Mathematical Analysis I: Lecture 59

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Start recording...

Annoucements

- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request
- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

Detailed answers to the simulation questions

(watch also the video, where I explain how to insert answers on Moodle)

Definite integral

- Primitive of f: a function F such that $f = F'$.
- Fundamental theorem of calculus: $\int_a^b f(x)dx = F(b) - F(a)$.
- Elementary integrals.
  - $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$ $(\alpha \neq -1)$
  - $\int e^x dx = e^x + C$
  - $\int \sin x dx = -\cos x + C$
  - $\int \cos x dx = \sin x + C$
  - $\int \frac{1}{x} dx = \log x + C$
  - $\int \frac{1}{x^2+1} dx = \arctan x + C$

More techniques for integral

- Substitution. $\int f'(\varphi(x))\varphi'(x)dx = f(\varphi(x)) + C$.
- Change of variables. $\int_a^b f(x)dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\varphi'(t)dt$.
- Integration by parts. $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx + C$.
- Partial fractions. $\frac{x}{x^2+3x+2} = \frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = -\frac{1}{x+1} + \frac{2}{x+2}$.

Consider

$$\int_{-1}^{1} \sqrt{1-x^2}dx.$$

With the change of variables $x = \sin t$, with $\frac{dx}{dt} = \cos t$

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2 t}\cos t\, dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\, dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2t + 1)\, dt \\
&= [\frac{1}{4}\sin 2t + \frac{t}{2}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}. \\
&= [\frac{1}{4}\sin 2t + \frac{t}{2}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}
\end{aligned}$$

Consider

$$\int x^2 e^{-x} dx.$$

By integration by parts, we have

$$\begin{aligned}
\int x^2 e^{-x} dx &= x^2(-e^{-x}) - \int 2x(-e^{-x}) dx + C \\
&= x^2(-e^{-x}) + \int 2x(e^{-x}) dx + C \\
&= x^2(-e^{-x}) + 2x(-e^{-x}) - \int 2(-e^{-x}) dx + C \\
&= x^2(-e^{-x}) + 2x(-e^{-x}) - 2e^{-x} + C
\end{aligned}$$

Consider

$$\int xe^{-x^2}dx.$$

By substitution $\varphi(x) = -x^2$,

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int(-2x)e^{-x^2}dx \\ &= -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

Improper integral

Consider an integral $\int_a^b f(x)dx$, where a is possibly $-\infty$, or b is possibly ∞, or $f(x)$ might be unbounded on the interval (a, b). This is defined by

$$\int_a^b f(x)dx = \lim_{\alpha \to a} \int_\alpha^c f(x)dx + \lim_{\beta \to b} \int_c^\beta f(x)dx,$$

where $a < \alpha < c < \beta < b$.
An improper integral $\int_a^b f(x)dx$ converges if there is a function $g(x)$ such that $g(x) \geq |f(x)|$ such that $\int_a^b g(x)dx$ converges.

Examples

- $\int_0^1 x^\gamma dx$ converges for $\gamma > -1$.
- $\int_1^\infty x^\gamma dx$ converges for $\gamma < -1$.
- $\int_0^\infty e^{\gamma x} dx$ converges for $\gamma < 0$.

Consider $\int_\beta^\infty xe^{\alpha x^2}dx$. $\frac{1}{2\alpha}e^{\alpha x^2}$ is a primitive of $xe^{\alpha x^2}$. Therefore,

$$\int_\beta^\infty xe^{\alpha x^2}dx = \lim_{\gamma\to\infty}\int_\beta^\gamma xe^{\alpha x^2}dx = \lim_{\gamma\to\infty}\frac{1}{2\alpha}[e^{\alpha x^2}]_\beta^\gamma$$
$$= \lim_{\gamma\to\infty}\frac{1}{2\alpha}(e^{\alpha\gamma^2} - e^{\alpha\beta^2})$$

This converges if $\alpha < 0$ to $-\frac{1}{2\alpha}e^{\alpha\beta^2}$
(if $\alpha = 0$, $\int_\beta^\infty xdx$ diverges).
As $\beta \to -\infty$, $-\frac{1}{2\alpha}e^{\alpha\beta^2} \to 0$, therefore, $\int_{-\infty}^\infty xe^{\alpha x^2}dx = 0$.

Consider $\int_1^\infty \frac{\log x}{(x-1)^\alpha x^{\frac{1}{3}}} dx$.

If $\gamma > -\alpha - \frac{1}{3}$, $\frac{\log x}{(x-1)^\alpha x^{\frac{1}{3}}} \frac{1}{x^\gamma}$ is bounded as $x \to \infty$. By comparison, the integral $\int_2^\infty \frac{\log x}{(x-1)^\alpha x^{\frac{1}{3}}} dx$ converges for $-\alpha - \frac{1}{3} < \gamma < -1$, hence $\alpha > \frac{2}{3}$.

If $\gamma \leq -\alpha$, $\frac{\log x}{(x-1)^\alpha x^{\frac{1}{3}}} \frac{1}{x^\gamma}$ is bounded as $x \to 1$. By comparison, the integral $\int_1^2 \frac{\log x}{(x-1)^\alpha x^{\frac{1}{3}}} dx$ converges for $-\alpha \geq \gamma > -1$, hence $\alpha < 1$.