Mathematical Analysis I: Lecture 58

Lecturer: Yoh Tanimoto

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Start recording...

Annoucements

- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request
- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

Detailed answers to the simulation questions

(watch also the video, where I explain how to insert answers on Moodle)

Study of functions

- Domain ($\frac{1}{g(x)}$ is defined for x such that $g(x) \neq 0$, $\log y$ is defined for $y > 0$, $\sqrt{y}$ is defined for $y \geq 0$...)
- Asymptotes
  - Vertical asymptote: look at the boundary points x_0 of the domain and see whether $\lim_{x \to x_0} |f(x)| \to \infty$.
  - Horizontal asymptote: see whether $\lim_{x \to \pm\infty} f(x)$ exists.
  - Oblique asymptote: see whether $\lim_{x \to \pm\infty} \frac{f(x)}{x}$ exists.
- (sign of the function)

Derivatives of functions

- Sign of the derivative: compute the derivative $f'(x)$ and determine where $f'(x) > 0$ (increasing), or $f'(x) < 0$ (decreasing).
- Stationary points (where $f'(x) = 0$). It is al local minimum if $f'(x)$ is increasing around that point, and a local maximum if $f'(x)$ is decreasing around that point.
  There might be other extremal poins where the definition of the function $f(x)$ changes.

Let us consider $f(x) = \log\left|\frac{1+x}{1-x}\right|$.

- Domain. $\log\left|\frac{1+x}{1-x}\right|$ is defined when $\left|\frac{1+x}{1-x}\right| > 0$ and $1 - x \neq 0$. So the excluded points are $x = 1$, and $\frac{1+x}{1-x} = 0$, that is $x = -1$. The domain is $\mathbb{R} \setminus \{-1, 1\}$.
- Asymptotes.
  - Vertical. $\lim_{x\to -1} \log\left|\frac{1+x}{1-x}\right| = -\infty$, so $x = -1$ is a vertical asymptote. $\lim_{x\to 1} \log\left|\frac{1+x}{1-x}\right| = \infty$, so $x = 1$ is a vertical asymptote.
  - Horizontal. $\lim_{x\to\infty} \log\left|\frac{1+x}{1-x}\right| = 0$ because $\frac{1+x}{1-x}$ tends to -1, and $|z|$ is continuous, and $\log z$ is continous around $z = 1$. $y = 0$ is a horizontal asymptote.
    $\lim_{x\to -\infty} \log\left|\frac{1+x}{1-x}\right| =$ because $\frac{1+x}{1-x}$ tends to -1, and $|z|$ is continuous, and $\log z$ is continous around $z = 1$. $y = 0$ is a horizontal asymptote.
  - Oblique. $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$. There are no oblique asymptotes.

Let us consider $f(x) = \log\left|\frac{1+x}{1-x}\right|$.

- Derivative. $f'(x) = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{(1-x)-(-(1+x))}{(1-x)^2} = \frac{2}{(1+x)(1-x)} = -\frac{2}{x^2-1}$
- Extremal points. There is no x such that $f'(x) = 0$. There are no other extremal points where the definition of f changes.
- The behaviour in $[-\frac{1}{2}, \frac{1}{2}]$. The derivative is positive, hence f is increasing.

Let us consider $f(x) = \arctan \sqrt{\left|\frac{1+x}{1-x}\right|}$.

- Domain. $\arctan \sqrt{\left|\frac{1+x}{1-x}\right|}$ is defined when $\left|\frac{1+x}{1-x}\right| \geq 0$ and $1 - x \neq 0$. So the excluded point is $x = 1$. The domain is $\mathbb{R} \setminus \{1\}$.
- Asymptotes.
  - Vertical. $\lim_{x\to 1} \arctan \sqrt{\left|\frac{1+x}{1-x}\right|} = \frac{\pi}{2}$, because $\lim_{z\to\infty} \arctan z = \frac{\pi}{2}$, so $x = 1$ is not vertical asymptote.
  - Horizontal. $\lim_{x\to\pm\infty} \arctan \sqrt{\left|\frac{1+x}{1-x}\right|} = \frac{\pi}{4}$ because $\sqrt{\left|\frac{1+x}{1-x}\right|}$ tends to 1, and $\arctan 1 = \frac{\pi}{4}$. $y = \frac{\pi}{4}$ is a horizontal asymptote.
  - Oblique. $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$. There are no oblique asymptotes.

Let us consider $f(x) = \arctan \sqrt{\left|\frac{1+x}{1-x}\right|}$.

- Derivative. For $-1 < x < 1$,
  $f'(x) = \frac{1}{(\frac{1+x}{1-x})^2+1} \cdot \frac{1}{2} \cdot \sqrt{\frac{1-x}{1+x}} \cdot \frac{(1-x)-(-(1+x))}{(1-x)^2} = \frac{1}{(1+x)^2+(1-x)^2}\sqrt{\frac{1-x}{1+x}}$
  Similarly,for $x < -1$ and $x > 1$,
  $f'(x) = \frac{1}{(\frac{1+x}{x-1})^2+1} \cdot \frac{1}{2} \cdot \sqrt{\frac{x-1}{1+x}} \cdot \frac{(x-1)-((1+x))}{(x-)^2} = -\frac{1}{(1+x)^2+(1-x)^2}\sqrt{\frac{1-x}{1+x}}$