Mathematical Analysis I: Lecture 53

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Annoucements

- Next lecture: 7 January 2021
- Today: Apostol Vol. 1, Chapter 8

Existence and uniqueness of solution

We have considered ordinary differential equations $y' = f(x, y)$ and found solutions to some of them. Yet, some differential equation does not have a solution for a given initial condition, and others have many solutions.

- Consider $(y')^2 - xy' + y + 1 = 0$: no solution with $y(0) = 0$, because then $y'(0)^2 + 1 = 0$, which is impossible because $y'(0)$ should be a real number.
- Consider $y' = 3y^{\frac{2}{3}}$: the initial condition $y(0) = 0$ has two solutions $y = 0$ and $y = x^3$.

Existence and uniqueness of solution

Yet, as we have seen, a differential equation gives a vector field, and it should be enough to "chase the arrows". For this to be possible, $f(x, t)$ should have certain nice properties. We only state the theorem, and leave the proof to a more advanced book.

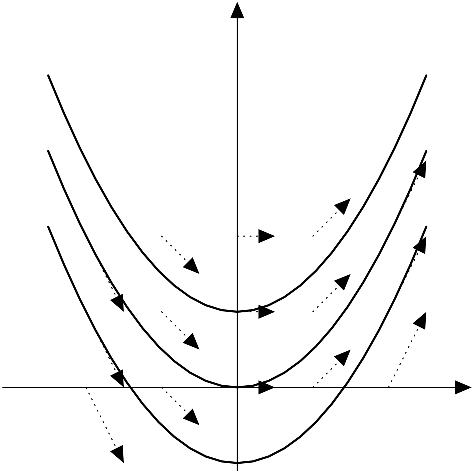

Figure: The integral curves of $y' = x$.

Existence and uniqueness of solution

For this purpose, we need the following concept: Let $f(x, y)$ be a function of two variables, that is, f gives a number for a given pair of numbers (x, y). For each fixed y, we can think of $f(x, y)$ as a function of x, and hence take the derivative with respect to x. This is called the **partial derivative** with respect to x, and denoted by $\frac{\partial f}{\partial x}$.

Example

- Let $f(x, y) = x^2 + y^2$. Then $\frac{\partial f}{\partial x} = 2x$.
- Let $f(x, y) = xy$. Then $\frac{\partial f}{\partial x} = y$.
- Let $f(x, y) = \sin(xy^2)$. Then $\frac{\partial f}{\partial x} = y^2 \cos(xy^2)$.

It is also possible to consider $\frac{\partial f}{\partial y}$. The detail will be explained in Mathematical Analysis II.

Existence and uniqueness of solution

Theorem

Suppose that $f(x,y)$ and $\frac{\partial f}{\partial x}$ are continuous in a rectangle

$$R = \{(x,y) : x_0 - \delta < x < x_0 + \delta, y_0 - \epsilon < y < y_0 + \epsilon\}.$$

Then there is δ_1 such that the equation $y' = f(x,y)$ has a unique solution $y(x)$ with initial condition $y(x_0) = y_0$ for $x_0 - \delta_1 < x < x_0 + \delta_1$.

Euler method

The proof of this theorem amounts to construct approximate solutions. At the end, for applications in science and engineering, we are satisfied with having sufficiently good approximate solutions.
There are many methods to obtain a numerical solution of a differential equation. One of the simplest of them is called the Euler's method, and it literally chase the vector field as follows.

Euler method

Let us consider the differential equation $y' = f(x, y)$ with the inizial condition $y(x_0) = y_0$, where $x_0, y_0 \in \mathbb{R}$. This means that the solution $y(x)$ passes the point (x_0, y_0). Furthermore, by "chasing the arrows", the slope of the curve $y(x)$ at the point (x_0, y_0) is $f(x_0, y_0)$. That is, if we take a small step ϵ, then the next point on the curve is close to $(x_0 + \epsilon, y_0 + f(x_0, y_0)\epsilon) = (x_1, y_1)$. Then, again at the point (x_1, y_1), the slope of the curve is $f(x_1, y_1)$, hence the next point on the curve is close to $(x_1 + \epsilon, y_1 + f(x_1, y_1)\epsilon) = (x_2, y_2)$, and so on. In this way, we obtain a union of segments which approximates the solution.
If we take smaller ϵ, the approximation gets better, while we need do more computations.

Euler method

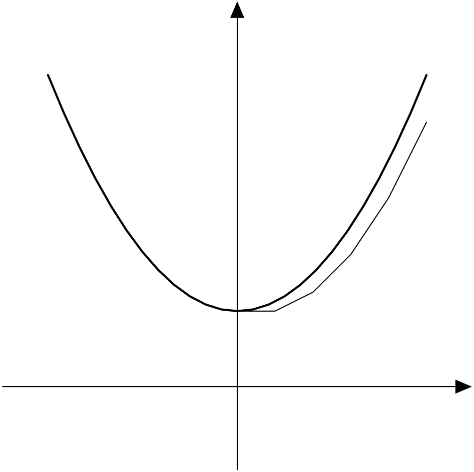

Figure: The Euler method to solve $y' = x$ with $(x_0, y_0) = (0, 1)$ with $\epsilon = 0.5$.

Numerical solutions

In the language Python, it is very easy to write a code to solve a differential equation. Let us see some examples[1]

[1] The plot part is taken from the book by Christian Hill, `https://scipython.com/book/chapter-8-scipy/additional-examples/the-sir-epidemic-model/`.

Exercises

- Try to use platform/programming language to solve differential equations and plot vector fields.