Mathematical Analysis I: Lecture 50

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Start recording...

Annoucements

- Office hour: Tuesday 11:30–12:30.
- Tutorial/more office hours?

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

- Today: Apostol Vol. 1, Chapter 8

Second-order linear differential equations

Let us next consider a linear differential equation of the form

$$y'' + P(x)y' + Q(x)y = R(x).$$

This contains the second derivative y'', hence it is a second-order differential equation. As in the case of first-order differential equations, we say that it is homogeneous if $R(x) = 0$.

Second-order linear differential equations

$$y'' + P(x)y' + Q(x)y = R(x).$$

If y_1, y_2 are two solutions of a homogeneous equation, then so is the linear combination $ay_1 + by_2$. If y_1, y_2 are two solutions of a linear equation, then the difference $y_1 - y_2$ is a solution of the equation where $R(x)$ is set to 0.

Second-order linear differential equations

A simplest of such equations is one where $P(x), Q(x), R(x)$ are constant:

$$y'' + ay' + by = 0.$$

As we will see, such equations appear naturally in physics.
An even simpler case is where $a = 0$:

$$y'' + by = 0.$$

Let us start with solutions of this type.

Second-order linear differential equations

$$y'' + by = 0.$$

- Case 1. $b = 0$. In this case, we have $y'' = 0$. This means that $y' = C_1$ (constant) and further $y = C_1 x + C_2$. It is easy to see that any solution is of this form.

$$y'' + by = 0.$$

- Case 2. $b < 0$. In this case, the equation can be written as $y'' = k^2 y$ where $b = -k^2$ and we can take easily check that $y(x) = C_1 e^{kx} + C_2 e^{-kx}$ is a solution for any constant C_1, C_2. Indeed, $y'(x) = kC_1 e^{kx} - kC_2 e^{-kx}$ and $y''(x) = k^2 C_1 e^{kx} + (-k)^2 C_2 e^{-kx} = k^2(C_1 e^{kx} + C_2 e^{-kx}) = k^2 y(x)$.

$$y'' + by = 0.$$

- Case 3. $b > 0$. In this case, the equation can be written as $y'' = -k^2 y$ where $b = k^2$. There are solutions of the form $y(x) = C_1 \sin(kx) + C_2 \cos(kx)$ is a solution for any constant C_1, C_2.

Second-order linear differential equations

Note that, in all these cases, there are two constants C_1, C_2. If we require an initial condition

- $y(a) = b_1$
- $y'(a) = b_2$

these constants are fixed.
For example, in Case 2 with $a = 0$, $y(x) = C_1 e^{kx} + C_2 e^{-kx}$, we should have $y(0) = C_1 + C_2 = b_1$ and $y'(0) = kC_1 - kC_2 = b_2$, hence $C_1 = \frac{1}{2}(b_1 + \frac{b_2}{k}), C_2 = \frac{1}{2}(b_1 - \frac{b_2}{k})$.
In general, for a second-order differential equation, we need to fix an inizial condition $y(a)$ and $y'(a)$.

Let us consider the general case

$$y'' + ay' + by = 0.$$

This can be reduced to the special case above as follows. We write $y(x) = u(x)e^{-\frac{ax}{2}}$, then

$$\begin{aligned}
y'(x) &= u'(x)e^{-\frac{ax}{2}} - \frac{a}{2}u(x)e^{-\frac{ax}{2}} = u'(x)e^{-\frac{ax}{2}} - \frac{a}{2}y(x) \\
y''(x) &= u''(x)e^{-\frac{ax}{2}} - \frac{a}{2}u'(x)e^{-\frac{ax}{2}} - \frac{a}{2}u'(x)e^{-\frac{ax}{2}} + \frac{a^2}{4}u(x)e^{-\frac{ax}{2}} \\
&= u''(x)e^{-\frac{ax}{2}} - au'(x)e^{-\frac{ax}{2}} + \frac{a^2}{4}u(x)e^{-\frac{ax}{2}}
\end{aligned}$$

Second-order linear differential equations

Therefore,, if y is a solution of this equation, it must hold that

$$\begin{aligned}
0 &= (u''(x)e^{-\frac{ax}{2}} - au'(x)e^{-\frac{ax}{2}} + \frac{a^2}{4}u(x)e^{-\frac{ax}{2}}) \\
&\quad + a(u'(x)e^{-\frac{ax}{2}} - \frac{a}{2}u(x)e^{-\frac{ax}{2}}) + bu(x)e^{-\frac{ax}{2}} \\
&= u''(x)e^{-\frac{ax}{2}} + \left(b - \frac{a^2}{4}\right)u(x)e^{-\frac{ax}{2}}
\end{aligned}$$

hence if u safisfies $u''(x) + \left(b - \frac{a^2}{4}\right)u(x) = 0$, then $y(x) = u(x)e^{-\frac{ax}{2}}$ satisfies $y'' + ay' + by = 0$. We know how to solve the former, hence so the latter.

