Mathematical Analysis I: Lecture 54

Lecturer: Yoh Tanimoto

07/01/2020

Start recording...

Annoucements

- A make-up session: 12 January (Tuesday) 11:30–13:00.
- Tutoring (by Mr. Lorenzo Panebianco): 12 January (Tuesday) 14:00–15:30.

- **Register for the exam calls on Delphi**
- Simulations are available on `https://esamionline.uniroma2.it`

- Today: Apostol Vol. 1, Chapter

Complex numbers

Consider the equation $x^2 + 1 = 0$. There is no real solution x that satisfies this equation because $x^2 > 0$, hence $x^2 + 1 > 0$. Yet, it is possible to *extend* the set of real numbers in such a way to include solutions to this equation. One of such solutions is denoted by i, that means $i^2 = -1$.

Complex numbers

We expect that i behaves in a way similar to the real numbers. We can consider $a + ib, a, b \in \mathbb{R}$, it should hold that $(a + ib)(c + id) = ac + i^2 bd + i(ad + bc) = ac - bd + i(ad + bc)$. This can be formulated as follows: A **complex number** is a pair (a, b) of real numbers, and we define

- (Equality) $(a, b) = (c, d)$ as complex numbers if and only if $a = c$ and $b = d$.
- (Sum) $(a, b) + (c, d) = (a + c, b + d)$.
- (Product) $(a, b)(c, d) = (ac - bd, ad + bc)$.

This is a formal definition, and it is customary to denote (a, b) by $a + ib$. a is called the **real part** and b is called the **imaginary part**.
The set of complex numbers is denoted by $\mathbb{C}$.

Complex numbers

Theorem

For $x, y, z \in \mathbb{C}$, we have the following.

- *(Commutativity)* $x + y = y + x, xy = yx$.
- *(Associativity)* $x + (y + z) = (x + y) + z, x(yz) = (xy)z$.
- *(Distributive law)* $x(y + z) = xy + xz$.

Proof.

All these properties can be shown by calculating both sides and using the properties of real numbers. For example, for
$x = a + ib = (a, b), y = c + id = (c, d)$,
$x + y = (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = y + x$, $xy = (a, b)(c, d) = (ac - bd, ad + bc) = (ca - db, da + cb) = (c, d)(a, b) = yx$.
Other properties are left as exercises. □

Complex numbers

Let us consider the complex number $(0, 0)$. For any other complex number (a, b), it holds that $(0, 0)(a, b) = (0, 0)$ and $(0, 0) + (a, b) = (a, b)$. That is, $(0, 0)$ plays the same role of $0 \in \mathbb{R}$.
Next, we consider $(1, 0)$. For any other complex number (a, b), it holds that $(1, 0)(a, b) = (a, b)$. That is, $(1, 0)$ plays the same role of $1 \in \mathbb{R}$.
For any $(a, b) \in \mathbb{C}$, it holds that $(a, b) + (-a, -b) = (0, 0)$. We write $-(a, b)$ for $(-a, -b)$.

For any $(a, b) \in \mathbb{C}, (a, b) \neq (0, 0)$, it holds that $(a, b)(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0)$. We write $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ for $1/(a, b)$, or $(a, b)^{-1}$.
All these properties tell that $\mathbb{C}$ is an object called a field.
For any number $a \in \mathbb{R}$, the complex number $(a, 0)$ behaves exactly as $a \in \mathbb{R}$. Indeed, $(a, 0) + (b, 0) = (a + b, 0), (a, 0)(b, 0) = (ab, 0)$.

Complex numbers

The complex number $(0,1)$ satisfies $(0,1)^2 = (-1,0)$. Indeed, $(0,1)^2 = (0-1, 0+0) = (-1,0)$. With the understanding that $(-1,0) = -1 \in \mathbb{R}$, we can interpret this as $(0,1) = i \in \mathbb{C}$.
For a real number a and a complex number (b,c), we have $a \cdot (b,c) = (a,0)(b,c) = (ab, ac)$.
With this understanding, any complex number (a,b) can be written as $a + ib$, where $a = (a,0), b = (b,0), i = (0,1)$. We can perform all the usual computations using $i^2 = -1$, for example,

$$(a+ib)(c+id) = ac + iad + ibc + i^2 bd = ac - bd + i(ad+bc),$$
$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2}.$$

As we represented a real number as a point on the line, we can represent a complex number on the plane:

