Mathematical Analysis I: Lecture 47

Lecturer: Yoh Tanimoto

11/12/2020

Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 10:00–11:30.
- Office hour: Tuesday 11:30–12:30.

- Basic Mathematics: first few lessons on
  - Tuesday (14:00 – 16:00 CET): Inequalities, Limits and Derivatives
  - Wednesday (14:00 – 16:00 CET): Study of function

  and then upon request.
- A make up session on 22 December 11:30

Exercises

Compute the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

Solution. We have $\sum_{k=1}^{n} a^k = a + a^2 + a^3 + \cdots + a^n = \frac{a-a^{n+1}}{1-a}$.

$$\sum_{n=1}^{\infty} \frac{3}{4^n} = 3 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = 3 \frac{\frac{1}{4}}{1-\frac{1}{4}} = 3 \frac{\frac{1}{4}}{\frac{3}{4}} = 1$$

Note: We have $\sum_{k=0}^{n} a^k = 1 + a + a^2 + a^3 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$.

Compute the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.
Solution. Note that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2(n+1)} \frac{(n+2)-n}{n(n+2)} = \frac{1}{2(n+1)} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ is a telescopic series with $b_n = \frac{1}{2n(n+1)}$, and $\lim_{n\to\infty} b_n = 0$, therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 = \frac{1}{4}$$

Compute the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.
Solution. Note that

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{1}{2}\frac{(n+1)-(n-1)}{(n+1)(n-1)} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

Therefore,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{1}{n^2-1} \\
&= \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \frac{1}{5^2-1} + \cdots \\
&= \frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10} + \frac{1}{8} - \frac{1}{12} + \cdots = \frac{1}{2} + \frac{1}{4} = \frac{3}{8}
\end{aligned}$$

Compute the series $\sum_{n=1}^{\infty} \frac{1+n}{n!}$.
Solution. Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

hence $e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$, and $\sum_{n=1}^{\infty} \frac{1}{n!} = (1 + \frac{1}{2} + \frac{1}{6} + \cdots) = e - 1$. Similarly, $\sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.
Therefore, the sum is $e - 1 + e = 2e - 1$.

Determine whether $\sum_n \frac{n}{n^2+1}$ converges.
Solution. This diverges because $\sum_n \frac{1}{n}$ diverges and

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

Determine whether $\sum_n \frac{(n!)^2}{(2n)!}$ converges.
Solution.
Put $a_n = \frac{(n!)^2}{(2n)!}$. Then

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{\frac{(n+1)!^2}{2(n+1)!}}{\frac{(n!)^2}{(2n)!}} = \lim_n \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

Hence by ratio test this converges.

Determine whether $\sum_n \frac{2^n+3^n}{5^n}$ converges.
Hint. Use root test, and $2^n + 3^n = 3^n((\frac{2}{3})^n + 1)$.
Or, compute two terms separately (geometric series).

Determine whether $\sum_n \frac{\log n}{n^2}$ converges.
Solution.
We know that $\sum_n \frac{1}{n^{\frac{3}{2}}}$ converges, and

$$\frac{\frac{\log n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}} = \frac{\log n}{n^{\frac{1}{2}}} \to 0.$$

Hence this converges by comparison.
Or one can use also the condensation principle.

Determine whether $\sum_n \frac{1}{(\log n)^2}$ converges.
Solution. Note that $(\log n)^2 < n$ for sufficiently large n, hence $\frac{1}{n} < \frac{1}{(\log n)^2}$, and we know that $\sum_n \frac{1}{n}$ diverges, hence also the given series diverges.
Or use the condensation principle.

Determine whether $\sum_n \frac{(-1)^n n}{n^2+1}$ converges.
Solution. This is an alternating series and with $a_n = \frac{n}{n^2+1}$, $a_n \to 0$ and a_n is decreasing (because $f(x) = \frac{x}{x^2+1}$ is decreasing for sufficiently large x, by computing the derivative).
Hence by Leibniz criterion this is convergent.

Determine whether $\sum_n \frac{(-1)^n}{n \log n}$ converges.
Solution. Use the Leibniz criterion with $a_n = \frac{1}{n \log n}$ and this is convergent.

Determine for which x, $\sum_n \frac{x^{2n}}{x^{2n}+1}$ converges.
Solution.
Put $a_n = \frac{x^{2n}}{x^{2n}+1}$.
If $|x| > 1$, then $a_n = \frac{1}{1+\frac{1}{x^{2n}}} \to 1$, hence the series does not converge.
If $|x| = 1$, then $a_n = \frac{1}{2}$ hence the series does not converge.
If $|x| < 1$, then $a_n < x^{2n}$ and $\sum_n x^{2n}$ converges (geometric series), hence also the given series converges.

Determine for which x, $\sum_n \frac{n^2 x^{2n}}{(2n)!}$ converges.

Solution. Converges for all x. Indeed, if we put $a_n = \frac{n^2 x^{2n}}{(2n)!}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2 x^{2(n+1)}}{(2(n+1)!}}{\frac{n^2 x^{2n}}{(2n)!}} = \frac{(n+1)^2 x^2}{n^2(2n+2)(2n+1)} \to 0$$

for all $x \in \mathbb{R}$. Therefore, by the ratio test, this converges for all $x \in \mathbb{R}$.

Determine for which x, $\sum_n \frac{(-1)^n x^n}{n}$ converges.

Solution. Let us put $a_n = \frac{|x|^n}{n}$. This is a positive sequence, and $\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1} n}{|x|^n (n+1)} \frac{|x|n}{n+1}$ to $|x|$. Therefore, if $|x| < 1$, then the series converges absolutely.

On the other hand, if $|x| > 1$, then a_n diverges and in particular the series does not converge.

Finally, if $x = 1$, the series is $\sum_n \frac{(-1)^n}{n}$ which converges by Leibniz criterion. If $x = -1$, then the series is $\sum_n \frac{(-1)^n (-1)^n}{n} = \sum_n \frac{1}{n}$ which diverges.

Altogether, the series converges if and only if $x \in (-1, 1]$.