Mathematical Analysis I: Lecture 52

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Start recording...

Annoucements

- A make up session on 22 December 11:30
- Today: Apostol Vol. 1, Chapter 8

General remarks

Many interesting differential equations are nonlinear: just for example, the motion in a gravitational field is given by

$$mx'' = -\frac{mMG}{x^2}$$

(in one-dimension). And it is difficult to solve such nonlinear equations.

General remarks

Let us consider a first-order differential equation

$$y' = f(x, y),$$

that is, f is a given function of two variables and the question is to find a function $y(x)$ such that $y'(x) = f(x, y(x))$ for x in a certain interval.
If from the differential equation $y' = f(x, y)$ we can derive a relation between x, y of the form

$$F(x, y, C) = 0,$$

where $F(x, y, C)$ is another two-variable function with a parameter C (hence 3-variables), then we say that the differential equation is solved, or integrated. This is because the relation $F(x, y, C) = 0$ for a fixed number C defines a function $y(x)$ implicitly: recall that, if $F(x, y) = x^2 + y^2 - C^2$, then it defines the function(s) $y(x) = \pm\sqrt{C^2 - x^2}$.

Separable differential equations

We call a first-order differential equation $y' = f(x, y)$ **separable** if it can be written in the form $y' = Q(x)R(y)$, where $Q(x)$ is a function of x alone and $R(y)$ is a function of y alone. For example,

- $y' = x^3$
- $y' = yx^2$
- $y' = \sin y \log x$.

When $R(y) \neq 0$, we can write this in the form $A(y)y' = Q(x)$.

Theorem

Let $G(y)$ be a primitive of $A(y)$ and $H(x)$ be a primitive of $Q(x)$. Then any differentiable function $y(x)$ which satisfies

$$G(y(x)) = H(x)$$

satisfies the differential equation $A(y(x))y'(x) = Q(x)$, and conversely, any solution $y(x)$ satisfies this equation for certain $H(x)$.

Proof.

Let $y(x)$ satisfies the equation above, then by the chain rule, we have $\frac{d}{dx}G(y(x)) = y'(x)A(y(x))$, while $\frac{d}{dx}H(x) = Q(x)$, hence we obtain $A(y(x))y'(x) = Q(x)$ by differentiating $G(y(x)) = H(x)$.
Conversely, if $y(x)$ is the solution of the differential equation, then by integrating both sides of $A(y(x))y'(x) = Q(x)$ by substitution, we have $G(y(x)) = H(x) + C$ for some constant C. Note that $H(x) + C$ is a primitive of $Q(x)$, hence we proved the claim. □

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Example

- Consider $y' = y^2 x$. This can be written as $\frac{y'}{y^2} = x$. Each sides can be integrated, and we obtain

$$-\frac{1}{y} = \frac{x^2}{2} + C,$$

or $y = -\frac{1}{\frac{x^2}{2}+C}$.

Separable differential equations

Example

- Consider $y' = \frac{x}{y}$. This can be written as $yy' = x$. Each sides can be integrated, and we obtain

$$\frac{y^2}{2} = \frac{x^2}{2} + C,$$

or $y = \pm\sqrt{x^2 + 2C}$.

Separable differential equations

Example

- Consider $xy' + y = y^2$.

Separable differential equations

Example

- Consider $xy' + y = y^2$. This can be written as $y' = \frac{y(y-1)}{x}$, or $\frac{y'}{y(y-1)} = \frac{1}{x}$. Each sides can be integrated, and we obtain

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$$\log\left|1 - \frac{1}{y}\right| = \log x + C,$$

or $\left|1 - \frac{1}{y}\right| = C'x$ for some constant $C' = e^C$.

Separable differential equations

Not many equations are separable, but some can be reduced to a separable equation. For example, consider the case where $y' = f(x, y)$ and

$$f(tx, ty) = f(x, y)$$

for any $t \neq 0$. In this case, we can introduce $v = \frac{y}{x}$, or $y = vx$ and hence $y' = v'x + v$. Therefore, if y' is the solution of the equation above, then it must hold that

$$v'x + v = y' = f(x, y) = f(1, y/x) = f(1, v).$$

This can be written as $v' = (f(1, v) - v)\frac{1}{x}$, hence is separable. Once v is obtained as a function of x, we can recover $y = vx$.

Example

Consider $y' = \frac{y-x}{x+y}$. $f(tx, ty) = \frac{ty-tx}{tx+ty} = \frac{y-x}{x+y} = f(x, y)$, hence this is can be solved by introducing $y = vx$.
We have $v' = (f(1, v) - v)\frac{1}{x} = (\frac{v-1}{1+v} - v)\frac{1}{x} = -\frac{1+v^2}{1+v}\frac{1}{x}$, and we have

$$\int \frac{1+v}{1+v^2} dv = \arctan v + \frac{1}{2}\log(1+v^2)$$
$$-\int \frac{1}{x} dx = -\log x + C$$

By bringing back $y = vx$, we have $\arctan \frac{y}{x} + \frac{1}{2}\log(x^2 + y^2) = C$.

Integral curves

As in previous example, it is typical that, by solving a differential equation, we obtain an implicit equation $F(x, y, C) = 0$. This means that, for each value of C, we have a relation between x, y, and in certain cases, it defines a function y of x. As this function $y(x)$ satisfies the differential equation $y'(x) = f(x, y(x))$, $y'(x)$ should mean the slope of the curve $y(x)$ at the point $(x, y(x))$.

Integral curves

For example, consider the equation $y' = x$. This can be integrated and $y = \frac{x^2}{2} + C$, and depending on the value of C, we have different parabolas.
On the other hand, at each point in the xy-plane, we can draw an arrow which goes from (x, y) to $(x + \epsilon, y + y'(x)\epsilon)$. These arrows are tangent to the curve which represents the solution.
This plot of arrows is called a vector field, and a solution is obtained by "connecting" these arrows.
(One can visualize the arrows by a command `VectorPlot[1,f(x,y)]` on Wolfram Alpha, and the stream by `StreamPlot[1,f(x,y)]`, where we took $\epsilon = 1$).

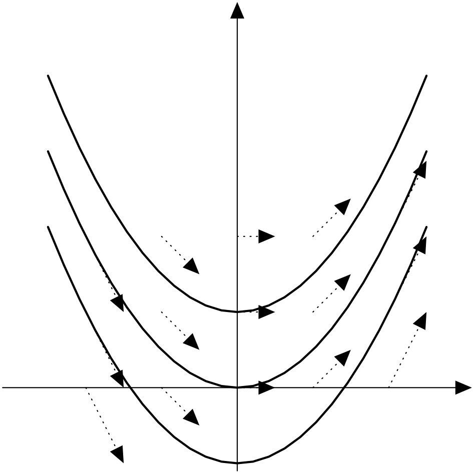

Figure: The integral curves of $y' = x$.

Exercises

- Find a relation between y and x for the following differential equation. $y' = \frac{x^3}{y^2}$.
- Find a relation between y and x for the following differential equation. $y' = (y-1)(y-2)$.
- Find a relation between y and x for the following differential equation. $y' = \frac{x^2+y^2}{xy}$.
- Find a relation between y and x for the following differential equation. $y' = 1 + \frac{y}{x}$.