Mathematical Analysis I: Lecture 48

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

- Today: Apostol Vol. 1, Chapter 8

(Ordinary) differential equations

Many scientific questions are expressed in terms of differential equation (equation about functions and their derivatives).

- The equation of motion in a gravitational field $m\frac{d^2x}{dt^2} = -\frac{mMG}{x^2}$
- The heat equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial u^2}{\partial x^2}$ (this is **partial differential equation**, because it contains partial derivatives, studied in Mathematical Analysis II) .
- The SIR model in epidemiology $\frac{dS}{dt} = -\frac{\beta S(t)I(t)}{N}, \frac{dI}{dt} = \frac{\beta S(t)I(t)}{N} - \gamma I(t), \frac{dR}{dt} = \gamma I(t)$

This is because the rate of change (the derivative) is often determined by the current status (the function). For example, in the equation of motion, the gravitational force $-\frac{mMG}{x^2}$ depends on the place of a particle $x(t)$, while the force determines the the rate of change of the speed (the acceleration), and the speed is $x'(t)$, hence the second derivative appears on the left-hand side.

Ordinary differential equations

Once the scientific problem is written in the form of differential equation, it is a mathematical problem to solve it, that is, to find functions that satisfies the given equation.
In the following, $y(x)$ will be a function of x and the derivatives are denoted by $y'(x), y''(x)$ and so on. Some more examples of differential equation are

- $y'(x) = y(x)$
- $y'(x) = x^3 y(x) + \sin(xy''(x))$
- Sometimes we just write this as $y' = x^3 y + \sin(xy'')$, keeping in mind that y is a function of x.

Ordinary differential equations

In a differential equation, certain higher derivative of y may appear. The highest order of the derivative of y is called the **order** of the differential equation. For example,

- $y'(x) = 2y(x)$ is a first-order differential equation.
- $y'(x) = x^3 y(x) + \sin(xy''(x))$ is a second-order differential equation.

We need to find functions $y(x)$ that satisfy the given equation. This is why it is called a differential equation. Compare it with an algebraic equation $x^2 + 3x - 4 = 0$, where we need to find numbers that satisfy this equation.

Ordinary differential equations

Let us consider first-order differential equations. In an abstract form, we can write it as

$$y' = f(x, y),$$

where f is explicitly written in examples, while y is the unknown functions which we need to find. In the example $y'(x) = 2y(x)$, we take $f(x, y) = 2y$. A **solution** of a differential equation is a (differentiable) function that satisfies this equation. For example, by taking $y(x) = Ce^{2x}$, we can check that this is a solution:

$$y'(x) = 2Ce^{2x} = 2y(x).$$

Some first-order differential equations

The simplest case is where f does not depend on y: that is,

$$y'(x) = f(x).$$

This means that f is the derivative of y, or y is a primitive of f. Therefore, y can be obtained by integrating f: $y(x) = \int f(x)dx + C$. Indeed, this y satisfies the given equation for any $C \in \mathbb{R}$, and there is no other solution.

Some first-order differential equations

Example

When a ball falls freely without drag, the speed $-gx$ is proportional to the time x. As the speed is the derivative of the position y, we have the equation

$$y'(x) = -gx.$$

This can be solved by integration, that is $y(x) = \int(-gx)dx = -\frac{gx^2}{2} + C$. The constant C depends on the position where the ball starts to fall.

As we see in this example, a differential equation may have many solutions. In practice, we are interested in one of them which satisfies additional conditions, the **initial conditions** or **boundary conditions**, that give the value of y, y' at a given time x.

Some first-order differential equations

Next, let us consider again the simplest differential equation $y' = f(x, y)$ where f depends on y.

Theorem

Let $a, C \in \mathbb{R}$. Then there is only one (differentiable) function y such that $y'(x) = ay(x)$ and $y(0) = C$.

Proof.

