# Mathematical Analysis I: Lecture 56

Lecturer: Yoh Tanimoto

11/01/2020

Start recording...

# Annoucements

- A make-up session: 12 January (Tuesday) 11:30–13:00.
- Tutoring (by Mr. Lorenzo Panebianco): 12 January (Tuesday) 14:00–15:30.
- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request

- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

# Exercises

Find a relation between $y$ and $x$ for the following differential equation.
$y' = \frac{x^3}{y^2}$.
*Solution.*
This equation is a separable equation with $Q(x) = x^3, R(y) = \frac{1}{y^2}$.
This is equivalent to

$$y^2 y' = x^3.$$

By integrathing these by $y$ and $x$ respectively,

$$\frac{y^3}{3} = \frac{x^4}{4} + C.$$

Explicitly, we have $y = (\frac{3}{4}x^4 + C)^{\frac{1}{3}}$.
(A formal way to remember this is to see $y' = \frac{dy}{dx}$, and

$$\frac{dy}{R(y)} = Q(x)dx.)$$

Find a relation between $y$ and $x$ for the following differential equation.
$y' = (y-1)(y-2)$.
*Solution.* This is a separable equation with
$Q(x) = 1, R(y) = (y-1)(y-2)$, and hence

$$\int \frac{1}{(y-1)(y-2)} dy = \int 1 dx + C$$

By $\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2} = \frac{-1}{y-1} + \frac{1}{y-2}$, we have
$\int (\frac{-1}{y-1} + \frac{1}{y-2}) dy = \log(y-2) - \log(y-1) = \log\frac{y-2}{y-1} = x + C$.
This can be solved explicitly as $\frac{y-2}{y-1} = C'e^x$ and solving this with respect to $y$: $y - 2 = (y-1)C'e^x$ and hence $y(1 - C'e^x) = -C'e^x + 2$, or
$y = \frac{-C'e^x + 2}{1 - C'e^x}$.

Find a relation between $y$ and $x$ for the following differential equation.
$y' = \frac{x^2+y^2}{xy}$.
*Solution.*
The right-hand side is a homogeneous function of $x, y$, therefore, by introducing $v = \frac{y}{x}$, or $y = xv$ and $y' = v + xv'$ and

$$v + xv' = \frac{1 + (\frac{y}{x})^2}{\frac{y}{x}} = \frac{1+v^2}{v},$$

or $v' = (\frac{1+v^2}{v} - v)\frac{1}{x} = \frac{1}{v}\frac{1}{x}$. Hence $vv' = \frac{1}{x}$ and $\frac{v^2}{2} = \log|x| + C$, or by $v = \frac{y}{x}$ we get $\frac{y^2}{2x^2} = \log|x| + C$

$$y^2 = 2x^2(\log|x| + C).$$

or $y = \pm|x|\sqrt{2(\log|x| + C)}$.

Find a relation between $y$ and $x$ for the following differential equation.
$y' = 1 + \frac{y}{x}$.
*Solution.*
The right-hand side is a homogeneous function of $x, y$, therefore, by introducing $v = \frac{y}{x}$, or $y = xv$ and $y' = v + xv'$ and

$$v + xv' = 1 + v,$$

or $v' = \frac{1}{x}$. This is separable, hence $v = \log x + C$, or $\frac{y}{x} = \log x + C$,
$y = x \log x + Cx$.

Calculate the product $3 + 2i$ and $1 - 2i$.
*Solution.*

$$(3 + 2i)(1 - 2i) = 3 \cdot 1 + 2i \cdot 1 - 3 \cdot 2i - 2i \cdot 2i = 3 + 2i - 6i - (-1) \cdot 2$$
$$= 7 - 4i$$

Calculate the inverse of $2+i$.
*Solution.*

$$\frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4+2i-2i-i^2} = \frac{2-i}{5} = \frac{2}{5} + \frac{2}{5}i.$$

(In general,

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+abi-abi-b^2i^2} = \frac{a-bi}{a^2+b^2}.)$$

Calculate the 3rd root of $i$.

*Solution.*

We have $i = (0, 1) = (1 \cos \frac{\pi}{2}, 1 \sin \frac{\pi}{2})$ and hence $i^{\frac{1}{3}} = (1 \cos \frac{\pi}{6}, 1 \sin \frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Calculate the 4th root of $-1$.

*Solution.*

We have $-1 = (-1, 0) = (1\cos\pi, 1\sin\pi)$ and hence
$(-1)^{\frac{1}{4}} = (1\cos\frac{\pi}{4}, 1\sin\frac{\pi}{4}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Solve the equation $x^2 + 2x + 5 = 0$.
*Solution.*
$(x+1)^2 + 4 = 0$, or $(x+1) = \pm\sqrt{-4} = \pm 2i$, hence $x = -1 \pm 2i$.

Solve the equation $x^3 + 1 = 0$.
*Hint.*
There is one solution $x = -1$, indeed, $(-1)^3 + 1 = -1 + 1 = 0$. Then we can divide $x^3 + 1$ by $x + 1$ and get $x^2 - x + 1$, hence we only have to solve $x^2 - x + 1 = 0$.

Represent $e^{\frac{\pi i}{2}}$ in the form of $a + ib$.
*Hint.* Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$.

Find $z \in \mathbb{C}$ such that $e^z = 1$.
*Hint.* If we take $z = i\theta$, then $e^z = \cos\theta + i\sin\theta$, and this is equal to 1 if $\theta = 2\pi n$, where $n \in \mathbb{Z}$.