Second-order linear differential equations

Example

Consider the equation $y'' + y' - y = 0$. Then we can write $y = ue^{-\frac{x}{2}}$ and then u should satisfy $u'' - \frac{5}{4}u = 0$. We know that $u(x) = C_1 e^{\frac{\sqrt{5}}{2}x} + C_2 e^{-\frac{\sqrt{5}}{2}x}$ is a solution of this, hence $y = C_1 e^{\frac{\sqrt{5}-1}{2}x} + C_2 e^{-\frac{\sqrt{5}+1}{2}x}$.

Second-order linear differential equations

Next, let us consider the inhomogeneous case, that is $y'' + ay' + by = R(x)$. In some cases we can find solutions.

Example

Take $R(x) = x^2$. Then $y(x) = \frac{1}{b}(x^2 - \frac{2ax}{b} + \frac{2a^2-2b}{b^2})$ is a solution. Indeed, $y'(x) = \frac{1}{b}(2x - \frac{2a}{b}), y''(x) = \frac{2}{b}$.
A general solution can be obtained by adding a solution of the homogeneous version $y'' + ay' + by = 0$ to this solution.

Physical examples

- Simple harmonic motion. Consider a mass m which is attached to a spring. Let us call $x(t)$ the position of the mass. When a spring is stretched by the distance r, then it pulls back the mass by the force kr. Similarly, when a spring is pressed by the distance r (hence the mass is displaced to $-r$), then it pushes back the mass by the force kr. Together with the direction of the force, it can be written as $-kx$. The equation of motion is about the variable $x(t)$ and the acceleration is $a(t) = x''(t)$, hence $F(x) = ma = mx''$ becomes

$$mx'' = F(x) = -kx.$$

That is, $x'' + \frac{k}{m}x = 0$.

Physical examples

- $x'' + \frac{k}{m}x = 0$. The general solution of this is

$$x(t) = C_1 \sin\sqrt{\frac{k}{m}}t + C_2 \cos\sqrt{\frac{k}{m}}t.$$

If we pull the mass to a and leave quietly at time $t = 0$, then the solution should have $x(0) = a, x'(0) = 0$. That is, $C_2 = a$ and $C_1 = 0$, and the special solutions is

$$x(t) = a\cos\sqrt{\frac{k}{m}}t.$$

This means that the mass oscilates between $-a$ and a.s
In general, if we specify the values $x(0)$ and $x'(0)$, then there is only one solution. These values are called the **initial conditions**.

Physical examples

- In addition to the previous example, let us consider the case where the mass lies on a floor hence receives the friction. The friction is proportional to the velocity and in the converse direction. Therefore, the equation of motion is

$$mx''(t) = -kx(t) - cx'(t),$$

or $x'' + \frac{c}{m}x + \frac{k}{m}x = 0$.

Physical examples

- $x'' + \frac{c}{m}x + \frac{k}{m}x = 0$.
  The solution is given by solving $u'' + (\frac{k}{m} - \frac{1}{4}(\frac{c}{m})^2)u = 0$. If $s^2 = \frac{k}{m} - \frac{1}{4}(\frac{c}{m})^2 < 0$, then we have
  $x(t) = C_1 e^{-(\frac{c}{2m} - s)t} + C_2 e^{-(\frac{c}{2m} + s)t}$.
  If $x(0) = a, x'(0) = 0$, then
  $C_1 + C_2 = a, -(\frac{c}{2m} - s)C_1 + -(\frac{c}{2m} + s)C_2 = 0$ hence $C_1 = C_2 = \frac{a}{2}$,

$$x(t) = \frac{a}{2}\left(e^{-(\frac{c}{2m} - s)t} + e^{-(\frac{c}{2m} + s)t}\right)$$

  Note that $\frac{c}{2m} > s$, hence this decays exponentially. This means that the mass arrives at 0 without going back and forth.
  We leave the remaining case $\frac{k}{m} - \frac{1}{4}(\frac{c}{m})^2 \geq 0$ as exercises.

Exercises

- Solve the following differential equation. $y'' + 4y = 0$ with $y(0) = 1, y'(0) = 1$.
- Solve the following differential equation. $y'' + 4y' = 0$ with $y(0) = 1, y'(0) = 1$.
- Solve the following differential equation. $y'' - y = x$ with $y(1) = 1, y'(1) = 1$.
- Solve the following differential equation. $y'' - y = x^2$ with $y(0) = 1, y'(0) = 1$.
- Solve the following differential equation. $y'' + y = e^x$ with $y(0) = 1, y'(0) = 1$.