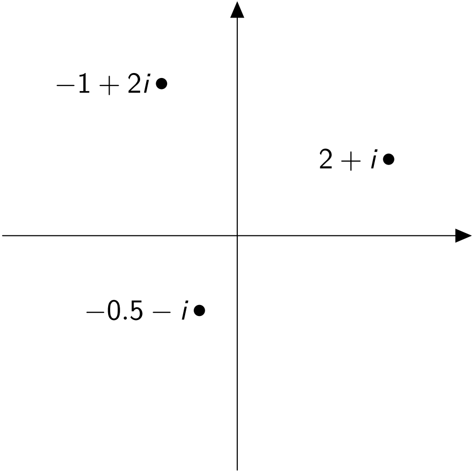

Figure: Various complex numbers on the plane

This is helpful especially when we consider various operations on complex numbers. For example, any complex number (a, b) can be considered as a segment from $(0, 0)$. Then the sum can be found by forming a parallelogram.

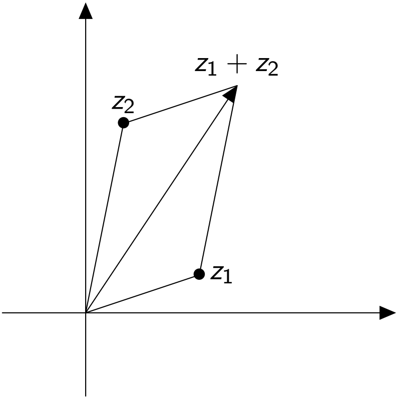

Figure: Sum of two complex numbers

As we identify a complex number with a point on the plane, for each complex number (a, b) the length of the segment $(0, 0)$–(a, b) is $\sqrt{a^2 + b^2}$ and the angle from the horizontal axis (the real numbers), and we can write this as $(a, b) = (r\cos\theta, r\sin\theta)$.

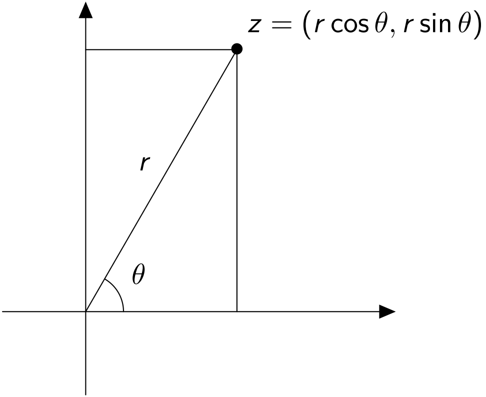

Figure: Various complex numbers on the plane

Complex numbers

With two complex numbers $(r_1 \cos\theta_1, r_1 \sin\theta_1), (r_2 \cos\theta_2, r_2 \sin\theta_2)$, we have

$$
\begin{aligned}
&(r_1 \cos\theta_1, r_1 \sin\theta_1)(r_2 \cos\theta_2, r_2 \sin\theta_2) \\
&= (r_1 r_2(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2), r_1 r_2(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)) \\
&= (r_1 r_2 \cos(\theta_1 + \theta_2), r_1 r_2 \sin(\theta_1 + \theta_2))
\end{aligned}
$$

Therefore, the product has the length $r_1 r_2$ and the angle $\theta_1 + \theta_2$. In particular, if we take $z = (r \cos\theta, r \sin\theta)$, then we have $z^n = (r^n \cos n\theta, r^n \sin n\theta)$.
From this, we can conlude that for any $z \in \mathbb{C}$ there is the n-th root in $\mathbb{C}$. Indeed, if $z = (r \cos\theta, r \sin\theta)$, then we can take $w = (r^{\frac{1}{n}} \cos\frac{\theta}{n}, r^{\frac{1}{n}} \sin\frac{\theta}{n})$.

Complex numbers

In complex numbers, we can solve the second degree equation $ax^2 + bx + c = 0$. Indeed, we can use the usual formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the square root is interpreted as a complex number, which always exists.
In general, a polynomial equation $a_n x^n + \cdots a_1 x + a_0 = 0$ has n solution in $\mathbb{C}$. This is called the fundamental theorem of algebra.

Exercises

- Calculate the product $3 + 2i$ and $1 - 2i$.
- Calculate the inverse of $2 + i$.
- Calculate the 3rd root of i.
- Calculate the 4th root of -1.
- Solve the equation $x^2 + 2x + 5 = 0$.
- Solve the equation $x^3 + 1 = 0$.