Mathematical Analysis I: Lecture 51

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

- Today: Apostol Vol. 1, Chapter 8

Exercises

Solve the following differential equation. $y' = 2y$ with $y(0) = 2$.
Solution. The general solution is $y(x) = Ce^{2x}$. Indeed, $y'(x) = 2Ce^{2x} = 2y(x)$. With the initial condition $y(0) = 2$, and $y(0) = Ce^0 = 2$, hence $C = 2$, and $y(x) = 2e^{2x}$.

Solve the following differential equation. $y' = -3y$ with $y(1) = -1$.
Solution. The general solution is $y(x) = Ce^{-3x}$. Indeed,
$y'(x) = -3Ce^{-3x} = -3y(x)$.
With the initial condition $y(1) = -1$, and $y(1) = Ce^{-3} = -1$, hence
$C = -e^3$, therefore, $y(x) = -e^3 e^{-3x}$.

Solve the following differential equation. $y' = x^3$ with $y(0) = 2$.
Solution. More precisely, we have $y'(x) = x^3$. Therefore, $y(x) = \frac{x^4}{4} + C$. With the initial condition $y(0) = 2$, we obtain $y(0) = \frac{0^4}{4} + C = 2$, therefore, $C = 2$, and $y(x) = \frac{x^4}{4} + 2$.

Solve the following differential equation. $y' = e^{2x}$ with $y(1) = -1$.
Solution. More precisely, we have $y'(x) = e^{2x}$. The general solution is $y(x) = \frac{1}{2}e^{2x} + C$. With the initial condition $y(1) = -1$, $y(1) = \frac{1}{2}e^2 + C = -1$, $C = -1 - \frac{1}{2}e^2$ and altogether $y(x) = \frac{1}{2}e^{2x} - 1 - \frac{1}{2}e^2$.

Solve the following differential equation. $y' + 2x^2 y = 0$ with $y(0) = 2$.
Solution. We can rewrite this as $y' = -2x^2 y$ and $D(\log y) = \frac{y'}{y} = -2x^2$.
Therefore,

$$\log y = \int (-2x^2) dx + C = -\frac{2x^3}{3} + C$$

and $y(x) = e^{-\frac{2x^3}{3} + C}$. With the given initial condition, $y(0) = e^C = 2$ and hence $C = \log 2$, $y(x) = 2e^{-\frac{2x^3}{3}}$.

Solve the following differential equation. $y' + xe^x y = 0$ with $y(1) = 1$.
Solution. The general solution is $\log y = \int -xe^x dx + C = -xe^x + e^x + C$.
With the initial condition $0 = \log y(1) = -e^1 + e^1 + C$, hence $C = 0$ and

$$y(x) = e^{-xe^x + e^x}.$$

Solve the following differential equation. $xy' - 3y = x^5$ with $y(1) = 1$.
Solution. The differential equation can be written as $y' - \frac{3}{x}y = x^4$. With $P(x) = -\frac{3}{x}$ and $Q(x) = x^4$, we have $A(x) = -\int_1^x \frac{3}{t}dt = -3\log x$.
Furthermore,

$$\int_1^x Q(t)e^{A(t)}dt = \int_1^x t^4 e^{-3\log t}dt = \int_1^x tdx = \frac{x^2}{2} - \frac{1}{2}.$$

Hence the general solution is

$$y(x) = Ce^{3\log x} + e^{3\log x}(\frac{x^2}{2} - \frac{1}{2}) = Cx^3 + x^3(\frac{x^2}{2} - \frac{1}{2})$$

With the initial condition $y(1) = 1$, we have $C = 1$ and $y(x) = x^3 + x^3(\frac{x^2}{2} - \frac{1}{2})$.

Solve the following differential equation. $y' + xy = x$ with $y(0) = 2$.
Hint. With $P(x) = x$ and $Q(x) = x$, we have $A(x) = \int_0^x \frac{x}{d}x = \frac{x^2}{2}$.
Furthermore, Furthermore,

$$\int_0^x Q(t)e^{A(t)}dt = \int_0^x te^{\frac{t^2}{2}}dt = e^{\frac{x^2}{2}} - 1.$$

Hence the general solution is

$$y(x) = Ce^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \cdot (e^{\frac{x^2}{2}} - 1) = (C-1)e^{-\frac{x^2}{2}} + 1$$

A thermometer is stored in a room whose temperature is $35^\circ\mathrm{C}$. Five minutes after being taken outdoor is $25^\circ\mathrm{C}$. After another five minutes, it reads $20^\circ\mathrm{C}$. Compute the outdoor temperature.

Solution. With T the outside temperature, we know that the temperature $y(x)$ of the thermometer obeys

$$y(x) = T + (35 - T)e^{-kx}$$

Since $y(0) = 30, y(5) = 25$ and $y(10) = 20$ we have that $(35 - T)(1 - e^{-5k}) = 10$ and $(35 - T)(e^{-5k} - e^{-10k}) = 5$, hence $e^{-5k} = \frac{1}{2}$ and $35 - T = 20$, that is, $T = 15$.

The half-life for Caesium-137 is about 30 years. Compute the percentage of a given quantity of Caesium that disintegrates in 10 years.
Solution. Let the initial quantity C, then the quantity at time x (years) is

$$y(x) = C2^{-x/30}.$$

With $x = 10$, $y(x) = C2^{-10/30} = C2^{-1/3} \cong 0.79C$ hence the quantity that disintegrates in the meantime is 21%.

Solve the following differential equation. $y'' + 4y = 0$ with $y(0) = 1, y'(0) = 1$.

Solution. The general solution is

$$y(x) = C_1 \sin(2x) + C_2 \cos(2x).$$

With the initial condition $y(0) = C_2 = 1$, $y'(0) = 2C_1 = 1$, hence $C_1 = \frac{1}{2}$. Altogether,

$$y(x) = \frac{1}{2}\sin(2x) + \cos(2x).$$

Solve the following differential equation. $y'' - 4y' + 3y = 0$ with $y(0) = 1, y'(0) = 1$.
Hint. The general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

With this, it is straightforward to determine C_1 and C_2.

Solve the following differential equation. $y'' - y = x$ with $y(1) = 1, y'(1) = 1$.
Hint. There is a solution $y(x) = -x$ to the differential equation. A general solution is given as the sum of $y(x) = -x$ and and a general solution of $y'' - y = 0$, that is, $C_1 e^x + C_2 e^{-x}$, hence the general solution of $y'' - y = x$ is

$$y(x) = -x + C_1 e^x + C_2 e^{-x}.$$

Solve the following differential equation. $y'' - y = x^2$ with $y(0) = 1, y'(0) = 1$.

Hint. There is a solution $y(x) = -x^2 - 2$ to the differential equation. A general solution is given as the sum of $y(x) = -x^2 - 2$ and and a general solution of $y'' - y = 0$, that is, $C_1 e^x + C_2 e^{-x}$, hence the general solution of $y'' - y = x^2$ is

$$y(x) = -x^2 - 2 + C_1 e^x + C_2 e^{-x}.$$

Solve the following differential equation. $y'' + y = e^x$ with $y(0) = 1, y'(0) = 1$.

Hint. There is a solution $y(x) = \frac{1}{2}e^x$ to the differential equation. A general solution is given as the sum of $y(x) = \frac{1}{2}e^x$ and and a general solution of $y'' + y = 0$, that is, $C_1 \sin x + C_2 \cos x$, hence the general solution of $y'' - y = x^2$ is

$$y(x) = \frac{1}{2}e^x + C_1 \sin x + C_2 \cos x.$$