Mathematical Analysis I: Lecture 49

Lecturer: Yoh Tanimoto

16/12/2020

Start recording...

Annoucements

- Office hour: Tuesday 11:30–12:30.
- Tutorial/more office hours?
- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30
- Today: Apostol Vol. 1, Chapter 8

First-order linear differential equations

Let us consider a differential equation

$$y' + P(x)y = 0.$$

Here, $P(x)$ is a known function and we have to find a function $y(x)$ which satisfies this equation. Such an equation is called **linear homogeneous differential equation** of first-order.
Linear means that there is no term containing $y^2, y^3, (y')^2$ etc.
Homogeneous means that the right-hand side (the term which does not depend on y) is 0.

First-order linear differential equations

$$y' + P(x)y = 0.$$

It is important to note that, if $y_1(x)$ and $y_2(x)$ are two solutions, then so is $y_3(x) = ay_1(x) + by_2(x)$, because

$$y_3'(x) = ay_1'(x) + by_2'(x) = -aP(x)y_1(x) - bP(x)y_2(x) = -P(x)y_3(x).$$

First-order linear differential equations

$$y' + P(x)y = 0.$$

This can be solved as follows:
Note that, as far as $y(x) \neq 0$, this can be written as

$$-P(x) = \frac{y'(x)}{y(x)} = D(\log y(x))$$

therefore, $\log y(x) = -\int P(x)dx - C$, or $y(x) = e^{-A(x)}$ with $A(x) = \int P(x)dx + C$.

Theorem

Assume that P is continuous in an interval I and $a \in I, b \in \mathbb{R}$. Then there is one and only one function $y(x)$ on I satisfying $y' + P(x)y = 0$ and $y(a) = b$.

Proof.

We put $y(x) = be^{-A(x)}$, where $A(x) = \int_a^x P(t)dt$. It is clear that $y(a) = b$ and $y'(x) = b(-A'(x))e^{-A(x)} = -P(x)be^{-A(x)} = -P(x)y(x)$ by the fundamental theorem of calculus.

If $h(x)$ is another solution, then consider $H(x) = h(x)e^{A(x)}$. We have

$$\begin{aligned} H'(x) &= h'(x)e^{A(x)} + h(x)P(x)e^{A(x)} \\ &= -P(x)h(x)e^{A(x)} + h(x)P(x)e^{A(x)} = 0 \end{aligned}$$

therefore, $H(x)$ is constant, and $H(a) = h(a) = b$, hence $h(x) = be^{-A(x)}$. $\square$

Next we us consider a differential equation

$$y' + P(x)y = Q(x),$$

which is called a **linear differential equation** of first-order. Note that, if $y_1(x)$ and $y_2(x)$ are two solutions, then $y_3(x) = y_1(x) - y_2(x)$ satisfies

$$\begin{aligned} y_3'(x) = y_1'(x) - y_2'(x) &= -P(x)y_1(x) + Q(x) - (-P(x)y_2 + Q(x)) \\ &= -P(x)y_3(x). \end{aligned}$$

First-order linear differential equations

Theorem

Assume that P, Q are continuous in an interval I and $a \in I, b \in \mathbb{R}$. Then there is one and only one function $y(x)$ on I satisfying $y' + P(x)y = Q(x)$ and $y(a) = b$.

Proof.

We put $y(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}dt$, where $A(x) = \int_a^x P(t)dt$. It is clear that $y(a) = b$ and

$$y'(x) = b(-P(x))e^{-A(x)} - P(x)e^{-A(x)} \int_a^x Q(t)e^{A(t)}dt + e^{-A(x)}Q(x)e^{A(x)}$$
$$= -P(x)y(x) + Q(x).$$

If $h(x)$ is another solution with $h(a) = b$, then consider $g(x) = h(x) - y(x)$ satisfies the equation $g'(x) = -P(x)g(x)$ and $g(a) = h(a) - y(a) = b - b = 0$, hence $g(x) = 0$ by the previous theorem. □

First-order linear differential equations

Example

- Find all solutions y of $y' + xy = 0$.

