Mathematical Analysis I: Lecture 57

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Start recording...

Annoucements

- A make-up session: 12 January (Tuesday) 11:30–13:00.
- Tutoring (by Mr. Lorenzo Panebianco): 12 January (Tuesday) 14:00–15:30.
- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request

- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

Detailed answers to the simulation questions

(watch also the video, where I explain how to insert answers on Moodle)

Taylor formula

Let f be a n-times differentiable function in an interval including x_0. Then the Taylor formula says that, for x in that interval,

$$\begin{aligned} & f(x) \\ & = f(x_0) + (x - x_0)f'(x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ & \qquad + o((x - x_0)^n) \end{aligned}$$

Here, $o((x - x_0)^n)$ is a term $g(x)$ such that $\lim_{x \to x_0} \frac{g(x)}{(x - x_0)^n} = 0$.

Complete the Taylor formula with $x_0 = 0, y_0 = 0$.

- $\frac{1}{1-y} = 1 + y + o(y)$ as $y \to 0$
- $\frac{1}{1-x^2} = 1 + x^2 + o(x^2)$ as $x \to 0$
- $\frac{1+x^2}{1-x^2} = (1 + x^2)(1 + x^2) + o(x^2) = 1 + 2x^2 + o(x^2)$ as $x \to 0$
- $\log(1 + y) = y + o(y)$
- $\log(1 + x^2) = x^2 + o(x^2)$
- $\log(1 - x^2) = -x^2 + o(x^2)$
- $\log(\frac{1+x^2}{1-x^2}) = \log(1 + x^2) - \log(1 - x^2) = 2x^2 + o(x^2)$
- $1 - \cos x = \frac{1}{2}x^2 + o(x^2)$.
- $\sin x = x(+0 \cdot x^2) + o(x^2)$.

Calculate the limit:

$$\lim_{x\to 0} \frac{\log(\frac{1+x^2}{1-x}) - \alpha \sin x}{1 - \cos x} = \lim_{x\to 0} \frac{2x^2 + o(x^2) - \alpha(x + o(x^2))}{\frac{1}{2}x^2 + o(x^2)}$$

This limit exits only if $\alpha = 0$, and in that case, the limit is 4.

Note: if $\alpha \neq 0$,

$$\frac{2x^2 + o(x^2) - \alpha(x + o(x^2))}{\frac{1}{2}x^2 + o(x^2)} = \frac{2x + o(x) - \alpha(1 + o(x))}{\frac{1}{2}x + o(x)}$$

this diverges as $x \to 0$.

Series and convergence

- Sequence of numbers: $a_1, a_2, \cdots, a_n, \cdots,$
- Series for the above sequence: $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$
- Question of convergence $\sum_{k=1}^{\infty} a_k$: use various tests
- Let $a_n, b_n > 0$.
  - Ratio test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, the series converges, if > 1 the series diverges.
  - Root test: If $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} < 1$, the series converges, if > 1 the series diverges.
  - Comparison test: if $a_n \le cb_n$ for some $c > 0$ and if $\sum b_n$ converges, then so does $\sum a_n$
- In general a_n, if $\sum |a_n|$ converges, then so does $\sum a_n$ (absolute convergence).
- If a_n is alternate ($a_{2n} > 0, a_{2n+1} < 0$ or the other case), then one may try the Leibniz criterion: if $|a_n|$ is decreasing and $\lim a_n = 0$, then the series $\sum a_n$ converges.

Consider the series $\sum \frac{n^2-1}{4^n} x^{2n}$.

- finite sum with $x = 1$: $\sum_{k=0}^{1} \frac{k^2-1}{4^k} x^{2k} = \frac{-1}{4^0} 1^0 + 0 = -1$.
- for convergence, root test with $a_n = \frac{n^2-1}{4^n}|x|^{2n}$, then $a_{n+1} = \frac{n^2+2n}{4 \cdot 4^n}|x|^{2n+2}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}|x|^2$
- Therefore, if $\frac{1}{4}|x|^2 < 1$, the series converges absolutely. Or, if $-2 < x < 2$.
- for $x = 2$, the series becomes $\sum(n^2 - 1)$, which diverges.