by $\int \frac{1}{Q(y)} dy = \int R(x)dx + C$.
- $y' = f(x, y)$ with homogeneous $f(x, y) = f(tx, ty)$. This can be reduced to a separable equation by $y = vx$: $y' = v'x + v$, hence $v' = (f(1, v) - v)\frac{1}{x}$.

Consider

$$y''(x) = \frac{1}{x}.$$

This means that $y'(x) = \int \frac{1}{x} dx = \log x + \tilde{C}_1$, and then $y(x) = x \log x - x + \tilde{C}_1 x + C_2$. By rewriting the constant, this is $y(x) = x \log x + C_1 x + C_2$.
With the initial condition $y(1) = 1, y'(1) = 1$, we should have $C_1 = 0$ and $C_2 = 1$.

Consider

$$y''(x) + 2y'(x) + 2y = 0.$$

This is a second-order linear homogeneous differential equation with constant coefficients.
We consider the equation $z^2 + 2z + 2 = 0$, and the solutions are $z = -1 \pm i$ because it is equivalent to $(z+1)^2 = -1$.
Therefore, the general solution is $y(x) = C_1 e^{-x} \sin x + C_2 e^{-x} \cos x$.
With the initial condition $y(0) = 1$, we have $y(0) = C_1 \sin 0 + C_2 \cos 0 = C_2$, therefore, $C_2 = 1$. From the condition $2 = y'(0)$ and $y'(x) = C_1(-e^{-x} \sin x + e^{-x} \cos x) + (-e^{-x} \cos x - e^{-x} \sin x)$, we have $y'(0) = C_1(-e^0 \sin 0 + e^0 \cos 0) + (-e^0 \cos 0 - e^0 \sin 0) = C_1 - 1$, hence $C_1 = 3$.

Consider

$$y' = \frac{\sin x}{y}$$

This is separable, and can be solved by

$$\int y dy = \int \sin x dx + C,$$

or $\frac{y^2}{2} = -\cos x + C$, or $y = \pm\sqrt{-2\cos x + 2C}$.

Consider

$$y'' + 4y' + 3y = 0$$

This first-order linear homogeneous differential equation with constant coefficients.

We consider the equation $z^2 + 4z + 3 = 0$, or $(z+1)(z+3) = 0$, hence the solutions are $z = -1, -3$.

Therefore, the general solution to the differential equation is

$$y(x) = C_1 e^{-3x} + C_2 e^{-x}.$$

Complex numbers

$z = a + bi = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
For example, $z = 1 + i\sqrt{3} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, hence $z^6 = 2^6(\cos 2\pi + i\sin 2\pi) = 64$.
On the other hand, we can find one n-th root from the expression $z = r(\cos\theta + i\sin\theta)$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$$

For example, $z^4 = -8 + i8\sqrt{3} = 16(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2^4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ therefore, we can take $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$.
Other roots are found by multiplying this by $\cos\frac{k}{n} + i\sin\frac{k}{n}$ for some k.