First-order linear differential equations

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- Find all solutions y of $y' + xy = 0$.
  With $P(x) = x$, we have $A(x) = \int_0^x x dx = \frac{x^2}{2}$ hence $y(x) = be^{-\frac{x^2}{2}}$.

First-order linear differential equations

Example

- Find all solutions y of $xy' + (1 - x)y = e^{2x}$.

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  $P(x) = \frac{1-x}{x}, Q(x) = \frac{e^{2x}}{x}$,

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  This can be written as $y' + \frac{1-x}{x}y = \frac{e^{2x}}{x}$, hence with $P(x) = \frac{1-x}{x}, Q(x) = \frac{e^{2x}}{x}$, we have $A(x) = \int_1^x \frac{1-x}{x}dx = \log x - x + 1$, hence $e^{-A(x)} = \frac{e^{x-1}}{x}$ and

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$$\int_1^x \frac{e^{2t}}{t} \cdot te^{1-t}dt = e(e^x - e),$$

  hence $y(x) = b\frac{e^{x-1}}{x} + \frac{e^{x-1}}{x}e(e^x - e) = \frac{e^{2x}}{x} + (\frac{b}{e} - e)\frac{e^x}{x}$.
  One can check that this satisfies the original equation.

First-order linear differential equations

- Consider a falling body in a resisting medium. For example, we drop a ball from a window. The gravitational force is constant g when the body moves the distance much shorter than the radius of the Earth. In addition, the ball is resisted by the air and the resistance is proportional to the velocity. To express this in a differential equation, let $v(t)$ be the velocity of the ball at time t, we leave it at time $t = 0$ from the height 0. Then

$$mv' = mg - kv,$$

with $v(0) = h$.

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with $v(0) = h$. This is of the form $v' + P(t)v = Q(t)$ with $P(t) = \frac{k}{m}, Q(t) = g$.
Its solution is $v(t) = e^{-\frac{kt}{m}} \int_0^t e^{\frac{kt}{m}} g dt = \frac{mg}{k}(1 - e^{-\frac{kt}{m}})$. Therefore, at time $t = 0$ the speed is 0, and it accelerates until the resistance and the gravitational force get to an equilibrium.

First-order linear differential equations

- Let us consider a small particle in a large medium. If the temperature of the particle and that of the medium is different, then the changing rate of the temperature is proportional to the difference of the temperature. As the medium is large, we may assume that only the temperature $y(t)$ changes with $y(0) = T$, while the medium remain in the same temperature M. In a differential equation,

$$y'(t) = k(M - y(t)).$$

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This is again of the form $y' + P(t)y = Q(t)$ with $P(t) = k, Q(t) = kM$. The solution is

$$y(t) = b(e^{-kt} - 1) + e^{kt} \int_0^t kMe^{-kt} dt = be^{-kt} + e^{kt} M(e^{-kt} - 1).$$

With $y(0) = T$ we have $b = T$, and altogether $y(t) = M + (T - M)e^{-kt}$. As $t \to \infty$, y tends to M.

Exercises

- Solve the following differential equation. $y' + 2x^2 y = 0$ with $y(0) = 2$.
- Solve the following differential equation. $y' + xe^x y = 0$ with $y(1) = 1$.
- Solve the following differential equation. $xy' - 3y = x^5$ with $y(0) = 1$.
- Solve the following differential equation. $y' + xy = x$ with $y(1) = 2$.
- A thermometer is stored in a room whose temperature is $35°\mathrm{C}$. Five minutes after being taken outdoor is $25°\mathrm{C}$. After another five minutes, it reads $20°\mathrm{C}$. Compute the outdoor temperature.
- The half-life for Caesium-137 is about 30 years. Compute the percentage of a given quantity of Caesium that disintegrates in 10 years.