# Mathematical Analysis I: Lecture 55

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# Annoucements

- A make-up session: 12 January (Tuesday) 11:30–13:00.
- Tutoring (by Mr. Lorenzo Panebianco): 12 January (Tuesday) 14:00–15:30.
- Basic Mathematics course: 12–15 January on functions, limits, integral and upon request

- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

- Today: Apostol Vol. 1, Chapter 9

# Complex numbers

Let us consider $x = a + ib \in \mathbb{C}$. We denote by $|x| = \sqrt{a^2 + b^2}$ the **absolute value** of $x$. For two complex numbers $x, y$, the **disntance** between $x, y$ is $|x - y|$. This distance coincide with the distance on the plane.

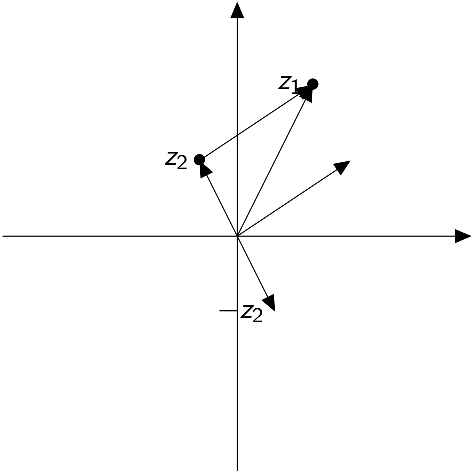


Figure: The distance of two complex numbers

We have the triangle inequality

$$|x - y| \leq |x - z| + |z - y|.$$

This is literally a triangle inequality, in the sense that $|x - y|, |y - z|, |z - x|$ are the lengths of the sides of the triangle formed by $x, y, z$ in $\mathbb{C}$.
For $x = a + ib$, let $\mathrm{Re}\, x = a, \mathrm{Im}\, x = b$. We have $|x| \geq |\mathrm{Re}\, x|, |\mathrm{Im}\, x|$, while $|x| \leq |\mathrm{Re}\, x| + |\mathrm{Im}\, x|$.

# Complex numbers

Let us consider a sequence of complex numbers $\{x_n\}$ and $x \in \mathbb{C}$. We say that $x_n$ converges to $x$ (and write $x_n \to x$) if $|x_n - x| \to 0$. Note that $\{|x_n - x|\}$ is a sequence of real numbers, hence we can use the definition and results there.

## Lemma

*$x_n$ converges to $x$ if and only if $\mathrm{Re}\, x_n$ and $\mathrm{Im}\, x_n$ converge to $\mathrm{Re}\, x$ and $\mathrm{Im}\, x$, respectively.*

## Proof.

If $|x_n - x| \to 0$, then $|\mathrm{Re}\, x_n - \mathrm{Re}\, x| = |\mathrm{Re}\, (x_n - x)| \le |x_n - x|$ hence $\mathrm{Re}\, x_n \to \mathrm{Re}\, x$. Similarly for $\mathrm{Im}\, x$.
If $|\mathrm{Re}\, x_n - \mathrm{Re}\, x|, |\mathrm{Im}\, x_n - \mathrm{Im}\, x| \to 0$, then
$|x_n - x| \le |\mathrm{Re}\, x_n - \mathrm{Re}\, x| + |\mathrm{Im}\, x_n - \mathrm{Im}\, x| \to 0$. □

In particular, we say that $\{x_n\}$ is a Cauchy sequence if for any $\epsilon > 0$ there is $N$ such that $|x_m - x_n| < \epsilon$ if $m, n > N$. In this case, $\{x_n\}$ is convergent.

# Complex numbers

If $\{z_n\}$ are complex numbers, we can also consider the series

$$\sum_{k=0}^{n} z_k = z_0 + z_1 + \cdots + z_n.$$

We can define the convergence of the series as the convergence of the sequence $\sum_{k=0}^{n} z_k$ just as with real numbers.
We say that the series $\sum_{k=0}^{n} z_k$ **converges absolutely** if $\sum_{k=0}^{n} |z_k|$ converges.

## Lemma

*If $\sum_{k=0}^{n} z_k$ converges absolutely, then the series $\sum_{k=0}^{n} z_k$ converges.*

## Proof.

If $\sum_{k=0}^{n} z_k$ converges absolutely, then $\sum_{k=0}^{n} |z_k|$ is a Cauchy sequence, and hence $\sum_{k=0}^{n} z_k$ is a Cauchy sequence (in the complex sense as above) because

$$\left|\sum_{k=0}^{n} z_k - \sum_{k=0}^{m} z_k\right| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \sum_{k=m+1}^{n} |z_k| < \epsilon$$

for sufficiently large $m, n$, by triangle inequality. Therefore, $\sum_{k=0}^{n} z_k$ converges. □

Recall that, for real number $x$, we have proved

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \end{aligned}$$

# Complex numbers

We can extend these functions by replacing $x$ by a complex number $z$. Indeed, the series

$$\sum_{n=0}^{N} \frac{z^n}{n!}$$

is absolutely convergent for any $z \in \mathbb{C}$, because

$$\sum_{n=0}^{N} \frac{|z|^n}{n!}$$

is convergent by the ratio test: with $a_n = \frac{|z|^n}{n!}$, $\frac{a_{n+1}}{a_n} = \frac{|z|}{n+1} \to 0$. Therefore, we can *define* the complex exponential function $e^z$ by $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Let us see what happens if $z = i\theta, \theta \in \mathbb{R}$.

$$
\begin{aligned}
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \\
&= \cos\theta + i \sin\theta.
\end{aligned}
$$

This last formula is called the Euler formula. In particlar with $\theta = \pi$, we have $e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.

# Complex numbers

We can also extend $\cos z, \sin z$ to complex variables (convergence is proven again by ratio test). As we have

$$
\begin{aligned}
e^{\theta} &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}, \qquad e^{-\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n!}, \\
\cos(i\theta) &= \sum_{n=0}^{\infty} \frac{(-1)^n (i\theta)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n \theta^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} = \frac{1}{2}(e^{\theta} + e^{-\theta}) = \cosh\theta, \\
\sin(i\theta) &= \sum_{n=0}^{\infty} \frac{(-1)^n (i\theta)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n i (-1)^n \theta^{2n+1}}{(2n+1)!} \\
&= i \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} = \frac{i}{2}(e^{\theta} - e^{-\theta}) = i \sinh\theta,
\end{aligned}
$$

# Complex numbers

Furthermore, this explains why the differential equation $y'' + y = 0$ has a general solution $y(x) = C_1 \sin x + C_2 \cos x$. By formally applying (this can be justified by the material in Mathematical Analysis II) the chain rule, we have $D(e^{ix}) = ie^{ix}, D^2(e^{ix}) = -e^{ix}$ and $D(e^{-ix}) = -ie^{-ix}, D^2(e^{-ix}) = -e^{-ix}$, hence they are formally two solutions of the equation $y'' + y$. Hence their linear combinations $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ satisfy the same equation.

# Exercises

- Represent $e^{\frac{\pi i}{2}}$ in the form of $a + ib$.
- Find $z \in \mathbb{C}$ such that $e^z = 1$.