We know that there is one such function: $y(x) = Ce^{ax}$. Indeed, we can check that $y'(x) = aCe^{ax} = ay(x)$ and $y(0) = Ce^0 = C$.
Suppose that there is $g(x)$ with the same condition. Let $h(x) = e^{-ax}g(x)$, then $h'(x) = -ae^{-ax}g(x) + e^{-ax}g'(x) = -ae^{-ax}g(x) + ae^{-ax}g(x) = 0$ for all $x \in \mathbb{R}$, hence $h(x)$ must be a constant. As $h(0) = e^0 g(0) = C$, $h(x) = C$ hence $g(x) = Ce^{ax}$. □

Some first-order differential equations

Let us consider when we see the equation $y' = ay$.

- A very typical example is radioactive atoms. Let $y(x)$ be the number of a single species of radioactive atoms at time x. It is known that each atom decays, independently from other atoms, in a certain time period by a certain probability. This means that, at each moment, the rate of decrease in numbers $y(x)$ is proportional to $y(x)$. With a constant a, we can write this as

$$y'(x) = -ay(x).$$

  If there are C atoms at time $x = 0$, we know that the solution is $y(x) = Ce^{-ax}$, hence the number of radioactive atoms decays exponentially. This can be written more conveniently as $y(x) = Ce^{-ax} = C2^{-ax/\log 2}$. Then with $T = \frac{\log 2}{a}$, we have $y(x) = C2^{-x/T}$, and it is clear that the number of atoms halves in time T. T is called the **half life** of this particular species of atom.

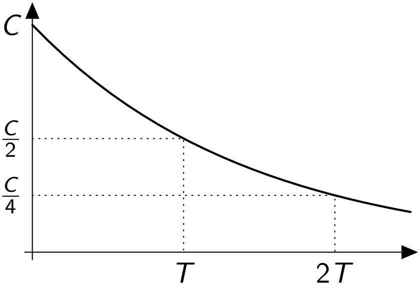

Figure: The exponential decay $y(x) = Ce^{-ax} = C2^{-ax/\log 2}$ The half-life is $T = \log 2/a$.

Some first-order differential equations

Let us consider when we see the equation $y' = ay$.

- Another instance is the SIR model in epidemiology. We consider the total population N, the numbers of $S(t)$ (succeptible), $I(t)$ (infected) and $R(t)$ (removed/recovered). It is assumed that each infected people has contact with a certain number of people in each day, hence this number is proportional to $\frac{S(t)}{I(t)}$, and assume that in each such contact transmission occurs by the rate β. On the other hand, each infected people lose infectivity by the rate γ.

$$\frac{dS}{dt} = -\frac{\beta S(t)I(t)}{N}$$
$$\frac{dI}{dt} = \frac{\beta S(t)I(t)}{N} - \gamma I(t)$$
$$\frac{dR}{dt} = \gamma I(t)$$

Some first-order differential equations

It is difficult to solve this set of equations. Yet, we can understand the behaviour when there are few infected people $I(t)$ compared to the total number N. When $I(t)$ is small, then $R(t)$ is also small and $S(t) = N - I(t) - R(t)$ is close to N. By putting $S(t) = N$, we have

$$\frac{dI}{dt} = (\beta - \gamma)I(t).$$

As a function of t, we know that $I(t) = Ce^{(\beta-\gamma)t}$, where C is the number of infected at day $t = 0$. This epidemic grows when $\beta - \gamma > 0$, and decays when $\beta - \gamma < 0$. $R_0 = \frac{\beta}{\gamma}$ is called the **basic reproduction number in the SIR model**. When $R_0 > 1$ the epidemic grows and when R_0 it decays.

Exercises

- Solve the following differential equation. $y' = 2y$ with $y(0) = 2$.
- Solve the following differential equation. $y' = -3y$ with $y(1) = -1$.
- Solve the following differential equation. $y' = x^3$ with $y(0) = 2$.
- Solve the following differential equation. $y' = e^{2x}$ with $y(1) = -1$.